Mathematical
Inequalities
& Applications
Volume 6, Number 3 (2003), 497–508

EQUIVALENT QUASINORMS FOR THE ANISOTROPIC NIKOL'SKIĬ–BESOV SPACES ON A CONE OF FUNCTIONS WITH A REGULAR FOURIER TRANSFORM

V. I. BURENKOV AND G. E. GARCÍA ALMEIDA

(*communicated by L. Pick*)

Abstract. We prove the equivalence of the quasinorms of the anisotropic Nikol'skiĭ-Besov spaces to simpler quasinorms on a cone of functions with positive Fourier transforms satisfying some regularity conditions.

1. Introduction

The main aim of the article is the proof of the equivalence of the quasinorms of the anisotropic Nikol'skiĭ-Besov spaces $B^{\vec{s}}_{p,\theta}\left(\mathbb{R}^N\right)$ to simpler quasinorms, which are weighted $L_\theta\left(\mathbb{R}^N\right)$ quasinorms of the Fourier transforms with power weights, on a cone of functions with sufficiently smooth positive Fourier transforms satisfying some further regularity conditions. The theorem proven here is an extension of the result, obtained by Batyrov & Burenkov [3], [4] for the isotropic Nikol'skiĭ-Besov spaces.

2. Definitions and basic properties

In this section we give main definitions, notation and basic properties that will be required to prove the main result in the next section.

2.1. Definition of the anisotropic Nikol'skiĭ-Besov spaces

Suppose[1] that $0 < p,\ \theta \leqslant \infty$, $\vec{s} = (s_1, \dots, s_N)$, where $-\infty < \vec{s} < \infty$, and all s_j have the same sign. Thus either $\vec{s} > 0$ or $\vec{s} < 0$ or $\vec{s} = 0$. If $\vec{s} > 0$ or $\vec{s} < 0$, let $\dfrac{1}{s} = \dfrac{1}{N}\sum_{j=1}^{N}\dfrac{1}{s_j}$. Furthermore, let $\vec{a} = \dfrac{s}{\vec{s}}$, i.e. $\vec{a} = (a_1, \dots, a_N)$, where

[1] We assume here and in the sequel that for $\vec{a} = (a_1, \dots, a_N)$ and $\vec{b} = (b_1, \dots, b_N)$ the inequalities $\vec{a} < \vec{b}$, $\vec{a} \leqslant \vec{b}$ mean that $a_j < b_j$, $a_j \leqslant b_j$ respectively, for all $j \in \{1, \dots, N\}$. If $b \in \mathbb{R}$, then $\vec{a} < b$, $\vec{a} \leqslant b$, $\vec{a} = b$ mean that $a_j < b$, $a_j \leqslant b$, $a_j = b$ respectively for all $j \in \{1, \dots, N\}$.

Mathematics subject classification (2000): 46E35; 42B25.
Key words and phrases: anisotropic function spaces, fractional smoothness, Nikol'skiĭ-Besov.

$a_j = \dfrac{s}{s_j}$, $j = 1,\dots,N$. Note that all $a_j > 0$ and $\sum_{j=1}^{N} a_j = N$. Thus, s is a certain mean smoothness and $\vec{a}$ measures the anisotropy. If $\vec{s} = 0$, then $s = 0$ and $\vec{a}$ is an arbitrary vector with positive components satisfying $\sum_{j=1}^{N} a_j = N$. (For different $\vec{a}$ one gets different spaces.) In the sequel $\vec{a}$ will be called the *anisotropy vector*. Define the anisotropic distance of $t = (t_1,\dots,t_N) \in \mathbb{R}^N$ from the origin as $|t|_{\vec{a}} = \left(\sum_{j=1}^{N} |t_j|^{\frac{2}{a_j}}\right)^{\frac{1}{2}}$ and the anisotropic ball $B_r = \left\{t \in \mathbb{R}^N : |t|_{\vec{a}} < r\right\}$. Next, for $k \in \mathbb{N}_0$, let $\varphi_k \in C_0^\infty\left(\mathbb{R}^N\right)$, $\varphi_k \geqslant 0$, and supp $\varphi_0 \subset B_2$; supp $\varphi_k \subset B_{2^{k+1}} \setminus B_{2^{k-1}}$ for all $k \in \mathbb{N}$. Moreover, let for every multi-index α

$$\sup_{k\in\mathbb{N}_0}\ \sup_{t\in\mathbb{R}^N} 2^{k\langle \vec{a},\alpha\rangle} \left|\left(D^\alpha \varphi_k\right)(t)\right| < \infty, \tag{1}$$

where $\langle \vec{a},\alpha\rangle = \sum_{k=1}^{N} a_k\alpha_k$, and for all $t \in \mathbb{R}^N$ $\sum_{k=0}^{\infty} \varphi_k(t) = 1$. One defines $\Phi^{\vec{a}}\left(\mathbb{R}^N\right)$ as the set of all the collections $\{\varphi_k\}_{k=0}^\infty$ with the properties listed above.

DEFINITION 1. (Nikol'skiĭ-Besov spaces). *One says that* $f \in B^{\vec{s}}_{p,\theta}\left(\mathbb{R}^N\right)$ *if* $f \in S'\left(\mathbb{R}^N\right)$*, the Schwartz space of tempered distributions, and*

$$\|f\|_{B^{\vec{s}}_{p,\theta}\left(\mathbb{R}^N\right)} = \left(\sum_{k=0}^{\infty}\left(2^{ks}\left\|F^{-1}\left(\varphi_k F(f)\right)\right\|^{\theta}_{L_p\left(\mathbb{R}^N\right)}\right)\right)^{\frac{1}{\theta}} < \infty \tag{2}$$

for $\theta < \infty$ *or*

$$\|f\|_{B^{\vec{s}}_{p,\infty}\left(\mathbb{R}^N\right)} = \sup_{k\in\mathbb{N}_0}\left(2^{ks}\left\|F^{-1}\left(\varphi_k F(f)\right)\right\|_{L_p\left(\mathbb{R}^N\right)}\right) < \infty \tag{3}$$

for $\theta = \infty$*, where* $\{\varphi_k\}_{k=0}^\infty \in \Phi^{\vec{a}}\left(\mathbb{R}^N\right)$.

If $1 \leqslant p,\ \theta \leqslant \infty$, then (2) and (3) are norms, in general case they are quasinorms. For different collections $\{\varphi_k\}_{k=0}^\infty \in \Phi^{\vec{a}}\left(\mathbb{R}^N\right)$ quasinorms (2) (or (3)) are equivalent[1]. The norm $\|f\|_{B^{\vec{s}}_2\left(\mathbb{R}^N\right)} := \|f\|_{B^{\vec{s}}_{2,2}\left(\mathbb{R}^N\right)}$ is equivalent to

$$\|f\|^{(1)}_{B^{\vec{s}}_2\left(\mathbb{R}^N\right)} = \left\|\left(1+|\omega|^2_{\vec{a}}\right)^{\frac{s}{2}}(Ff)(\omega)\right\|_{L_2\left(\mathbb{R}^N\right)},$$

which for $\vec{s} \geqslant 0$ is equivalent to

$$\|f\|^{(2)}_{B^{\vec{s}}_2\left(\mathbb{R}^N\right)} = \left\|\left(1+|\omega_1|^{s_1}+\cdots+|\omega_N|^{s_N}\right)(Ff)(\omega)\right\|_{L_2\left(\mathbb{R}^N\right)},$$

[1] Let X be a quasinormed space with the quasinorm $\|\cdot\|_1$, and let the quasinorm $\|\cdot\|_2$ be defined on a subset $Y \subset X$. One says that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent on Y (briefly $\|\cdot\|_1 \sim \|\cdot\|_2$ on Y) if there exist two constants $a,\ b > 0$ such that $a\|x\|_1 \leqslant \|x\|_2 \leqslant b\|x\|_1$ for all $x \in Y$. If $Y = X$, one says that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent briefly $\left(\|\cdot\|_1 \sim \|\cdot\|_2\right)$.

and for $\vec{s} \leqslant 0$ to

$$\|f\|^{(3)}_{B_2^{\vec{s}}(\mathbb{R}^N)} = \left\| \left(1 + |\omega_1|^{|s_1|} + \cdots + |\omega_N|^{|s_N|}\right)^{-1} (Ff)(\omega) \right\|_{L_2(\mathbb{R}^N)}.$$

If $\vec{s} > 0$ and $1 \leqslant p,\ \theta \leqslant \infty$, then for $\theta < \infty$ norm (2) is equivalent to

$$\|f\|^{(4)}_{B^{\vec{s}}_{p,\theta}(\mathbb{R}^N)} = \|f\|_{L_p(\mathbb{R}^N)} + \sum_{j=1}^{N} \left(\int_0^\infty \left(h^{-\gamma_j} \left\| \Delta^2_{h,j} \left(\frac{\partial^{r_j} f}{\partial x_j^{r_j}} \right) \right\|_{L_p(\mathbb{R}^N)} \right)^{\theta} \frac{dh}{h} \right)^{\frac{1}{\theta}}$$

and norm (3) is equivalent to

$$\|f\|^{(4)}_{B^{\vec{s}}_{p,\infty}(\mathbb{R}^N)} = \|f\|_{L_p(\mathbb{R}^N)} + \sum_{j=1}^{N} \sup_{h>0} \left(h^{-\gamma_j} \left\| \Delta^2_{h,j} \left(\frac{\partial^{r_j} f}{\partial x_j^{r_j}} \right) \right\|_{L_p(\mathbb{R}^N)} \right).$$

Here r_j is the greatest integer which is less than s_j, $\gamma_j = s_j - r_j$; $\Delta^2_{h,j} = f(x + 2he_j) - 2f(x + he_j) + f(x)$, where $h \in \mathbb{R}$ and e_j is the j^{th} unit vector in $\mathbb{R}^N$, the one with 1 in the j^{th} place and 0 in the other places; and $\dfrac{\partial^{r_j} f}{\partial x_j^{r_j}}$ are weak derivatives.

When $s_1 = \cdots = s_N = s$ we have the isotropic case, and we write $B^s_{p,\theta}\left(\mathbb{R}^N\right)$ for $B^{(s,\dots,s)}_{p,\theta}\left(\mathbb{R}^N\right)$.

2.2. Embedding theorems for the anisotropic Nikol'skiĭ-Besov spaces

Let $0 < p,\ \theta \leqslant \infty,\ -\infty < \vec{s} < \infty$ and all the components of $\vec{s}$ have the same sign. Then the following embeddings are valid:

$$B^{\vec{s}}_{p,\theta}\left(\mathbb{R}^N\right) \hookrightarrow B^{\vec{s}}_{p,\theta_1}\left(\mathbb{R}^N\right) \tag{4}$$

where $\theta < \theta_1 \leqslant \infty$;

$$B^{\vec{s}+\vec{\varepsilon}}_{p,\theta_1}\left(\mathbb{R}^N\right) \hookrightarrow B^{\vec{s}}_{p,\theta}\left(\mathbb{R}^N\right) \hookrightarrow B^{\vec{s}-\vec{\varepsilon}}_{p,\theta_2}\left(\mathbb{R}^N\right), \tag{5}$$

where $\vec{\varepsilon} > 0$ is proportional to $\vec{s}$, $0 < \theta_1,\ \theta_2 \leqslant \infty$;

$$B^{\vec{s}}_{p,\theta}\left(\mathbb{R}^N\right) \hookrightarrow B^{\vec{\rho}}_{q,\theta}\left(\mathbb{R}^N\right), \tag{6}$$

where $p < q \leqslant \infty$, $\vec{\rho} = \varkappa\vec{s}$, $\varkappa = 1 - \left(\frac{1}{p} - \frac{1}{q}\right)\frac{N}{s}$. (If $\vec{s} = 0$, then $\vec{\rho} = -\left(\frac{1}{p} - \frac{1}{q}\right)\frac{N}{\vec{a}}$, where $\vec{a}$ is the anisotropy vector defining $B^{\vec{0}}_{p,\theta}\left(\mathbb{R}^N\right)$. If $\vec{s} \neq 0$ and $\varkappa = 0$, hence $\vec{\rho} = 0$, then the anisotropy vector defining $B^{\vec{0}}_{p,\theta}\left(\mathbb{R}^N\right)$ is equal to $\frac{s}{\vec{s}}$.)

Here continuous embedding $\hookrightarrow$ is defined as follows:

DEFINITION 2. *Let X, Y be two quasinormed spaces. If $X \subset Y$ and in addition there exists $c > 0$ such that*

$$\|x\|_Y \leqslant c\,\|x\|_X\,,$$

by all $x \in X$, where $\|\cdot\|_Y$, $\|\cdot\|_X$ are the quasinorms in Y, X respectively, then it is said that the continuous embedding

$$X \hookrightarrow Y$$

holds.

For more details about Nikol'skiĭ-Besov spaces see Nikol'skiĭ [6], Besov, Il'in and Nikol'skiĭ [1], Triebel [8]–[11], Schmeisser and Triebel [7], Burenkov [2].

3. Equivalent quasinorms on a cone of functions with a regular Fourier transform

THEOREM. *Assume that $0 < p,\ \theta \leqslant \infty,\ -\infty < \vec{s} < \infty$, and all the components of $\vec{s}$ have the same sign. Let $\vec{a} = \frac{s}{\vec{s}}$ if $\vec{s} \neq 0$. If $\vec{s} = 0$, let $\vec{a}$ be an arbitrary vector with positive components, satisfying $\sum_{j=1}^{N} a_j = N$. Moreover, for $\lambda > 1$, let $\Lambda(\lambda, \vec{a}, p)$ denote the cone of all functions $f \in S'\left(\mathbb{R}^N\right)$ satisfying the following conditions:*

1) $(Ff)(\xi) > 0$ *for all* $\xi \in \mathbb{R}^N$,

2) *for* $\xi,\ \eta \in \mathbb{R}^N$

$$\frac{1}{2} \leqslant \frac{|\xi|_{\vec{a}}}{|\eta|_{\vec{a}}} \leqslant 2 \Rightarrow \frac{1}{\lambda} \leqslant \frac{(Ff)(\xi)}{(Ff)(\eta)} \leqslant \lambda, \tag{7}$$

3) *for all* $\alpha \in \mathbb{N}_0^N$ *satisfying*

$$|\alpha| \leqslant \frac{1}{p} + 1 \tag{8}$$

the derivatives $D^\alpha Ff$ are continuous on $\mathbb{R}^N$ and

$$|(D^\alpha Ff)(\xi)| \leqslant \lambda \left(1 + |\xi|_{\vec{a}}^2\right)^{-\frac{\langle \vec{a}, \alpha\rangle}{2}} (Ff)(\xi)\,. \tag{9}$$

Then for all $\lambda > 1$

$$\|f\|_{B^{\vec{s}}_{p,\theta}\left(\mathbb{R}^N\right)} \sim \left\| \left(1 + |\xi|_{\vec{a}}^2\right)^{\frac{1}{2}\left(s - \frac{N}{p} + \frac{N}{\theta'}\right)} (Ff)(\xi) \right\|_{L_\theta\left(\mathbb{R}^N\right)} \tag{10}$$

on $\Lambda(\lambda, \vec{a}, p)$, where $\frac{1}{\theta} + \frac{1}{\theta'} = 1$.

In the isotropic case $\vec{s}=(s,\ldots,s)$, hence $\vec{a}=\vec{1}\equiv(1,\ldots,1)$, and (10) reduces to the equivalence

$$\|f\|_{B^s_{p,\theta}(\mathbb{R}^N)}\sim\left\|\left(1+|\xi|^2\right)^{\frac{1}{2}\left(s-\frac{N}{p}+\frac{N}{\theta'}\right)}(Ff)(\xi)\right\|_{L_\theta(\mathbb{R}^N)} \tag{11}$$

on $\Lambda\left(\lambda,\vec{1},p\right)$, which has been established in Batyrov & Burenkov [3], [4].

Proof.

Step 1. Note that without loss of generality we can choose $\{\varphi_k\}_{k\in\mathbb{N}_0}\in\Phi^{\vec{a}}\left(\mathbb{R}^N\right)$ given in the definition of the anisotropic Nikol'skiĭ-Besov spaces satisfying the following conditions:

$\varphi_k(x)=1$ for $x\in B_{2^{k+\frac{1}{3}}}\setminus B_{2^{k-\frac{1}{3}}}$ and $\operatorname{supp}\varphi_k\subset B_{2^{k+\frac{2}{3}}}\setminus B_{2^{k-\frac{2}{3}}}$ for $k\in\mathbb{N}$;

$\varphi_0(x)=1$ for $x\in B_{2^{\frac{1}{3}}}$ and $\operatorname{supp}\varphi_0\subset B_{2^{\frac{2}{3}}}$.

Step 2. Let $f\in\Lambda(\lambda,\vec{a},p)$, then for all $c_1\geqslant 2$ there exists $c_2>0$, depending only on c_1, λ and $\vec{a}$, such that

$$c_1^{-1}\leqslant\frac{|\xi|_{\vec{a}}}{|\eta|_{\vec{a}}}\leqslant c_1\Rightarrow c_2^{-1}\leqslant\frac{(Ff)(\xi)}{(Ff)(\eta)}\leqslant c_2. \tag{12}$$

This follows by condition $2)$. First, by induction one can prove that for all $m\in\mathbb{N}$

$$2^{-m}\leqslant\frac{|\xi|_{\vec{a}}}{|\eta|_{\vec{a}}}\leqslant 2^m\Rightarrow\lambda^{-m}\leqslant\frac{(Ff)(\xi)}{(Ff)(\eta)}\leqslant\lambda^m. \tag{13}$$

Indeed, for $m=1$ this is condition $2)$. Assume that (13) holds for $m=k$ and let

$$2^{-k-1}\leqslant\frac{|\xi|_{\vec{a}}}{|\eta|_{\vec{a}}}\leqslant 2^{k+1}.$$

If $|\eta|_{\vec{a}}\leqslant|\xi|_{\vec{a}}$, take $\eta_1\in\mathbb{R}^N$ such that $|\eta_1|_{\vec{a}}=2|\eta|_{\vec{a}}$. Then

$$\frac{1}{2}\leqslant\frac{|\xi|_{\vec{a}}}{|\eta_1|_{\vec{a}}}\leqslant 2^k$$

and by (13) with $m=k$ and $m=1$

$$\lambda^{-k-1}(Ff)(\eta)\leqslant\lambda^{-k}(Ff)(\eta_1)\leqslant(Ff)(\xi)\leqslant\lambda^k(Ff)(\eta_1)\leqslant\lambda^{k+1}(Ff)(\eta).$$

If $|\eta|_{\vec{a}}>|\xi|_{\vec{a}}$, then a similar argument works if one takes $\eta_2\in\mathbb{R}^N$ such that $|\eta_2|_{\vec{a}}=\frac{1}{2}|\eta|_{\vec{a}}$.

For arbitrary $c_1\geqslant 2$ we choose $m\in\mathbb{N}$ such that $2^m\leqslant c_1<2^{m+1}$, i.e. $m=\left\lfloor\frac{\ln c_1}{\ln 2}\right\rfloor$. Then (12) follows with $c_2=\lambda^{m+1}$. Here $\lfloor x\rfloor$ denotes the integer part of x.

Step 3. Let $f\in\Lambda(\lambda,\vec{a},p)$. Then for all $c_1\geqslant 2$ there exists $c_2>0$, depending only on c_1, λ, $\vec{a}$ and N, such that

$$|\xi|_{\vec{a}}\,, |\eta|_{\vec{a}} \leqslant c_1 \Rightarrow c_2^{-1} \leqslant \frac{(Ff)(\xi)}{(Ff)(\eta)} \leqslant c_2. \tag{14}$$

This follows by condition 3) with $|\alpha| = 1$. Indeed, $|\xi_j - \eta_j| \leqslant 2c_1^a$, $j = 1, \ldots, N$, where $a = \max\limits_{j=1,\ldots,N} a_j$ and

$$\begin{aligned}
\left|\ln \frac{(Ff)(\xi)}{(Ff)(\eta)}\right| &= |(\ln Ff)(\xi) - (\ln Ff)(\eta)| \\
&= \left| \sum_{j=1}^{N} (\xi_j - \eta_j) \int_0^1 \left(\frac{\partial \ln Ff}{\partial x_j}\right)(\eta + t(\xi - \eta))\, dt \right| \\
&\leqslant 2c_1^a \sum_{j=1}^{N} \sup_{x\in\mathbb{R}^N} \left| \left(\frac{\partial \ln Ff}{\partial x_j}\right)(x) \right| \\
&\leqslant 2Nc_1^a \max_{j=1,\ldots,N} \sup_{x\in\mathbb{R}^N} \frac{\left|\left(\frac{\partial Ff}{\partial x_j}\right)(x)\right|}{(Ff)(x)} \leqslant 2N\lambda c_1^a.
\end{aligned}$$

Hence

$$e^{-2N\lambda c_1^a} \leqslant \frac{(Ff)(\xi)}{(Ff)(\eta)} \leqslant e^{2N\lambda c_1^a}.$$

Step 4. There exist c_3, $c_4 > 0$ such that

$$\left|\left(F^{-1}\varphi_k Ff\right)(x)\right| \leqslant c_3 2^{kN} (Ff)\left(2^{k\vec{a}}\right) \tag{15}$$

for all $f \in \Lambda(\lambda, \vec{a}, p)$, $k \in \mathbb{N}_0$ and $x \in \mathbb{R}^N$, where $2^{k\vec{a}} = \left(2^{ka_1}, \ldots, 2^{ka_N}\right)$, and

$$\left|\left(F^{-1}\varphi_k Ff\right)(x)\right| \geqslant c_4 2^{kN} (Ff)\left(2^{k\vec{a}}\right) \tag{16}$$

for all $f \in \Lambda(\lambda, \vec{a}, p)$, $k \in \mathbb{N}_0$ and $x \in \mathbb{R}^N$ satisfying

$$|x|_{\vec{a}} \leqslant N^{-\frac{1}{b}} 2^{-k-1}, \tag{17}$$

where $b = \min\limits_{j=1,\ldots,N} a_j$.

Indeed, by the properties of the functions φ_k (see Section 2.1 and Step 1) for all $x \in \mathbb{R}^N$

$$\begin{aligned}
\left|\left(F^{-1}\varphi_k Ff\right)(x)\right| &= (2\pi)^{-\frac{N}{2}} \left| \int_{\mathbb{R}^N} e^{ix\cdot\xi} \varphi_k(\xi) (Ff)(\xi)\, d\xi \right| \\
&\leqslant (2\pi)^{-\frac{N}{2}} \int_{A_k} (Ff)(\xi)\, d\xi,
\end{aligned}$$

where $A_0 = B_2$ and $A_k = B_{2^{k+1}} \setminus B_{2^{k-1}}$ for $k \in \mathbb{N}$. Since

$$\xi \in A_0 \implies |\xi|_{\overrightarrow{a}}\,, |\overrightarrow{1}|_{\overrightarrow{a}} \leqslant \max\left\{2, \sqrt{N}\right\},$$

where $\vec{1} = (1, \dots, 1)$, and for $k \in \mathbb{N}$

$$\xi \in A_k \Rightarrow \frac{1}{2\sqrt{N}} \leqslant \frac{|\xi|_{\vec{a}}}{\left|2^{k\vec{a}}\right|_{\vec{a}}} = \frac{|\xi|_{\vec{a}}}{\sqrt{N}2^k} \leqslant \frac{2}{\sqrt{N}},$$

by Steps 2 and 3 it follows that, for some $c_5 > 0$

$$c_5^{-1}(Ff)\left(2^{k\vec{a}}\right) \leqslant (Ff)(\xi) \leqslant c_5 (Ff)\left(2^{k\vec{a}}\right) \tag{18}$$

for all $k \in \mathbb{N}_0$ and $\xi \in A_k$. Hence (15) follows.

On the other hand

$$\begin{aligned} &\left|\left(F^{-1}\varphi_k Ff\right)(x)\right| \\ &\quad = (2\pi)^{-\frac{N}{2}} \left| \int_{A_k} \cos x\cdot\xi \, \varphi_k(\xi)(Ff)(\xi)\, d\xi + i \int_{A_k} \sin x\cdot\xi \, \varphi_k(\xi)(Ff)(\xi)\, d\xi \right| \\ &\quad \geqslant (2\pi)^{-\frac{N}{2}} \left| \int_{A_k} \cos x\cdot\xi \varphi_k(\xi)(Ff)(\xi)\, d\xi \right|. \end{aligned}$$

If $|x|_{\overrightarrow{a}} \leqslant N^{-\frac{1}{b}} 2^{-k-1}$ and $\xi \in A_k$, hence $|\xi|_{\overrightarrow{a}} \leqslant 2^{k+1}$, then $|x_j| \leqslant N^{-\frac{a_j}{b}} 2^{-a_j(k+1)}$, $|\xi_j| \leqslant 2^{a_j(k+1)}$ and

$$|x\cdot\xi| \leqslant \sum_{j=1}^{N} |x_j|\,|\xi_j| \leqslant \sum_{j=1}^{N} N^{-\frac{a_j}{b}} \leqslant 1.$$

Hence $\cos x\cdot\xi \geqslant \cos 1$ and by applying (18) we get

$$\begin{aligned} \left|\left(F^{-1}\varphi_k Ff\right)(x)\right| &\geqslant (2\pi)^{-\frac{N}{2}} \cos 1 \int_{A_k} \varphi_k(\xi)(Ff)(\xi)\, d\xi \\ &\geqslant c_6 (Ff)\left(2^{k\vec{a}}\right) \int_{A_k} \varphi_k(\xi)\, d\xi, \end{aligned}$$

where $c_6 = c_5^{-1}(2\pi)^{-\frac{N}{2}} \cos 1$.

Since $\varphi_k(\xi) = 1$ for $\xi \in B_{2^{k+\frac{1}{3}}} \setminus B_{2^{k-\frac{1}{3}}}$, we have

$$\int_{A_k} \varphi_k(\xi)\, d\xi \geqslant \operatorname{meas} B_{2^{k+\frac{1}{3}}} - \operatorname{meas} B_{2^{k-\frac{1}{3}}} = 2^{kN}\left(2^{\frac{N}{3}} - 2^{-\frac{N}{3}}\right) \operatorname{meas} B_1,$$

because, by the scaling argument, it follows that

$$\operatorname{meas} B_r = r^{a_1+\cdots+a_N} \operatorname{meas} B_1 = r^N \operatorname{meas} B_1, \ r > 0.$$

Hence inequality (16) follows.

Step 5. There exist c_5, $c_6 > 0$ such that

$$c_5 2^{\frac{kN}{p'}} (Ff)\left(2^{k\vec{a}}\right) \leqslant \left\|F^{-1}\varphi_k Ff\right\|_{L_p\left(\mathbb{R}^N\right)} \leqslant c_6 2^{\frac{kN}{p'}} (Ff)\left(2^{k\vec{a}}\right) \tag{19}$$

for all $f \in \Lambda(\lambda, \vec{a}, p)$ and $k \in \mathbb{N}_0$.

From (15), (16) and (17) it follows that for some c_7, $c_8 > 0$

$$c_7 2^{\frac{kN}{p'}} (Ff)\left(2^{k\vec{a}}\right) \leqslant \left\|F^{-1}\varphi_k Ff\right\|_{L_p\left(|x|_{\vec{a}} \leqslant N^{-\frac{1}{b}} 2^{-k-1}\right)} \leqslant c_8 2^{\frac{kN}{p'}} (Ff)\left(2^{k\vec{a}}\right)$$

for all $f \in \Lambda(\lambda, \vec{a}, p)$ and $k \in \mathbb{N}_0$.

So it remains to estimate $\left\|F^{-1}\varphi_k Ff\right\|_{L_p\left(|x|_{\vec{a}} \geqslant N^{-\frac{1}{b}} 2^{-k-1}\right)}$ above.

Step 6. Note that

$$|x_j|^{\frac{1}{a_j}} < \frac{r}{\sqrt{N}}, \quad j = 1, \ldots, N \Rightarrow |x|_{\vec{a}} < r.$$

Hence if we define cuboids Q_r as

$$Q_r = \left\{x \in \mathbb{R}^N : |x_j| < r^{a_j},\ j = 1, \ldots, N\right\},$$

then

$$Q_{\frac{r}{\sqrt{N}}} \subset B_r.$$

In particular

$$Q_{\sigma 2^{-k}} \subset B_{N^{-\frac{1}{b}} 2^{-k-1}},$$

where $\sigma = \dfrac{1}{2} N^{-\frac{1}{b}-\frac{1}{2}}$. For $\nu = (\nu_1, \ldots, \nu_N)$, where $\nu_j = 0$ or 1, $j = 1, \ldots, N$, we define cuboids

$$Q^\nu = \left\{x \in \mathbb{R}^N : |x_j| < \sigma_j 2^{-ka_j} \text{ if } \nu_j = 0;\ |x_j| \geqslant \sigma_j 2^{-ka_j} \text{ if } \nu_j = 1\right\},$$

where $\sigma_j = \sigma^{a_j}$, $j = 1, \ldots, N$. Then $Q^0 = Q_{\sigma 2^{-k}}$ and

$$\mathbb{R}^N \setminus B_{N^{-\frac{1}{b}} 2^{-k-1}} \subset \mathbb{R}^N \setminus Q^0 = \bigcup_{\substack{0 \leqslant \nu \leqslant 1 \\ |\nu| > 0}} Q^\nu.$$

Therefore

$$\left\|F^{-1}\varphi_k Ff\right\|_{L_p\left(|x|_{\vec{a}} \geqslant N^{-\frac{1}{b}} 2^{-k-1}\right)} \leqslant \left(2^{N\left(\frac{1}{p}-1\right)_+}\right) \sum_{\substack{0 \leqslant \nu \leqslant 1 \\ |\nu| > 0}} \left\|F^{-1}\varphi_k Ff\right\|_{L_p(Q^\nu)}, \tag{20}$$

where $(a)_+ = \max\{a, 0\}$.

Step 7. For all $m \in \mathbb{N}$ and $0 \leqslant \nu \leqslant 1$, $|\nu| > 0$, there exist $c_9 > 0$ such that for all $f \in \Lambda(\lambda, \vec{a}, p)$, $k \in \mathbb{N}$ and $x \in Q^\nu$

$$\left|\left(F^{-1}\varphi_k Ff\right)(x)\right| \leqslant c_9 \left|x^{-m\nu}\right| 2^{k(N - \langle \vec{a}, m\nu \rangle)} (Ff)\left(2^{k\vec{a}}\right). \tag{21}$$

Indeed (1) and condition 3) imply that

$$\begin{aligned}
\left|\left(F^{-1}\varphi_k Ff\right)(x)\right| &= \left|(-ix)^{-m\nu}\left(F^{-1}\left(D^{m\nu}\left(\varphi_k Ff\right)\right)(x)\right)\right| \\
&\leqslant c_{10}\left|x^{-m\nu}\right| \int\limits_{A_k} \left|D^{m\nu}\left(\varphi_k Ff\right)(\xi)\right| d\xi \\
\sum_{0\leqslant\beta\leqslant m\nu} &\leqslant c_{11}\left|x^{-m\nu}\right| \int\limits_{A_k} \left|\left(D^{\beta}\varphi_k\right)(\xi)\right| \cdot \left|\left(D^{m\nu-\beta}Ff\right)(\xi)\right| d\xi \\
&\leqslant c_{12}\left|x^{-m\nu}\right| \sum_{0\leqslant\beta\leqslant m\nu} \int\limits_{A_k} 2^{-k\langle \vec{a},\beta\rangle}\left(1+|\xi|_{\vec{a}}^{2}\right)^{-\frac{\langle \vec{a},m\nu-\beta\rangle}{2}} (Ff)(\xi)\, d\xi \\
&\leqslant c_{12}\left|x^{-m\nu}\right| \sum_{0\leqslant\beta\leqslant m\nu} 2^{-k\langle \vec{a},\beta\rangle}\left(2^{2(k-1)}\right)^{-\frac{\langle \vec{a},m\nu-\beta\rangle}{2}} \int\limits_{A_k} (Ff)(\xi)\, d\xi \\
&\leqslant c_{13}\left|x^{-m\nu}\right| 2^{-k\langle \vec{a},m\nu\rangle} \int\limits_{A_k} (Ff)(\xi)\, d\xi,
\end{aligned}$$

where c_{10}, c_{11}, c_{12}, $c_{13} > 0$ are independent of f, k and x. Since for $\eta = 2^{k\vec{a}}$ and all $\xi \in A_k$ the left-hand side inequality in (14) for $k=0$ and the left-hand side inequality in (12) for $k \in \mathbb{N}$ are satisfied with some $c_1 \geqslant 2$ independent of k, inequality (21) follows by applying (14) and (12).

Step 8. Let $m = \left\lfloor \frac{1}{p} \right\rfloor + 1$. Then

$$\left\|x^{-m\nu}\right\|_{L_p(Q^\nu)} \leqslant c_{14} 2^{k\left(\langle \vec{a},m\nu\rangle - \frac{N}{p}\right)}, \tag{22}$$

where $c_{14} > 0$ is independent of $k \in \mathbb{N}_0$.

Indeed, if $p = \infty$, then $m = 1$ and

$$\left\|x^{-m\nu}\right\|_{L_\infty(Q^\nu)} = \prod_{j:\nu_j=1} \max_{x_j \geqslant \sigma_j 2^{-ka_j}} x_j^{-1} = c_{15} \prod_{j:\nu_j=1} 2^{ka_j} = c_{15} 2^{k\langle \vec{a},\nu\rangle},$$

where $c_{15} > 0$ is independent of k. If $p < \infty$, then $mp > 1$ and

$$\begin{aligned}
\left\|x^{-m\nu}\right\|_{L_p(Q^\nu)}^{p} &= 2^N \prod_{j:\nu_j=0} \int\limits_{0}^{\sigma_j 2^{-ka_j}} dx_j \prod_{j:\nu_j=1} \int\limits_{\sigma_j 2^{-ka_j}}^{\infty} x_j^{-mp} dx_j \\
&= c_{16} 2^{-k(a_1+\cdots+a_N)} \prod_{j:\nu_j=1} 2^{ka_j mp} = c_{16} 2^{k(\langle \vec{a},m\nu\rangle\, p - N)},
\end{aligned}$$

where $c_{16} > 0$ is independent of k.

Inequalities (20), (21) and (22) imply that

$$\left\|F^{-1}\varphi_k Ff\right\|_{L_p\left(|x|_{\vec{a}} \geqslant N^{-\frac{1}{b}} 2^{-k-1}\right)} \leqslant c_{17} 2^{\frac{kN}{p'}} (Ff)\left(2^{k\vec{a}}\right),$$

where $c_{17} > 0$ is independent of f and k, and inequality (19) follows.

Step 9. If $\theta < \infty$, inequality (19) implies that

$$\|f\|_{B^{\vec{s}}_{p,\theta}(\mathbb{R}^N)} \sim \left(\sum_{k=0}^{\infty}\left(2^{k\left(s+\frac{N}{p'}\right)}(Ff)\left(2^{k\vec{a}}\right)\right)^{\theta}\right)^{\frac{1}{\theta}} \tag{23}$$

on $\Lambda(\lambda,\vec{a},p)$.

To accomplish the proof it suffices to verify that also

$$\left\|\left(1+|\xi|^2_{\vec{a}}\right)^{\frac{1}{2}\left(s-\frac{N}{p}+\frac{N}{\theta'}\right)}(Ff)(\xi)\right\|_{L_\theta\left(\mathbb{R}^N\right)} \sim \left(\sum_{k=0}^{\infty}\left(2^{k\left(s+\frac{N}{p'}\right)}(Ff)\left(2^{k\vec{a}}\right)\right)^{\theta}\right)^{\frac{1}{\theta}} \tag{24}$$

on $\Lambda(\lambda,\vec{a},p)$.

Indeed,

$$\left\|\left(1+|\xi|^2_{\vec{a}}\right)^{\frac{1}{2}\left(s-\frac{N}{p}+\frac{N}{\theta'}\right)}(Ff)(\xi)\right\|^{\theta}_{L_\theta\left(\mathbb{R}^N\right)} = \sum_{k=0}^{\infty}\int_{D_k}\left(\left(1+|\xi|^2_{\vec{a}}\right)^{\frac{1}{2}\left(s-\frac{N}{p}+\frac{N}{\theta'}\right)}(Ff)(\xi)\right)^{\theta}d\xi,$$

where $D_0 = B_1$, $D_k = B_{2^k}\setminus B_{2^{k-1}}$, $k\in\mathbb{N}$. Moreover, there exist c_{18}, $c_{19} > 0$ such that for all $\xi\in D_k$, $k\in\mathbb{N}_0$

$$c_{18}2^k \leqslant \left(1+|\xi|^2_{\vec{a}}\right)^{\frac{1}{2}} \leqslant c_{19}2^k$$

and, by Steps 2, 3,

$$c_{18}(Ff)\left(2^{k\vec{a}}\right) \leqslant (Ff)(\xi) \leqslant c_{19}(Ff)\left(2^{k\vec{a}}\right).$$

This implies (24).

The case $\theta = \infty$ is similar.

Thus, the theorem is proved. □

REMARK. For $0 < p,\ \theta \leqslant \infty,\ -\infty < \vec{s} < \infty$ let us consider another space describing anisotropic smoothness:

$$G^{\vec{s}}_{p,\theta}\left(\mathbb{R}^N\right) = \left\{f\in S'\left(\mathbb{R}^N\right) : \left\|\left(1+|\xi|^2_{\vec{a}}\right)^{\frac{1}{2}\left(s-\frac{N}{p}+\frac{N}{\theta'}\right)}(Ff)(\xi)\right\|_{L_\theta\left(\mathbb{R}^N\right)} < \infty\right\}.$$

Then the statement of the theorem proved above may be written as

$$B^{\vec{s}}_{p,\theta}\left(\mathbb{R}^N\right)\cap\Lambda(\lambda,\vec{a},p) = G^{\vec{s}}_{p,\theta}\left(\mathbb{R}^N\right)\cap\Lambda(\lambda,\vec{a},p) \tag{25}$$

for $\lambda > 1$.

Consider the embedding theorem for the anisotropic Nikol'skiĭ-Besov spaces

$$B^{\vec{s}}_{p,\theta}\left(\mathbb{R}^N\right) \hookrightarrow B^{\vec{\rho}}_{q,\theta}\left(\mathbb{R}^N\right), \tag{26}$$

where $0 < p < q \leqslant \infty$, $0 < \theta \leqslant \infty$, $\vec{\rho} = \varkappa \overrightarrow{s}$ and $\varkappa = 1 - \frac{N}{s}\left(\frac{1}{p} - \frac{1}{q}\right)$. (See Section 2.2.) The theorem implies that under the same assumptions on the parameters for any $\lambda > 1$

$$B^{\vec{s}}_{p,\theta}\left(\mathbb{R}^N\right) \cap \Lambda\left(\lambda, \vec{a}, p\right) = B^{\vec{\rho}}_{q,\theta}\left(\mathbb{R}^N\right) \cap \Lambda\left(\lambda, \vec{a}, p\right), \tag{27}$$

where $\vec{a} = \frac{s}{\vec{s}} \left(= \frac{\rho}{\vec{\rho}}\right)$.

Indeed, first assume that $\vec{s} \neq 0$ and $\vec{\rho} \neq 0$. Then $G^{\vec{s}}_{p,\theta}\left(\mathbb{R}^N\right) = G^{\vec{\rho}}_{q,\theta}\left(\mathbb{R}^N\right)$ since $s - \frac{N}{p} = \rho - \frac{N}{q}$ and $\frac{s}{\vec{s}} = \frac{\rho}{\vec{\rho}}$. Moreover, $\Lambda\left(\lambda, \vec{a}, p\right) \subset \Lambda\left(\lambda, \vec{a}, q\right)$ since $p < q$.

Therefore

$$\begin{aligned} B^{\vec{s}}_{p,\theta}\left(\mathbb{R}^N\right) \cap \Lambda\left(\lambda, \vec{a}, p\right) &= G^{\vec{s}}_{p,\theta}\left(\mathbb{R}^N\right) \cap \Lambda\left(\lambda, \vec{a}, p\right) \\ &= G^{\vec{\rho}}_{q,\theta}\left(\mathbb{R}^N\right) \cap \Lambda\left(\lambda, \vec{a}, q\right) \cap \Lambda\left(\lambda, \vec{a}, p\right) \\ &= B^{\vec{\rho}}_{q,\theta}\left(\mathbb{R}^N\right) \cap \Lambda\left(\lambda, \vec{a}, p\right). \end{aligned}$$

If $\vec{s} = 0$ or $\vec{\rho} = 0$ the proof is similar. One should take into account the comments in Section 2.2. related to these cases.

Note that relation (27) immediately implies that the embedding (26) is sharp, i.e. it is impossible to replace $\vec{\rho}$ by $\vec{\varrho} + \vec{\varepsilon}$ where $\vec{\varepsilon} \geqslant 0$ and $\varepsilon_j > 0$ for at least one j.

Finally we recall one of the statements of type (25) obtained previously by C. S. Herz [5]. For the cone C of positive decreasing functions of $|x|$, in [5] Proposition 1.8, it was proved for the isotropic case that

$$B^{s}_{p,\theta}\left(\mathbb{R}^N\right) \cap C = L_{p,\theta}\left(\mathbb{R}^N\right) \cap C$$

for $0 < s < \frac{1}{p}$, where $L_{p,\theta}\left(\mathbb{R}^N\right)$ is the Lorentz space.

Acknowledgments

This work was supported by grants of the *INTAS (project 99-01080)*, of the *Russian Foundation for Basic Research (RFBR) (project 02-01-00602)*, of the *Consejo Nacional de Ciencia y Tecnología (CONACYT), México* and by the *Universidad Autónoma de Yucatán (UADY), Mérida, Yucatán, México*.

REFERENCES

[1] O. V. BESOV, V. P. IL'IN, AND S. M. NIKOL'SKIĬ, *Integral representations of functions and imbedding theorems*, "Nauka", Moscow, 1975; English transl., Vols. 1, 2, Wiley, 1979.
[2] V. I. BURENKOV, *Sobolev spaces on domains*, B. G. Teubner, Stuttgart-Leipzig, 1998.
[3] B. E. BATYROV AND V. I. BURENKOV, *On estimates of convolutions in Nikol'skiĭ-Besov spaces*, Russian Akad. Nauk Dokl. Mat., Vol. 330, No. 1, 1993; English transl., Russian Acad. Sci. Dokl. Math., Vol. 47, No. 3, 1993.
[4] B. E. BATYROV AND V. I. BURENKOV, *On estimates of convolutions in Nikol'skiĭ-Besov spaces*, Vestnik of People's Friendship University of Russia, Mathematics, Vol. 1, 6–24, 1994 (In Russian).
[5] C. S. HERZ, *Lipschitz spaces and Bernstein's theorem on absolutely convergent Fourier transforms*, Journal of Mathematics and Mechanics, Vol. 18, No. 4, 1968.
[6] S. M. NIKOL'SKIĬ, *Approximation of functions of several variables and imbedding theorems*, "Nauka", Moscow, 1969; English transl., Springer-Verlag, 1975.
[7] H. J. SCHMEISSER AND H. TRIEBEL, *Topics in Fourier analysis and function spaces*, Wiley, 1987.
[8] H. TRIEBEL, *Theory of function spaces*, Birkhäuser Verlag, 1983.
[9] H. TRIEBEL, *Theory of function spaces II*, Birkhäuser Verlag, 1992.
[10] H. TRIEBEL, *Interpolation theory, function spaces, differential operators*, 2^{nd} ed., Johann Ambrosius Barth Verlag, 1995.
[11] H. TRIEBEL, *The structure of functions*, Birkhäuser Verlag, 2001.

(Received August 27, 2002)

V. I. Burenkov
School of Mathematics
Cardiff University
PO Box 926
Cardiff CF24 4YH, UK
e-mail: burenkov@cardiff.ac.uk

G. E. García Almeida
Facultad de Matemáticas
Universidad Autónoma de Yucatán
Periférico Norte, Tab. 13615
Parque Industrial, Chuburná Hidalgo
Mérida, Yucatán
México
e-mail: galmeida@tunku.uady.mx