Mathematical
Inequalities
& Applications
Volume 6, Number 3 (2003), 521–530

SPECHT RATIO $S(1)$ CAN BE EXPRESSED BY KANTOROVICH CONSTANT $K(p)$: $S(1) = \exp[K'(1)]$ AND ITS APPLICATION

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Dedicated to Professor Hisaharu Umegaki
on his 77 th birthday
with respect and affection

(*communicated by J. Pečarić*)

Abstract. In what follows, an operator means a bounded linear operator on a Hilbert space H. We show a very interesting new relation between *Specht ratio* $S(1)$ and *Kantorovich constant* $K(p)$: $S(1) = e^{K'(1)}$ and several applications of this relation are given.

1. Introduction

An operator T is said to be positive (denoted by $T \geqslant 0$) if $(Tx, x) \geqslant 0$ and also T is said to be strictly positive (denoted by $T > 0$) if T is positive and invertible.

DEFINITION 1. Let $h > 1$. *The Kantorovich constant* $K(h, p)$ is defined by

$$K(h,p) = \frac{(h^p - h)}{(p-1)(h-1)} \left(\frac{(p-1)}{p} \cdot \frac{(h^p - 1)}{(h^p - h)} \right)^p \quad \text{for any } p > 1 \text{ or } p < 0 \tag{1.1}$$

and especially $K(h, p)$ for $p > 1$ can be usually written by

$$K(h,p) = \frac{(p-1)^{p-1}}{p^p} \cdot \frac{(h^p - 1)^p}{(h-1)(h^p - h)^{p-1}} \quad \text{for any } p > 1. \tag{1.2}$$

$K(h, p)$ is sometimes denoted by $K(p)$ briefly. Also $K_+(m, M, p)$ is defined by

$$K_+(m,M,p) = \frac{(p-1)^{p-1}}{p^p} \cdot \frac{(M^p - m^p)^p}{(M-m)(mM^p - Mm^p)^{p-1}} \quad \text{for } M > m > 0 \text{ and } p > 1. \tag{1.3}$$

We remark that $K_+(m, M, p)$ in (1.3) just coincides with (1.2) by putting $h = \frac{M}{m} > 1$ in (1.3).

Mathematics subject classification (2000): 47A63.
Key words and phrases: Specht ratio, Kantorovich constant.

DEFINITION 2. Let $h > 1$. $S(h,p)$ is defined by

$$S(h,p) = \frac{h^{\frac{p}{h^p-1}}}{e\log h^{\frac{p}{h^p-1}}} \quad \text{for any real number } p \tag{1.4}$$

and $S(h,p)$ is sometimes denoted by $S(p)$ briefly. Especially $S(1) = S(h,1) = \dfrac{h^{\frac{1}{h-1}}}{e\log h^{\frac{1}{h-1}}}$ is said to be *Specht ratio* and $S(1) > 1$ is well known (see (iv) and (x) of Proposition 1). We remark that $S(h^p,1) = S(h,p)$ for any real number p by Definition (1.4).

DEFINITION 3. The determinant $\Delta_x(A)$ for strictly positive operator A at a unit vector x in Hilbert space H is defined by

$$\Delta_x(A) = \exp\langle\, ((\log A)x,x)\rangle\,. \tag{1.5}$$

Let A be strictly positive operator satisfying $MI \geqslant A \geqslant mI > 0$, where $M > m > 0$. The celebrated Kantorovich inequality asserts that

$$(Ax,x)(A^{-1}x,x) \leqslant \frac{(m+M)^2}{4mM}$$

holds for every unit vector x and this inequality is just equivalent to the following one

$$(A^2x,x) \leqslant \frac{(m+M)^2}{4mM}(Ax,x)^2$$

holds for every unit vector x. We remark that $K_+(m,M,p)$ in (1.3) is an extension of $\dfrac{(m+M)^2}{4mM}$, in fact, $K(m,M,2) = \dfrac{(m+M)^2}{4mM}$ holds.

Many papers on Kantorovich inequality have been published. Among others, there is a long research series by Mond-Pečarić, we cite [6] and [7] for examples. An extension of Kantorovich inequality is given in [Theorem 1.5, 3] and [Theorem 3, 5] as follows.

THEOREM A. (Kantorovich type inequality). *Let A be strictly positive operator satisfying $MI \geqslant A \geqslant mI > 0$, where $M > m > 0$. Also let $h = \frac{M}{m} > 1$. Then the following inequality holds for every unit vector x:*

$$K(h,p)(Ax,x)^p \geqslant (A^px,x) \geqslant (Ax,x)^p \quad \textit{for any } p > 1 \textit{ or } p < 0. \tag{1.6}$$

The latter half inequality in (1.6) of Theorem A is called *Hölder-McCarthy inequality* and the former one can be considered as the reverse inequality of this latter one.

The following interesting result is shown in [Corollary, 2].

THEOREM B. *Let A be strictly positive operator satisfying $MI \geqslant A \geqslant mI > 0$, where $M > m > 0$. Also let $h = \frac{M}{m} > 1$. Then the following inequality holds for every unit vector x:*

$$S(h,p)\Delta_x(A^p) \geqslant (A^px,x) \geqslant \Delta_x(A^p) \quad \textit{for any real number } p. \tag{1.7}$$

We state the following result [Theorem 4, 10] related to Theorem A.

THEOREM C. *Let* $K_+(m,M,p)$ *be defined in* (1.3). *Then*

$$F(p,r,m,M) = K_+\left(m^r, M^r, \frac{p+r}{r}\right)$$

is an increasing function of p, r *and* M *and also decreasing function of* m *for* $p > 0$ *and* $r > 0$.

2. Statement of main results

We state the following Theorem 1 which is our main result and Corollary 2 as an immediate consequence of Theorem 1.

THEOREM 1. *Let* A *be strictly positive operator satisfying* $MI \geqslant A \geqslant mI > 0$, *where* $M > m > 0$ *and* $h = \frac{M}{m} > 1$. *Then the following inequalities hold for every unit vector* x:

(i) $(Ax,x)\log(Ax,x) + [\log S(1)](Ax,x) \geqslant ((A\log A)x,x) \geqslant (Ax,x)\log(Ax,x)$;

(ii) $((\log A)x,x) + \log S(1) \geqslant \log(Ax,x) \geqslant ((\log A)x,x)$;

(iii) $S(1)(Ax,x) \geqslant e^{\frac{((A\log A)x,x)}{(Ax,x)}} \geqslant (Ax,x)$;

(iv) $S(1)\Delta_x(A) \geqslant (Ax,x) \geqslant \Delta_x(A)$;

where $S(1) = S(h,1)$ *is Specht ratio defined in Definition 2.*

The former inequality of (i) in Theorem 1 is considered as the reverse inequality of the latter inequality of (i) which is obtained by convex function $t\log t$ on $[0,\infty)$ and also the former inequality of (ii) in Theorem 1 is considered as the reverse inequality of the latter inequality of (ii) which is obtained by concave function $\log t$ on $[0,\infty)$ (see Remark 3).

COROLLARY 2. *Let* A *be strictly positive operator satisfying* $MI \geqslant A \geqslant mI > 0$, *where* $M > m > 0$ *and* $h = \frac{M}{m} > 1$. *Then the following inequalities hold for every unit vector* x:

(i) $(A^px,x)\log(A^px,x) + [\log S(h,p)](A^px,x) \geqslant ((A^p\log A^p)x,x) \geqslant (A^px,x)\log(A^px,x)$ *for any* $p > 0$;

(ii) $((\log A^p)x,x) + \log S(h,p) \geqslant \log(A^px,x) \geqslant ((\log A^p)x,x)$ *for any* $p > 0$;

(iii) $S(h,p)(A^px,x) \geqslant e^{\frac{((A^p\log A^p)x,x)}{(A^px,x)}} \geqslant (A^px,x)$ *for any* $p > 0$;

(iv) $S(h,p)\Delta_x(A^p) \geqslant (A^px,x) \geqslant \Delta_x(A^p)$ *for any* $p > 0$;

where $S(h,p)$ *is defined in Definition 2.*

We remark that (iv) of Theorem 1 and (iv) of Corollary 2 are both equivalent to (1.7) of Theorem B (see Remark 1).

3. Basic properties on $K(p)$ and $S(p)$ and especially $S(1) = \exp\langle\, [\frac{dK(p)}{dp}]_{p=1}\rangle$

Before giving proofs of results in § 2, we state the following fundamental relation between $K(p)$ and $S(p)$, especially $S(1) = e^{K'(1)} = e^{-K'(0)}$ which is the central basic result to give a proof of Theorem 1.

PROPOSITION 1. *The following properties on $K(p)$ and $S(p)$ hold:*

(i) $K(-p) = K(p+1)$ *for any* $p \geqslant 0$*;*

(ii) $K(0) = K(1) = 1$*;*

(iii) $S(-p) = S(p)$ *for any* $p \geqslant 0$*;*

(iv) $S(0) = 1$*;*

(v) $K'(-p) = -K'(p+1)$ *for any* $p \geqslant 0$*;*

(vi) $\lim_{r\to+0} K_+\left(m^r, M^r, 1+\frac{p}{r}\right) = S(h,p)$ *for* $M > m > 0$ *and* $h = \frac{M}{m} > 1$*;*

(vii) $\lim_{n\to\infty} K\left(\frac{1+\frac{1}{n}\log M}{1+\frac{1}{n}\log m}, np\right) = S(h,p)$ *for* $M > m > 0$ *and* $h = \frac{M}{m} > 1$*;*

(viii) $S(1) = e^{K'(1)} = e^{-K'(0)}$*;*

(ix) $K(p)$ *is increasing for* $p > 1$ *and decreasing for* $p < 0$*;*

(x) $S(p)$ *is increasing for* $p > 0$ *and decreasing for* $p < 0$*.*

Proof. (i) By an easy calculation, we have for any $p \geqslant 0$

$$\begin{aligned}
K(-p) &= \frac{(h^{-p}-h)}{(-p-1)(h-1)}\left(\frac{(-p-1)}{-p}\cdot\frac{(h^{-p}-1)}{(h^{-p}-h)}\right)^{-p} \\
&= \frac{(h^{p+1}-1)}{(p+1)(h-1)h^p}\left(\frac{p}{(p+1)}\cdot\frac{(h^{p+1}-1)}{(h^p-1)}\right)^{p} \\
&= \frac{(h^{p+1}-h)}{p(h-1)}\left(\frac{p}{(p+1)}\cdot\frac{(h^{p+1}-1)}{(h^{p+1}-h)}\right)^{p+1} \\
&= K(p+1).
\end{aligned}$$

(ii) By L. Hopital's theorem.

(iii) Obvious.

(iv) By L. Hopital's theorem.

(v) Differentiate $K(p)$ by p, we obtain $K'(p)$ as follows:

$$\begin{aligned}
K'(p) =& \frac{\left(\frac{(p-1)}{p}\cdot\frac{(h^p-1)}{(h^p-h)}\right)^p}{(h-1)(h^p-1)(p-1)}\{h^p(h^p-1+p-hp)\log h \\
&+(h^p-1)(h^p-h)\log\frac{(p-1)(h^p-1)}{p(h^p-h)}\bigg\}.
\end{aligned} \tag{$*$}$$

By $(*)$ we have

$$\begin{aligned}
K'(-p) =& \frac{\left(\frac{(-p-1)}{-p}\cdot\frac{(h^{-p}-1)}{(h^{-p}-h)}\right)^{-p}}{(h-1)(h^{-p}-1)(-p-1)}\Big\{h^{-p}(h^{-p}-1-p+hp)\log h \\
&+(h^{-p}-1)(h^{-p}-h)\log\frac{(-p-1)(h^{-p}-1)}{-p(h^{-p}-h)}\Big\} \\
=& -\frac{h^{-p}\left(\frac{(p+1)}{p}\cdot\frac{(h^p-1)}{(h^{p+1}-1)}\right)^{-p}}{(h-1)(h^p-1)(p+1)}\Big\{-\log h - h^p(hp-p-1)\log h
\end{aligned}$$

$$
\begin{aligned}
&- (h^p-1)(h^{p+1}-1)\log\frac{(p+1)(h^p-1)}{p(h^{p+1}-1)}\Big\} \\
=& -\frac{h^{-p}\left(\frac{(p+1)}{p}\cdot\frac{(h^p-1)}{(h^{p+1}-1)}\right)^{-p}}{(h-1)(h^p-1)(p+1)}\Big\{h^p(h^{p+1}+p-hp-h)\log h \\
&+ (h^p-1)(h^{p+1}-1)\log\frac{p(h^{p+1}-1)}{h(p+1)(h^p-1)}\Big\} \\
=& -\frac{\left(\frac{p(h^{p+1}-1)}{h(p+1)(h^p-1)}\right)^{p}}{(h-1)(h^p-1)(p+1)}\Big\{h^p(h^{p+1}+p-hp-h)\log h \\
&+ (h^p-1)(h^{p+1}-1)\log\frac{p(h^{p+1}-1)}{h(p+1)(h^p-1)}\Big\} \\
=& -\frac{\left(\frac{p}{(p+1)}\cdot\frac{(h^{p+1}-1)}{(h^{p+1}-h)}\right)^{p+1}}{(h-1)(h^{p+1}-1)p}\Big\{h^{p+1}(h^{p+1}+p-hp-h)\log h \\
&+ (h^{p+1}-1)(h^{p+1}-h)\log\frac{p(h^{p+1}-1)}{(p+1)(h^{p+1}-h)}\Big\} \\
=& -K'(p+1)
\end{aligned}
$$

because the third equality holds by the following obvious relation:

$$
-\log h - h^p(hp-p-1)\log h = h^p(h^{p+1}+p-hp-h)\log h - (h^p-1)(h^{p+1}-1)\log h.
$$

(vi) is shown in [Lemma 11, 10].
(vii) is shown in [Proposition 2, 4] and [Proposition 2, 5].
(viii) We recall $K'(p)$ by $(*)$ as follows:

$$
K'(p)=\frac{\left(\frac{(p-1)}{p}\cdot\frac{(h^p-1)}{(h^p-h)}\right)^{p}}{(h-1)(h^p-1)}\left\{\frac{h^p(h^p-1+p-hp)\log h+(h^p-1)(h^p-h)\log\frac{(p-1)(h^p-1)}{p(h^p-h)}}{p-1}\right\}. \tag{$*$}
$$

By using L. Hopital's theorem to $(*)$, we have

$$
\begin{aligned}
\lim_{p\to 1}K'(p) &= \frac{h-1}{h\log h}\cdot\frac{1}{(h-1)^2}\Big\{h\log h(h\log h+1-h)+(h-1)h\log h\log\Big[\frac{h-1}{h\log h}\Big]\Big\} \\
&= \frac{h}{h-1}\log h - 1 + \log\Big[\frac{h-1}{h\log h}\Big] \\
&= \log\left[\frac{h^{\frac{1}{h-1}}}{e\log h^{\frac{1}{h-1}}}\right] \\
&= \log S(1)
\end{aligned}
$$

so that we have $S(1)=e^{K'(1)}=e^{-K'(0)}$ since $K'(0)=-K'(1)$ holds by (v), so we have (viii).

(ix) Put $r=1$ in Theorem C. Then $F(p,1,m,M)=K_+(m,M,p+1)=K(h,p+1)=K(p+1)$ is increasing for $p>0$, that is, $K(p)$ is increasing for $p>1$ and this fact ensures that $K(p)$ is decreasing for $p<0$ by (v).

(x)

$$\begin{aligned} S(h,p) &= \lim_{n\to\infty} K\left(\frac{1+\frac{1}{n}\log M}{1+\frac{1}{n}\log m}, np\right) \quad \text{by (vii)} \\ &\leqslant \lim_{n\to\infty} K\left(\frac{1+\frac{1}{n}\log M}{1+\frac{1}{n}\log m}, np'\right) \quad \text{for } p'>p>0 \text{ such that } np'>np>1 \\ &\leqslant S(h,p') \quad \text{by (vii)} \end{aligned}$$

since the first inequality holds by (ix), so that $S(p)$ is increasing for $p>0$ and this result ensures that $S(p)$ is decreasing for $p<0$ by (iii).

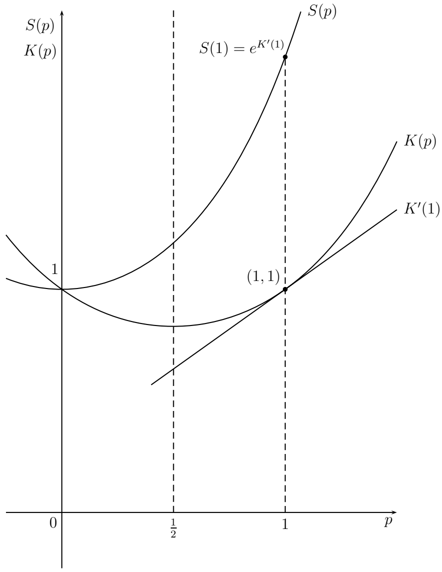

Figure 1. Relation between $K(p)$ and $S(p)$

4. Proofs of Theorem 1 and Corollary 2 in § 2

Proof of Theorem 1.

(i) Define $f_1(p)$ as follows: $f_1(p)=(A^px,x)-(Ax,x)^p$ for $p\geqslant 1$. Obviously $f_1(1)=0$ holds. As $f_1(p)\geqslant 0$ for any $p>1$ by the latter half inequality of (1.6) of Theorem A, we have $f_1'(1)\geqslant 0$, that is,

$$f_1'(1)=((A\log A)x,x)-(Ax,x)\log(Ax,x)\geqslant 0$$

so that we have the latter half inequality.

Also define $g_1(p)$ as follows: $g_1(p) = K(p)(Ax,x)^p - (A^p x,x)$ for $p \geqslant 1$. Then we have

$$g_1(1) = K(1)(Ax,x) - (Ax,x) = 0$$

holds since $K(1) = 1$ by (ii) of Proposition 1.

As $g_1(p) \geqslant 0$ for any $p > 1$ by the former half inequality of (1.6) of Theorem A, we have $g_1'(1) \geqslant 0$, that is,

$$g_1'(1) = K(1)(Ax,x)\log(Ax,x) + K'(1)(Ax,x) - ((A\log A)x,x) \geqslant 0$$

so that we have the former half inequality because $K(1) = 1$ and $K'(1) = \log S(1)$ by (ii) and (viii) of Proposition 1 respectively.

(ii) Define $f_2(p)$ as follows: $f_2(p) = (A^p x,x) - (Ax,x)^p$ for $p \leqslant 0$. Obviously $f_2(0) = 0$ holds. As $f_2(p) \geqslant 0$ for any $p < 0$ by the latter half inequality of (1.6) of Theorem K, we have $f_2'(0) \leqslant 0$, that is,

$$f_2'(0) = ((\log A)x,x) - \log(Ax,x) \leqslant 0$$

so that we have the latter half inequality.

Also define $g_2(p)$ as follows: $g_2(p) = K(p)(Ax,x)^p - (A^p x,x)$ for $p \leqslant 0$. Then we have

$$g_2(0) = K(0)(Ax,x)^0 - (A^0 x,x) = 0$$

holds since $K(0) = 1$ by (ii) of Proposition 1.

As $g_2(p) \geqslant 0$ for any $p < 0$ by the former half inequality of (1.6) of Theorem K, we have $g_2'(0) \leqslant 0$, that is,

$$\begin{aligned} g_2'(0) &= K(0)(Ax,x)^0\log(Ax,x) + K'(0)(Ax,x)^0 - ((A^0\log A)x,x) \\ &= \log(Ax,x) - \log S(1) - ((\log A)x,x) \leqslant 0 \end{aligned}$$

so that we have the former half inequality because $K(0) = 1$ and $K'(0) = -\log S(1)$ by (ii) and (viii) of Proposition 1 respectively.

(iii) Deviding the each term of (i) by (Ax,x), we have

$$\log(Ax,x) + \log S(1) \geqslant \frac{((A\log A)x,x)}{(Ax,x)} \geqslant \log(Ax,x)$$

so that we have (iii).

(iv) (ii) of Theorem 1 easily yields (iv) by $\Delta_x(A)$ in (1.5) of Definition 3.

Whence the proof of Theorem 1 is complete.

Proof of Corollary 2.

The hypotheses imply that $M^p I \geqslant A^p \geqslant m^p I > 0$ and also $h^p = \frac{M^p}{m^p} > 1$ for any $p > 0$. Applying Theorem 1 for A^p, we have Corollary 2 since $S(h^p,1) = S(h,p)$ holds stated in the definition 2.

REMARK 1. We remark that (iv) of Theorem 1 and (iv) of Corollary 2 are equivalent to (1.7) of Theorem B, that is,

$$S(h,1)\Delta_x(A) \geqslant (Ax,x) \geqslant \Delta_x(A). \qquad (p_1)$$

$$S(h,p)\Delta_x(A^p) \geqslant (A^p x,x) \geqslant \Delta_x(A^p) \quad \text{for any } p>0. \qquad (p_+)$$

$$S(h,p)\Delta_x(A^p) \geqslant (A^p x,x) \geqslant \Delta_x(A^p) \quad \text{for any } p<0. \qquad (p_-)$$

In fact $(p_+) \Longrightarrow (p_1)$ is obvious, so we show $(p_1) \Longrightarrow (p_+)$. Replacing A by A^p in (p_1), then $M^p I \geqslant A^p \geqslant m^p I > 0$ and

$$S(h^p,1)\Delta_x(A^p) \geqslant (A^p x,x) \geqslant \Delta_x(A^p) \quad \text{for any } p>0 \text{ by } (p_1)$$

so we have (p_+) since $S(h^p,1) = S(h,p)$, that is, $(p_1) \Longrightarrow (p_+)$.

Next, replacing A by A^{-1} in (p_+), then $m^{-1}I \geqslant A^{-1} \geqslant M^{-1}I > 0$ and

$$S\left(\frac{m^{-1}}{M^{-1}},p\right)\Delta_x(A^{-p}) \geqslant (A^{-p}x,x) \geqslant \Delta_x(A^{-p}) \quad \text{for any } p>0 \text{ by } (p_+)$$

so we have (p_-) since $S\left(\frac{m^{-1}}{M^{-1}},p\right) = S(h,p) = S(h,-p)$ by (iii) of Proposition 1, that is, $(p_+) \Longrightarrow (p_-)$ and the reverse implication is trivial. Remark 1 is essentially shown in the proof of [Corollary, 2].

REMARK 2. We remark that (iii) of Theorem 1 can be rewritten as follows:

(iii_1) $S(1)(Ax,x) \geqslant \Delta_y(A) \geqslant (Ax,x)$ where y is unit vector such that $y = \frac{A^{\frac{1}{2}}x}{||A^{\frac{1}{2}}x||}$.

It is interesting contrast between (iii_1) and (iv) of Theorem 1:

$$S(1)\Delta_x(A) \geqslant (Ax,x) \geqslant \Delta_x(A).$$

Also (iii) of Corollary 2 can be rewritten as follows:

(iii_p) $S(h,p)(A^p x,x) \geqslant \Delta_y(A^p) \geqslant (A^p x,x)$ where y is unit vector such that $y = \frac{A^{\frac{p}{2}}x}{||A^{\frac{p}{2}}x||}$.

It is interesting contrast between (iii_p) and (iv) of Corollary 2:

$$S(h,p)\Delta_x(A^p) \geqslant (A^p x,x) \geqslant \Delta_x(A^p).$$

5. Relations between Theorem 1, convex functions and concave functions

In this section, we discuss some relations between Theorem 1, convex functions and and concave functions. First of all we state the following result in [1, p. 281]:

THEOREM D. *Let A be positive operator and also let x be unit vector. Then*
(i) $(g(A)x,x) \geqslant g((Ax,x))$ *for convex function $g(t)$ on $[0,\infty)$.*
(ii) $h((Ax,x)) \geqslant (h(A)x,x)$ *for concave function $h(t)$ on $[0,\infty)$.*

We recall the latter half inequality of (i) in Theorem 1 and also the latter half inequality of (ii), that is, let A be strictly positive operator and also let x be unit vector.
(i') $((A\log A)x,x) \geqslant (Ax,x)\log(Ax,x)$.
(ii') $\log(Ax,x) \geqslant ((\log A)x,x)$.

In fact, by using Theorem D, we easily obtain (i') since $f(t) = t\log t$ is convex function on $[0,\infty)$ and also we have (ii') since $f(t) = \log t$ is concave function on $[0,\infty)$.

REMARK 3. We recall (i) and (ii) of Theorem 1:

Let A be strictly positive operator satisfying $MI \geqslant A \geqslant mI > 0$, where $M > m > 0$ and $h = \frac{M}{m} > 1$. Then the following inequalities hold for every unit vector x:

(i) $(Ax,x)\log(Ax,x) + [\log S(1)](Ax,x) \geqslant ((A\log A)x,x) \geqslant (Ax,x)\log(Ax,x)$.

(ii) $((\log A)x,x) + \log S(1) \geqslant \log(Ax,x) \geqslant ((\log A)x,x)$.

The former inequality in (i) can be considered as *the reverse inequality* of the latter obtained in (i') one and also the former inequality in (ii) can be considered as *the reverse inequality* of the latter one obtained in (ii'). Similar results to Theorem 1 for convex functions are shown in [6].

6. Another proof of Theorem B via Theorem A.

As stated in Remark 1 in § 4, we show a proof of Theorem B via Corollary 2 obtained by using Theorem A, here we shall show a direct and simple proof of Theorem B via Theorem A faithfully along the definition $\Delta_x(A)$ in (1.5).

Direct proof of Theorem B via Theorem A. The hypothesis $MI \geqslant A \geqslant mI > 0$ implies $M_1 I \geqslant A_1 \geqslant m_1 I > 0$ where $M_1 = I + \frac{\log M}{n}$, $A_1 = I + \frac{\log A}{n}$, $m_1 = I + \frac{\log m}{n}$ for sufficiently large nutural number n respectively, so that replacing M by M_1, A by A_1 and m by m_1 in Theorem A, we have the following inequalities for $p > 0$ such that $np > 1$

$$K\left(\frac{M_1}{m_1}, np\right)(A_1 x,x)^{np} \geqslant (A_1^{np}x,x) \geqslant (A_1 x,x)^{np}$$

that is,

$$\begin{aligned} K\left(\frac{1+\frac{1}{n}\log M}{1+\frac{1}{n}\log m}, np\right)\left(\left(I+\frac{\log A}{n}\right)x,x\right)^{np} &\geqslant \left(\left(I+\frac{\log A}{n}\right)^{np}x,x\right) \\ &\geqslant \left(\left(I+\frac{\log A}{n}\right)x,x\right)^{np}. \end{aligned} \tag{6.1}$$

We recall

$$\left(\left(I+\frac{\log A}{n}\right)x,x\right)^{np} = \left(1+\frac{((\log A)x,x)}{n}\right)^{np} \Longrightarrow e^{((\log A)x,x)p} = \Delta_x(A^p)$$
$$\text{as } n\to\infty \text{ by } (1.5)$$

$$\left(\left(I+\frac{\log A}{n}\right)^{np}x,x\right) \Longrightarrow (A^p x,x) \quad \text{as } n\to\infty$$

and

$$K\left(\frac{1+\frac{1}{n}\log M}{1+\frac{1}{n}\log m}, np\right) \Longrightarrow S(h,p) \quad \text{as } n\to\infty \text{ by (vii) of Proposition 1}$$

so that (1.7) for $p > 0$ is complete by (6.1) and a proof for $p < 0$ is also obtained by Remark 1, so the proof of (1.7) is complete since the case $p = 0$ is obvious by (iv) of Proposition 1.

We remark that the proof stated above is a direct proof faithfully along the definition (1.5) by tracing the idea of [Theorem 2, 4] based on [8] and another proof of Theorem B via Theorem A is given in [9].

Ackowledgement

We would like to our cordial thanks to Professor Josip Pečarić for his useful advice to our first version to improve our careless mistake.

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(Received May 26, 2002)

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