Mathematical
Inequalities
& Applications
Volume 6, Number 3 (2003), 531–543

AN OPERATOR INEQUALITY ASSOCIATED WITH THE OPERATOR CONCAVITY OF OPERATOR ENTROPY $A \log A^{-1}$

TAKAYUKI FURUTA AND JOSIP PEČARIĆ

Dedicated to Professor Masahiro Nakamura
with respect and affection

(*communicated by S. Saitoh*)

Abstract. In what follows, an operator means a bounded linear operator on a Hilbert space H. We show an operator inequality associated with the operator concavity of operator entropy $A \log A^{-1}$ and also we discuss related upper bound.

Let A and B be strictly positive operator satisfying $MI \geqslant A, B \geqslant mI > 0$, where $M > m > 0$, $h = \frac{M}{m} > 1$ and $S(1) = \dfrac{h^{\frac{1}{h-1}}}{e \log h^{\frac{1}{h-1}}}$ which is said to be Specht ratio. Then the following inequalities (i) *and* (ii) *hold for all $\lambda \in [0,1]$:*

(i)

$$\begin{aligned}
&[\log S(1)]((1-\lambda)A+\lambda B)+((1-\lambda)A+\lambda B)\log((1-\lambda)A+\lambda B)\\
&\qquad \geqslant (1-\lambda)A\log A+\lambda B\log B\\
&\qquad \geqslant ((1-\lambda)A+\lambda B)\log((1-\lambda)A+\lambda B),
\end{aligned}$$

(ii)

$$\begin{aligned}
&\frac{M\log h}{h-1}(S(1)-1)+((1-\lambda)A+\lambda B)\log((1-\lambda)A+\lambda B)\\
&\qquad \geqslant (1-\lambda)A\log A+\lambda B\log B\\
&\qquad \geqslant ((1-\lambda)A+\lambda B)\log((1-\lambda)A+\lambda B).
\end{aligned}$$

Further extensions of (i) *and* (ii) *are obtained as follows: let A_j be strictly positive operator satisfying $MI \geqslant A_j \geqslant mI > 0$ for $j = 1, 2, \ldots, n$, where $M > m > 0$ and $h = \frac{M}{m} > 1$. Also λ_1, $\lambda_2, \ldots, \lambda_n$ be any positive numbers such that $\sum_{j=1}^{n} \lambda_j = 1$. Then the following inequalities* (iii), (iv) *and related result* (v) *hold:*

Mathematics subject classification (2000): 47A63.
Key words and phrases: Operator entropy, operator concave, operator convex.

(iii)

$$[\log S(1)]\sum_{j=1}^{n}\lambda_j A_j+\Big(\sum_{j=1}^{n}\lambda_j A_j\Big)\log\Big(\sum_{j=1}^{n}\lambda_j A_j\Big)$$
$$\geqslant\sum_{j=1}^{n}\lambda_j A_j\log A_j\geqslant\Big(\sum_{j=1}^{n}\lambda_j A_j\Big)\log\Big(\sum_{j=1}^{n}\lambda_j A_j\Big),$$

(iv)

$$\frac{M\log h}{h-1}(S(1)-1)+\Big(\sum_{j=1}^{n}\lambda_j A_j\Big)\log\Big(\sum_{j=1}^{n}\lambda_j A_j\Big)$$
$$\geqslant\sum_{j=1}^{n}\lambda_j A_j\log A_j\geqslant\Big(\sum_{j=1}^{n}\lambda_j A_j\Big)\log\Big(\sum_{j=1}^{n}\lambda_j A_j\Big),$$

(v) $$[\log S(1)]+\sum_{j=1}^{n}\lambda_j\log A_j\geqslant\log\Big(\sum_{j=1}^{n}\lambda_j A_j\Big)\geqslant\sum_{j=1}^{n}\lambda_j\log A_j.$$

Firstly we shall show elementary proofs of these results and secondary we shall give alternative simple proofs.

1. Introduction

An operator T is said to be positive (denoted by $T\geqslant 0$) if $(Tx,x)\geqslant 0$ and also T is said to be strictly positive (denoted by $T>0$) if T is positive and invertible.

DEFINITION 1. Let $h>1$. $S(h,p)$ is defined by

$$S(h,p)=\frac{h^{\frac{p}{h^p-1}}}{e\log h^{\frac{p}{h^p-1}}}\quad\text{for any real number } p \tag{1.1}$$

and $S(h,p)$ is sometimes denoted by $S(p)$ briefly. Especially $S(1)=S(h,1)=\dfrac{h^{\frac{1}{h-1}}}{e\log h^{\frac{1}{h-1}}}$ is said to be *Specht ratio* and $S(1)>1$ is well known.

Let $h>1$. *The generalized Kantorovich constant* $K(h,p)$ is defined by

$$K(h,p)=\frac{(h^p-h)}{(p-1)(h-1)}\left(\frac{(p-1)}{p}\cdot\frac{(h^p-1)}{(h^p-h)}\right)^p\quad\text{for any } p>1 \text{ or } p<0 \tag{1.2}$$

and $K(h,p)$ is sometimes denoted by $K(p)$ briefly. Many papers on Kantorovich inequality have been published. Among others, there is a long research series by Mond-Pecaric, we cite [6] and [7] for examples. The following result is shown in [4].

Let A be strictly positive operator satisfying $MI\geqslant A\geqslant mI>0$, where $M>m>0$. Also let $h=\frac{M}{m}>1$. Then the following inequality holds for every unit vector x:

$$K(p)(Ax,x)^p \geqslant (A^p x,x) \geqslant (Ax,x)^p \quad \textit{for any } p>1 \textit{ or } p<0. \tag{1.3}$$

On the other hand, the following interesting relation among $S(1)$, $K'(1)$ and $K'(0)$ is shown in [Proposition 1, 5] such that $S(1) = \exp\langle [\frac{dK(p)}{dp}]_{p=1}\rangle = \exp\langle [\frac{-dK(p)}{dp}]_{p=0}\rangle$.

PROPOSITION A. [5]. *The following properties on $K(p)$ and $S(p)$ hold:*
(i) $K(-p) = K(p+1)$ *for any $p \geqslant 0$,*
(ii) $K(0) = K(1) = 1$,
(iii) $S(-p) = S(p)$ *for any $p \geqslant 0$,*
(iv) $S(0) = 1$,
(v) $K'(-p) = -K'(p+1)$ *for any $p \geqslant 0$,*
(vi) $S(1) = e^{K'(1)} = e^{-K'(0)}$.

By applying this relation (vi) of Proposition A to (1.3), we have the following result in [Theorem 1, 5] which is the definitive role to give a proof of Theorem 1.

THEOREM B. [5]. *Let A be strictly positive operator satisfying $MI \geqslant A \geqslant mI > 0$, where $M > m > 0$ and $h = \frac{M}{m} > 1$. Then the following inequalities hold for every unit vector x:*
(i) $[\log S(1)](Ax,x) + (Ax,x)\log(Ax,x) \geqslant ((A\log A)x,x) \geqslant (Ax,x)\log(Ax,x)$.
(ii) $[\log S(1)] + ((\log A)x,x) \geqslant \log(Ax,x) \geqslant ((\log A)x,x)$.

We need the following result [6] by Mond and Pečarić to give a proof of Theorem 2.

THEOREM C. [6]. *Let A be a strictly positive operator satisfying $MI \geqslant A \geqslant mI > 0$, where $M > m > 0$ and $f \in C[m,M]$ be a convex function. Then*
(i) $(f(A)x,x) \geqslant f((Ax,x))$ *holds for every unit vector x. Moreover, let $g \in C([m,M])$. Then for any real number α*
(ii) $\alpha g((Ax,x)) + \beta \geqslant (f(A)x,x)$ *holds for every unit vector x, where* $\beta = \max\limits_{m\leqslant t\leqslant M}\{f(m) + \frac{f(M)-f(m)}{M-m}(t-m) - \alpha g(t)\}$.

2. An operator inequality and the upper bound related to the operator convexity of $A\log A$

At first we shall state an operator inequality associated with the operator concavity of operator entropy $A\log A^{-1}$, equivalently the operator convexity of $A\log A$.

THEOREM 1. *Let A and B be strictly positive operator satisfying $MI \geqslant A$, $B \geqslant mI > 0$, where $M > m > 0$ and $h = \frac{M}{m} > 1$. Then the following inequality holds:*

$$\begin{aligned}[\log S(1)]((1-\lambda)A+\lambda B) &+ ((1-\lambda)A+\lambda B)\log((1-\lambda)A+\lambda B)\\ &\geqslant (1-\lambda)A\log A + \lambda B\log B\\ &\geqslant ((1-\lambda)A+\lambda B)\log((1-\lambda)A+\lambda B)\end{aligned}$$

for all $\lambda \in [0,1]$, where $S(1)$ is defined in (1.1).

Secondary we shall state the upper bound associated with the operator concavity of operator entropy $A\log A^{-1}$, equivalently the operator convexity of $A\log A$.

THEOREM 2. *Let A and B be strictly positive operator satisfying $MI \geqslant A$, $B \geqslant mI > 0$, where $M > m > 0$ and $h = \frac{M}{m} > 1$. Then the following inequality holds*:

$$\begin{aligned}\frac{M\log h}{h-1}(S(1)-1) &+ ((1-\lambda)A+\lambda B)\log((1-\lambda)A+\lambda B)\\ &\geqslant (1-\lambda)A\log A + \lambda B\log B\\ &\geqslant ((1-\lambda)A+\lambda B)\log((1-\lambda)A+\lambda B)\end{aligned}$$

for all $\lambda \in [0,1]$, where $S(1)$ is defined in (1.1).

REMARK 1. Let $S\nabla_\lambda T$ is defined by

$$S\nabla_\lambda T = (1-\lambda)S + \lambda T \quad \text{for operators } S \text{ and } T. \tag{2.1}$$

Although the arithmetic mean $S\nabla_\lambda T$ is usually defined for positive operators A and B, we employ $S\nabla_\lambda T$ for *not necessary positive operators* A and B for convenience. By using this notation, Theorem 1 and Theorem 2 can be briefly expressed as follows respectively.

THEOREM 1'. *Let A and B be strictly positive operator satisfying $MI \geqslant A$, $B \geqslant mI > 0$, where $M > m > 0$ and $h = \frac{M}{m} > 1$. Then the following inequality holds*:

$$\begin{aligned}[\log S(1)]A\nabla_\lambda B + (A\nabla_\lambda B)\log(A\nabla_\lambda B) &\geqslant (A\log A)\nabla_\lambda(B\log B)\\ &\geqslant (A\nabla_\lambda B)\log(A\nabla_\lambda B)\end{aligned}$$

for all $\lambda \in [0,1]$, where $S(1)$ are $S\nabla_\lambda T$ are defined in (1.1) *and* (2.1) *respectively.*

THEOREM 2'. *Let A and B be strictly positive operator satisfying $MI \geqslant A$, $B \geqslant mI > 0$, where $M > m > 0$ and $h = \frac{M}{m} > 1$. Then the following inequality holds*:

$$\begin{aligned}\frac{M\log h}{h-1}(S(1)-1) + (A\nabla_\lambda B)\log(A\nabla_\lambda B) &\geqslant (A\log A)\nabla_\lambda(B\log B)\\ &\geqslant (A\nabla_\lambda B)\log(A\nabla_\lambda B)\end{aligned}$$

for all $\lambda \in [0,1]$, where $S(1)$ are $S\nabla_\lambda T$ are defined in (1.1) *and* (2.1) *respectively.*

3. Proofs of Theorem 1 and Theorem 2

Proof of Theorem 1. First of all, we state the following well known operator inequality as the operator concavity of the operator entropy $A\log A^{-1}$, equivalently the operator convexity of $A\log A$ ([1][8] and [3]):

$$(1-\lambda)A\log A + \lambda B\log B \geqslant ((1-\lambda)A+\lambda B)\log((1-\lambda)A+\lambda B). \tag{3.1}$$

In Theorem B, we replace A by $A \oplus B$, and also replace x by $\sqrt{\lambda}x \oplus \sqrt{1-\lambda}x$ for some fixed $\lambda \in [0,1]$. Then we have

$$\begin{aligned}
&\log S(1)((1-\lambda)A+\lambda B)x,x) \\
&\quad = \log S(1)[((1-\lambda)(Ax,x)+\lambda(Bx,x)] \\
&\quad \geqslant (1-\lambda)(A\log Ax,x)+\lambda(B\log Bx,x) \\
&\quad - [(1-\lambda)(Ax,x)+\lambda(Bx,x)]\log[(1-\lambda)(Ax,x)+\lambda(Bx,x)] \text{ by Theorem B} \\
&\quad = (((1-\lambda)A\log A+\lambda B\log B)x,x) \\
&\quad - ((1-\lambda)A+\lambda B)x,x)\log(((1-\lambda)A+\lambda B)x,x) \\
&\quad \geqslant (((1-\lambda)A\log A+\lambda B\log B)x,x) \\
&\quad - ([((1-\lambda)A+\lambda B)\log((1-\lambda)A+\lambda B)]x,x) \\
&\quad \geqslant 0 \quad \text{by (3.1)}
\end{aligned}$$

and the second inequality follows by $(T\log Tx,x) \geqslant (Tx,x)\log(Tx,x)$ which is derived from (i) of Theorem C since $t\log t$ is convex function. Whence the proof is complete.

We state the following result to give a proof of Theorem 2.

PROPOSITION 3. *Let A be a strictly positive operator satisfying $MI \geqslant A \geqslant mI > 0$, where $M > m > 0$ and $h = \frac{M}{m} > 1$. Then the following inequality holds for every unit vector x:*

$$\begin{aligned}
\frac{M\log h}{h-1}(S(1)-1)+(Ax,x)\log(Ax,x) &\geqslant (A\log Ax,x) \\
&\geqslant (Ax,x)\log(Ax,x),
\end{aligned}$$

where $S(1)$ is defind in (1.1).

Proof of Proposition 3. We put $f(t) = g(t) = t\log t$ which is a convex function in Theorem C. Also put $\alpha = 1$ in Theorem C. Then we have

$$\begin{aligned}
\beta &= \max_{m\leqslant t\leqslant M}\left\{f(m)+\frac{f(M)-f(m)}{M-m}(t-m)-\alpha g(t)\right\} \\
&= \max_{m\leqslant t\leqslant M}\left\{m\log m+\frac{M\log M-m\log m}{M-m}(t-m)-t\log t\right\} \\
&= \max_{m\leqslant t\leqslant M}\{h(t)\}
\end{aligned}$$

where $h(t) = \dfrac{h(t-m)\log h+(h-1)t(\log m-\log t)}{h-1}$. By an easy differential calculation we have $g'(t_0) = 0$ and $g''(t_0) = -\frac{e}{m}h^{\frac{-h}{h-1}} < 0$ for $t_0 = \frac{m}{e}h^{\frac{h}{h-1}}$, so that

$$\beta = h(t_0) = \frac{M}{e}\left(h^{\frac{1}{h-1}} - e\log h^{\frac{1}{h-1}}\right) = \frac{M\log h}{h-1}(S(1)-1).$$

Whence the proof is complete by (i) and (ii) of Theorem C.

Proof of Theorem 2. By the same way as in the proof of Theorem 1 by Theorem B, we have Theorem 2 by Proposition 3, that is, we have only to replace $[\log S(1)((1-\lambda)A+\lambda B]$ by $\frac{M\log h}{h-1}(S(1)-1)$ in the proof of Theorem 1, swe omit its proof in detail.

4. Further extensions of Theorem 1 and Theorem 2

In this section, we shall state further extensions of Theorem 1 and Theorem 2 as follows.

THEOREM 4. *Let* A_j *be strictly positive operator satisfying* $MI \geqslant A_j \geqslant mI > 0$ *for* $j = 1,2,\dots,n$, *where* $M > m > 0$ *and* $h = \frac{M}{m} > 1$. *Also* $\lambda_1,\ \lambda_2,\dots,\lambda_n$ *be any positive numbers such that* $\sum_{j=1}^{n} \lambda_j = 1$. *Then the following inequalities hold*:

(i)

$$[\log S(1)] \sum_{j=1}^{n} \lambda_j A_j + \Big(\sum_{j=1}^{n} \lambda_j A_j\Big) \log\Big(\sum_{j=1}^{n} \lambda_j A_j\Big) \geqslant \sum_{j=1}^{n} \lambda_j A_j \log A_j \geqslant \Big(\sum_{j=1}^{n} \lambda_j A_j\Big) \log\Big(\sum_{j=1}^{n} \lambda_j A_j\Big)$$

(ii) $[\log S(1)] + \sum_{j=1}^{k} \lambda_j \log A_j \geqslant \log\Big(\sum_{j=1}^{k} \lambda_j A_j\Big) \geqslant \sum_{j=1}^{k} \lambda_j \log A_j$

where $S(1) = S(h,1)$ *is Specht ratio defined in Definition* (1.1).

THEOREM 5. *Let* A_j *be strictly positive operator satisfying* $MI \geqslant A_j \geqslant mI > 0$ *for* $j = 1,2,\dots,n$, *where* $M > m > 0$ *and* $h = \frac{M}{m} > 1$. *Also* $\lambda_1,\ \lambda_2,\dots,\lambda_n$ *be any positive numbers such that* $\sum_{j=1}^{n} \lambda_j = 1$. *Then the following inequalities hold*:

(i)

$$\frac{M \log h}{h-1}(S(1)-1) + \Big(\sum_{j=1}^{n} \lambda_j A_j\Big) \log\Big(\sum_{j=1}^{n} \lambda_j A_j\Big) \geqslant \sum_{j=1}^{n} \lambda_j A_j \log A_j \geqslant \Big(\sum_{j=1}^{n} \lambda_j A_j\Big) \log\Big(\sum_{j=1}^{n} \lambda_j A_j\Big)$$

where $S(1) = S(h,1)$ *is Specht ratio defined in Definition* (1.1).

In order to give proofs of Theorem 4 and Theorem 5, we prepare the following results.

THEOREM B'. *Let A_j be strictly positive operator satisfying $MI \geqslant A_j \geqslant mI > 0$ for $j = 1, 2, \ldots, n$, where $M > m > 0$ and $h = \frac{M}{m} > 1$ and also let $x_1, x_2, \ldots, x_n$ be any finite number of vectors such that $\sum_{j=1}^{n} ||x_j||^2 = 1$. Then the following inequality holds*:

(i)

$$
\begin{aligned}
[\log S(1)] \sum_{j=1}^{n} (A_j x_j, x_j) + \Big(\sum_{j=1}^{n} (A_j x_j, x_j)\Big) \log\Big(\sum_{j=1}^{n} (A_j x_j, x_j)\Big) \\
\geqslant \sum_{j=1}^{n} ((A_j \log A_j) x_j, x_j) \\
\geqslant \Big(\sum_{j=1}^{n} (A_j x_j, x_j)\Big) \log\Big(\sum_{j=1}^{n} (A_j x_j, x_j)\Big)
\end{aligned}
$$

(ii) $\log S(1) + \sum_{j=1}^{n} (\log A_j x_j, x_j) \geqslant \log\Big(\sum_{j=1}^{n} A_j x_j, x_j\Big) \geqslant \sum_{j=1}^{n} (\log A_j x_j, x_j)$.

Proof of Theorem B'. We have only to put $A = A_1 \oplus A_2 \oplus \ldots \oplus A_n$ and $x = x_1 \oplus x_2 \oplus \ldots \oplus x_n$ for $\sum_{j=1}^{n} ||x_j||^2 = 1$ in (i) and (ii) of Theorem B.

PROPOSITION 3'. *Let A_j be strictly positive operator satisfying $MI \geqslant A_j \geqslant mI > 0$ for $j = 1, 2, \ldots, n$, where $M > m > 0$ and $h = \frac{M}{m} > 1$ and also let $x_1, x_2, \ldots, x_n$ be any finite number of vectors such that $\sum_{j=1}^{n} ||x_j||^2 = 1$. Then the following inequality holds*:

(i')

$$
\begin{aligned}
\frac{M \log h}{h-1}(S(1) - 1) + \Big(\sum_{j=1}^{n} (A_j x_j, x_j)\Big) \log\Big(\sum_{j=1}^{n} (A_j x_j, x_j)\Big) \\
\geqslant \sum_{j=1}^{n} ((A_j \log A_j) x_j, x_j) \\
\geqslant \sum_{j=1}^{n} (A_j x_j, x_j) \log\Big(\sum_{j=1}^{n} (A_j x_j, x_j)\Big).
\end{aligned}
$$

Proof of Proposition 3'. We have only to put $A = A_1 \oplus A_2 \oplus \ldots \oplus A_n$ and $x = x_1 \oplus x_2 \oplus \ldots \oplus x_n$ for $\sum_{j=1}^{n} ||x_j||^2 = 1$ in Proposition 3, so the proof is complete.

PROPOSITION D. *Let $f(t)$ be an operator convex function and $\lambda_1, \lambda_2, \ldots, \lambda_n$ be any positive numbers such that $\sum_{j=1}^{n} \lambda_j = 1$. Then*

$$\sum_{j=1}^{n} \lambda_j f(A_j) \geqslant f\Big(\sum_{j=1}^{n} \lambda_j A_j\Big). \tag{4.1}$$

Although Proposition D is almost well known, we state a proof for the sake of completeness and reader's convenience.

Proof. Assume (4.1). Let $\lambda_1, \lambda_2, \ldots, \lambda_{n+1}$ be any positive numbers such that $\sum_{j=1}^{n+1} \lambda_j = 1$. Also let $\mu_j = \frac{\lambda_j}{1-\lambda_1}$ for $j = 2, \ldots, n, n+1$. Then $\sum_{j=2}^{n+1} \mu_j = 1$ and

$$\begin{aligned} &f(\lambda_1 A_1 + \lambda_2 A_2 + \ldots + \lambda_{n+1} A_{n+1}) \\ &\qquad = f(\lambda_1 A_1 + (1-\lambda_1)(\mu_2 A_2 + \ldots + \mu_{n+1} A_{n+1})) \\ &\qquad \leqslant \lambda_1 f(A_1) + (1-\lambda_1)(f(\mu_2 A_2 + \ldots + \mu_{n+1} A_{n+1})) \\ &\qquad \leqslant \lambda_1 f(A_1) + (1-\lambda_1)(\mu_2 f(A_2) + \ldots + \mu_{n+1} f(A_{n+1}))) \\ &\qquad = \lambda_1 f(A_1) + \lambda_2 f(A_2) + \ldots + \lambda_{n+1} f(A_{n+1}) \end{aligned}$$

since the first inequality follows by the operator convexity and the second one follows by the assumption of (4.1) and the last equality follows by $\lambda_j = (1-\lambda_1)\mu_j$ for $j = 2, \ldots, n+1$, so the proof is complete by mathematical induction.

Proof of Theorem 4. Proof of (i). Although we have only to trace a proof of Theorem 1, we state its proof for the sake of completeness. As $(T \log Tx, x) \geqslant (Tx, x)\log(Tx, x)$ holds for every unit vector x. Since $t \log t$ is a convex function, so we put $T = \sum_{j=1}^{n} \lambda_j A_j$. Then we have the following (4.2)

$$\Big(\Big[\Big(\sum_{j=1}^{n} \lambda_j A_j\Big) \log\Big(\sum_{j=1}^{n} \lambda_j A_j\Big)\Big] x, x\Big) \geqslant \Big(\sum_{j=1}^{n} (\lambda_j A_j x, x)\Big) \log\Big(\sum_{j=1}^{n} (\lambda_j A_j x, x)\Big). \tag{4.2}$$

Put $\mu_j = \sqrt{\lambda_j}$ for $j = 1, 2, \ldots, n$. We recall that $\sum_{j=1}^{n} ||\mu_j x||^2 = \sum_{j=1}^{n} \lambda_j ||x||^2 = 1$ for every unit vector x. Then we have

$$\begin{aligned} &[\log S(1)]\Big(\sum_{j=1}^{n} \lambda_j A_j x, x\Big) \\ &\qquad = [\log S(1)] \sum_{j=1}^{n} (A_j \mu_j x, \mu_j x) \\ &\qquad \geqslant \sum_{j=1}^{n} (A_j \log A_j \mu_j x, \mu_j x) - \Big(\sum_{j=1}^{n} (A_j \mu_j x, \mu_j x)\Big) \log\Big(\sum_{j=1}^{n} (A_j \mu_j x, \mu_j x)\Big) \end{aligned}$$

by (i) of Theorem B'

$$= \left(\left(\sum_{j=1}^{n} \lambda_j A_j \log A_j\right)x, x\right) - \left(\sum_{j=1}^{n}(\lambda_j A_j x, x)\right) \log\left(\sum_{j=1}^{n}(\lambda_j A_j x, x)\right)$$

$$\geqslant \left(\left(\sum_{j=1}^{n} \lambda_j A_j \log A_j\right)x, x\right) - \left(\left[\left(\sum_{j=1}^{n} \lambda_j A_j\right) \log\left(\sum_{j=1}^{n} \lambda_j A_j\right)\right]x, x\right)$$

$$\geqslant 0$$

and the second inequality follows by (4.2) and the last inequality follows by (4.1) of Proposition D since $t\log t$ is an operator convex function by [1][8] and [3].

Proof of (ii). As the reverse inequality to (4.1) of Proposition D holds for an operator concave function by tracing the proof of Proposition D, so that the latter inequality of (ii) holds since $f(t) = \log t$ is an operator concave function. Next we show the former half inequality of (ii). We recall the following inequality by (ii) of Theorem B':

$$\log S(1) + \sum_{j=1}^{n}(\log A_j x_j, x_j) \geqslant \log\left(\sum_{j=1}^{n} A_j x_j, x_j\right). \tag{4.3}$$

Put $x_j = \sqrt{\lambda_j}x$ for $j = 1, 2, \ldots, n$. We recall that $\sum_{j=1}^{n} ||x_j||^2 = \sum_{j=1}^{n} \lambda_j ||x||^2 = 1$ for every unit vector x. Then we have the following (4.4) by refining (4.3)

$$\log S(1) + \left(\left(\sum_{j=1}^{n} \lambda_j \log A_j\right)x, x\right) \geqslant \log\left(\left(\sum_{j=1}^{n} \lambda_j A_j\right)x, x\right) \geqslant \left(\left(\log \sum_{j=1}^{n} \lambda_j A_j\right)x, x\right) \tag{4.4}$$

and the last inequality of (4.4) holds since $f(t) = \log t$ is concave function. (4.4) implies the former inequality of (ii). Whence the proof is complete.

REMARK 2. It is interesting to point out interesting contrast between (i) of Theorem 4 and (ii) of Theorem 4, that is, (ii) of Theorem 4 is derived from the operator concavity of $\log t$, (i) of Theorem 4 is derived from the operator concavity of $t\log t^{-1}$, equivalently the operator convexity of $t\log t$.

We remark that (ii) of Theorem 4 in the case $n = 2$ is shown in [2] by nice considering of operator concavity of $\log t$.

Proof of Theorem 5. We have only to trace a proof of Theorem 2, that is, by applying Proposition 3' instead of Proposition 3, we have only to replace $[\log S(1)(\sum_{j=1}^{n} \lambda_j A_j)]$ by $\frac{M\log h}{h-1}(S(1) - 1)$ in the proof of Theorem 4, swe omit its proof in detail.

5. Alternative simple proof of Theorem 4

We state an elementary proof of Theorem 4 in § 4, here we shall give an alternative simple proof of Theorem 4. We need following result in [9] by using Mond-Pecaric method.

THEOREM E. [9]. *Let A be strictly positive operator satisfying $MI \geqslant A \geqslant mI > 0$, where $M > m > 0$. Also let $h = \frac{M}{m} > 1$ and $\lambda_j \in R_+$ such that $\sum_{j=1}^{n} \lambda_j = 1$. Then the following inequality holds for $p \in R$:*

$$\alpha_2\Big(\sum_{j=1}^{n}\lambda_j A_j\Big)^p \leqslant \sum_{j=1}^{n}\lambda_j A_j^p \leqslant \alpha_1\Big(\sum_{j=1}^{n}\lambda_j A_j\Big)^p. \tag{5.1}$$

with

$$\alpha_2 = \begin{cases} K(p)^{-1} & \text{if } p < -1 \text{ or } p > 2 \\ 1 & \text{if } -1 \leqslant p < 0 \text{ or } 1 \leqslant p \leqslant 2 \\ K(p) & \text{if if } 0 < p < 1 \end{cases}$$

and

$$\alpha_1 = \begin{cases} K(p) & \text{if } p < 0 \text{ or } p > 1 \\ 1 & \text{if if } 0 < p \leqslant 1 \end{cases}$$

where $K(p)$ is defined by in Definition (1.1) as follows :

$$K(p) = K(h,p) = \frac{(h^p - h)}{(p-1)(h-1)} \cdot \left(\frac{(p-1)}{p} \cdot \frac{(h^p-1)}{(h^p-h)}\right)^p.$$

Alternative proof of Theorem 4. At first we recall (5.1) of Theorem E:

$$\alpha_2\Big(\sum_{j=1}^{n}\lambda_j A_j\Big)^p \leqslant \sum_{j=1}^{n}\lambda_j A_j^p \leqslant \alpha_1\Big(\sum_{j=1}^{n}\lambda_j A_j\Big)^p. \tag{5.1}$$

Let $f(p)$ and $g(p)$ be defined respectively as follows:

$$f(p) = \alpha_1\Big(\sum_{j=1}^{n}\lambda_j A_j\Big)^p - \sum_{j=1}^{n}\lambda_j A_j^p$$

and

$$g(p) = \sum_{j=1}^{n}\lambda_j A_j^p - \alpha_2\Big(\sum_{j=1}^{n}\lambda_j A_j\Big)^p.$$

Then we have the following (5.2) and (5.3) by easy differential calculus:

$$f'(p) = \frac{d\alpha_1}{dp}\Big(\sum_{j=1}^{n}\lambda_j A_j\Big)^p + \alpha_1\Big(\sum_{j=1}^{n}\lambda_j A_j\Big)^p \log\Big(\sum_{j=1}^{n}\lambda_j A_j\Big) - \sum_{j=1}^{n}\lambda_j A_j^p \log A_j \tag{5.2}$$

and

$$g'(p) = \sum_{j=1}^{n} \lambda_j A_j^p \log A_j - \frac{d\alpha_2}{dp}\Big(\sum_{j=1}^{n} \lambda_j A_j\Big)^p - \alpha_2\Big(\sum_{j=1}^{n} \lambda_j A_j\Big)^p \log\Big(\sum_{j=1}^{n} \lambda_j A_j\Big). \quad (5.3)$$

Proof of (i). We consider p such that $2 \geqslant p \geqslant 1$. In this case, it turns out that $\alpha_1 = K(p)$ and $\alpha_2 = 1$ by Theorem E:

Obviously $f(1) = K(1)\sum_{j=1}^{n} \lambda_j A_j - \sum_{j=1}^{n} \lambda_j A_j = 0$ holds since $K(1) = 1$ by (ii) of Proposition A. As $f(p) \geqslant 0$ for any p such that $2 \geqslant p \geqslant 1$ by the latter half inequality of (5.1), we have $f'(1) \geqslant 0$, that is, we have

$$f'(1) = K'(1)\Big(\sum_{j=1}^{n} \lambda_j A_j\Big) + K(1)\Big(\sum_{j=1}^{n} \lambda_j A_j\Big)\log\Big(\sum_{j=1}^{n} \lambda_j A_j\Big) - \sum_{j=1}^{n} \lambda_j A_j \log A_j \geqslant 0. \quad (5.4)$$

So that (5.4) implies the former inequality of (i) in Theorem 4 since $K'(1) = \log S(1)$ by (vi) of Proposition A and $K(1) = 1$ by (ii) of Proposition A.

Obviously $g(1) = \sum_{j=1}^{n} \lambda_j A_j - \sum_{j=1}^{n} \lambda_j A_j = 0$. As $g(p) \geqslant 0$ for any p such that $2 \geqslant p \geqslant 1$ by the former half inequality of (5.1), we have $g'(1) \geqslant 0$, that is, we have the following (5.5) since $\frac{d\alpha_2}{dp} = 0$ holds by $\alpha_2 = 1$:

$$g'(1) = \sum_{j=1}^{n} \lambda_j A_j \log A_j - \Big(\sum_{j=1}^{n} \lambda_j A_j\Big)\log\Big(\sum_{j=1}^{n} \lambda_j A_j\Big) \geqslant 0. \quad (5.5)$$

(5.5) is the latter inequality of (i) in Theorem 4.

Proof of (ii). We consider p such that $1 > p > 0$. In this case it turns out that $\alpha_1 = 1$ and $\alpha_2 = K(p)$ by Theorem E.

Obviously $f(0) = 1 - \sum_{j=1}^{n} \lambda_j = 0$. As $f(p) \geqslant 0$ for any p such that $1 > p > 0$ by the latter half inequality of (5.1), we have $f'(0) \geqslant 0$, that is, we have we have the following (5.6) since $\frac{d\alpha_1}{dp} = 0$ holds by $\alpha_1 = 1$:

$$f'(0) = \log\Big(\sum_{j=1}^{n} \lambda_j A_j\Big) - \sum_{j=1}^{n} \lambda_j \log A_j \geqslant 0 \quad (5.6)$$

so that (5.6) implies the latter inequality of (ii) in Theorem 4.

Obviously $g(0)=\sum_{j=1}^{n}\lambda_j-K(0)=0$ since $K(0)=1$ by (ii) of Proposition A. As $g(p)\geqslant 0$ for any p such that $1>p>0$ by the former half inequality of (5.1), we have $g'(0)\geqslant 0$, that is,

$$g'(0)=\sum_{j=1}^{n}\lambda_j\log A_j-K'(0)-K(0)\log\Big(\sum_{j=1}^{n}\lambda_jA_j\Big)\geqslant 0 \tag{5.7}$$

and (5.7) is the former inequality of (ii) in Theorem 4 since $-K'(0)=\log S(1)$ by (vi) of Proposition A and $K(0)=1$ by (ii) of Proposition A. Whence an alternative proof of Theorem 4 is complete.

REMARK 3. Although we state a proof of (i) of Theorem 4 by considering the case $2\geqslant p\geqslant 1$ and also we state a proof of (ii) of Theorem 4 by considering the case $1>p>0$, it suffices to consider the case $1>p>0$ to give a proof of (i), that is, we have only to consider the case $1>p>0$ to show both (i) and (ii). In fact, $\alpha_1=1$ and $\alpha_2=K(p)$ in case $1>p>0$, $f(p)\geqslant 0$ for $1>p>0$ and $f(1)=0$, so that $f'(1-0)\leqslant 0$, that is,

$$f'(1-0)=\Big(\sum_{j=1}^{n}\lambda_jA_j\Big)\log\Big(\sum_{j=1}^{n}\lambda_jA_j\Big)-\sum_{j=1}^{n}\lambda_jA_j\log A_j\leqslant 0 \tag{5.8}$$

and (5.8) is the latter half inequality of (i). Also $g(p)\geqslant 0$ for $1>p>0$ and $g(1)=0$ since $K(1)=1$ by (ii) of Proposition A, so that $g'(1-0)\leqslant 0$, that is,

$$g'(1-0)=\sum_{j=1}^{n}\lambda_jA_j\log A_j-K'(1)\Big(\sum_{j=1}^{n}\lambda_jA_j\Big)-K(1)\Big(\sum_{j=1}^{n}\lambda_jA_j\Big)\log\Big(\sum_{j=1}^{n}\lambda_jA_j\Big)\leqslant 0 \tag{5.9}$$

and (5.9) is the former half inequality of (ii) since $K'(1)=\log S(1)$ by (vi) of Proposition A and $K(1)=1$ by (ii) of Proposition A.

REMARK 4. It is interesting to remark that an alternative simple proof of (ii) of Theorem 4 in § 5 is a direct one of the operator concavity of *the operator entropy* $f(A)=-A\log A$ shown in [1][8] and [3].

REFERENCES

[1] CH. DAVIS, *Operator-valued entropy of a quantum mechanical measurement*, Proc. Japan Acad., **37** (1961), 533-538.
[2] M. FUJII, Y. SEO AND M. TOMINAGA, *Golden-Thompson type inequalities related to a geometric mean via Specht's ratio*, Math. Inequal. and Appl., **5** (2002), 573–582.
[3] T. FURUTA, *Simple proof of the concavity of operator entropy* $f(A)=-A\log A$, Math. Inequal. and Appl., **3** (2000), 305–306.
[4] T. FURUTA, *Operator inequalities associated with Hölder-McCarthy and Kantorovich inequalities*, J. Inequal. and Appl., **2** (1998), 137–148.

[5] T. FURUTA, *Specht ratio* $S(1)$ *can be expressed by Kantorovich constant* $K(p)$: $S(1) = \exp\langle\, [\frac{dK(p)}{dp}]_{p=1}\rangle$ *and its application*, to appear in Math. Inequal. and Appl.
[6] B. MOND AND J. E. PEČARIĆ, *Convex inequalities in Hilbert space*, Houston J. Math., **19** (1993), 405–420.
[7] B. MOND AND J. E. PEČARIĆ, *A matrix version of Ky Fan Generalization of the Kantorovich inequality*, Linear and Multilinear Algebra, **36** (1994), 217–221.
[8] M. NAKAMURA AND H. UMEGAKI, *A note on the entropy for operator algebras*, Proc. Japan Acad., **37** (1961), 149–154.
[9] J. E. PEČARIĆ AND J. MIĆIĆ, *Chaotic order among means of positive operators*, Scienticae Mathematicae Online, **7** (2002), 97–106.

(Received October 1, 2002)

Takayuki Furuta
Department of Mathematical Information Science
Tokyo University of Science
1-3 Kagurazaka, Shinjukuku Tokyo 162-8601
Japan
e-mail: furuta@rs.kagu.tus.ac.jp

Josip Pecaric
Faculty of Textile Technology
University of Zagreb
Pierottijeva 6
10000 Zagreb
Croatia
e-mail: pecaric@hazu.hr