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# STABILITY OF IMPLICIT RESOLVENT DYNAMICAL SYSTEMS

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*Abstract.* In this paper, we propose and analyze implicit resolvent dynamical systems associated with mixed quasi variational inequalities by using the techniques of the resolvent operators. We prove that the globally asymptotic stability of these dynamical systems requires monotonicity of the operator. We also discuss some special cases, which can be obtained from our main results.

## 1. Introduction

Variational inequalities theory has witnessed an explosive growth in theoretical advances, algorithmic development and applications across all disciplines of pure and applied sciences. This theory provides a unified and novel treatment of equilibrium problems arising in economics, finance, transportation, elasticity and structural analysis, see [1–27]. It combines novel theoretical and algorithmic advances with new domain of applications. As a result of interaction between different branches of mathematical and engineering sciences, we now have a variety of techniques to suggest and analyze various iterative algorithms for solving variational inequalities and related optimization problems. Though the problems in each of these areas may look completely different, the resulting algorithms can be very closely related. Using the projection technique, one can show that the variational inequalities are equivalent to the fixed-point problem. This equivalence has been used [5–8, 20, 21, 25–27] to suggest and analyzed a projected dynamical system, in which the right-hand side of the ordinary differential equation is a projection operator. Projected dynamical systems are characterized by a discontinuous right-hand side. The discontinuity arises from the constraints governing the applications in question. The novel feature of the projected dynamical system is that the set of the stationary points of the dynamical systems corresponds to the set of the solutions of the variational inequalities. Consequently, the equilibrium problems which can be formulated in the setting of variational inequalities can now be studied in the more general setting of the dynamical systems. They are intrinsically as framework for creating behavioral model for describing disequilibrium trajectories of real economics and physical process prior to reaching steady states. Furthermore, Xia and Wang [26] have shown that the projected dynamical systems can be used effectively in designing

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neural network for solving variational inequalities. The neural network are computing systems composed of highly interconnected simple information processing units, and thus can solve variational inequalities and optimization problems in execution times at the order of magnitude mush faster than most iterative algorithms for general purpose digital computers.

In recent years variational inequalities have been extended and generalized in several directions using novel and innovative techniques. A useful and an important generalization is called the mixed quasi variational inequalitiy introduced and studied in [1, 4, 12, 13, 15, 16, 19] to tackle the complicated and complex problems arising in fluid flow through porous media, elasticity and structural analysis. Due to the presence of the bifunction in the formulation of these mixed quasi variational inequalities, projection, resolvent methods and their variant forms can't be extended and modified for mixed quasi variational inequalities. There are no such type of dynamical systems for mixed quasi variational inequalities. In this paper, we show that such type of dynamical systems can be suggested for mixed quasi variational inequalities. It is known [19] that if the bifunction involving the variational inequalities is proper, convex and lower-semicontinuous in the first argument, then mixed quasi variational inequalities are equivalent to the fixed-points. We use this alternative equivalent formulation to suggest implicit dynamical systems associated with mixed quasi variational inequalities. We use these dynamical systems to prove the uniqueness of a solution of mixed quasi variational inequalities, which requires only the Lipschitz continuity of the operator. This approach does not need any type of monotonicity. Secondly, if the bifunction is skew-symmetric and the operator is monotone, then we show that the implicit resolvent dynamical systems have globally asymptotic stability property. Since the mixed quasi variational inequalities include (quasi) variational inequalities and several optimization problems as special cases, our result continue to hold for these problems. Our results can be viewed as significant and unified extensions of the known results in this area.

## 2. Formulations and basic facts

Let $\mathbf{R}^n$ be a Euclidean space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ respectively. Let $K$ be a closed convex set in $\mathbf{R}^n$ and $A : \mathbf{R}^n \longrightarrow \mathbf{R}^n$ be a nonlinear operator. Let $\varphi(.,.) : \mathbf{R}^n \times \mathbf{R}^n \longrightarrow \mathbf{R} \cup \{+\infty\}$ be a continuous bifunction. We consider the problem of finding $u \in \mathbf{R}^n$ such that

$$\langle A(u), v-u \rangle + \varphi(v,u) - \varphi(u,u) \geqslant 0, \quad \forall v \in \mathbf{R}^n. \tag{1}$$

Problem (2.1) is called the *mixed quasi variational inequality*. It has been shown that a large class of obstacle, unilateral, contact, free, moving, and equilibrium problems arising in regional, physical, mathematical, engineering and applied sciences can be studied in the unified and general framework of the mixed quasi variational inequalities (2.1), see [1, 4, 8, 12–14, 18–20].

For $\varphi(v,u) = \varphi(v)$, $\forall u \in \mathbf{R}^n$, problem (2.1) reduces to finding $u \in \mathbf{R}^n$ such that

$$\langle A(u), v-u \rangle + \varphi(v) - \varphi(u) \geqslant 0, \quad \forall v \in \mathbf{R}^n, \tag{2}$$

which is called the mixed variational inequality or variational inequality of the second kind. For the recent state-of-the-art, see [1, 4, 9–13, 17–19] and the references therein.

If $\varphi$ is an indicator function of a closed convex set $K$ in $\mathbf{R}^n$, then problem (2.2) is equivalent to finding $u \in K$ such that

$$\langle Au, v-u\rangle \geqslant 0, \quad \forall v \in K, \tag{3}$$

which is known as the classical variational inequality introduced and studied by Stampacchia [24] in 1964. For the recent state-of-the-art, see [1–27] and the references therein.

We also need the following well known results and concepts.

DEFINITION 2.1. For all $u, v \in \mathbf{R}^n$, the operator $A : \mathbf{R}^n \longrightarrow \mathbf{R}^n$ is said to be

(a) *monotone*, if

$$\langle A(u) - A(v), u - v\rangle \geqslant 0.$$

(b) *Lipschitz continuous*, if there exists a constant $\beta > 0$ such that

$$\|Au - Av\| \leqslant \beta\|u - v\|.$$

DEFINITION 2.2. The bifunction $\varphi(.,.)$ is said to be *skew-symmetric*, if,

$$\varphi(u,u) - \varphi(u,v) - \varphi(v,u) + \varphi(v,v) \geqslant 0, \quad \forall u, v \in \mathbf{R}^n. \tag{4}$$

Clearly, if the bifunction $\varphi(.,.)$ is linear in both arguments, then,

$$\varphi(u,u) - \varphi(u,v) - \varphi(v,u) + \varphi(v,v) = \varphi(u-v, u-v) \geqslant 0, \quad \forall u, v \in \mathbf{R}^n,$$

which shows that the bifunction $\varphi(.,.)$ is nonnegative.

DEFINITION 2.3. Let $A$ be a maximal monotone operator, then the resolvent operator associated with $A$ is defined

$$J_A(u) = (I + \rho A)^{-1}(u), \quad \forall u \in \mathbf{R}^n,$$

where $\rho > 0$ is a constant and $I$ is the identity operator.

REMARK 2.1. It is well known that the subdifferential $\partial\varphi(.,.)$ of a convex, proper and lower-semicontinuous function $\varphi(.,.) : \mathbf{R}^n \times \mathbf{R}^n \longrightarrow \mathbf{R} \cup \{+\infty\}$ is a maximal monotone with respect to the first argument. We define its resolvent by

$$J_{\varphi(u)} = (I + \rho\partial\varphi(.,u))^{-1}) \equiv (I + \rho\partial\varphi(u))^{-1}, \tag{5}$$

where $\partial\varphi(u) \equiv \partial\varphi(.,u)$, unless otherwise specified. For more details, see [19].

The resolvent operator $J_{\varphi(u)}$ defined by (2.5) has the following characterization, which has important and significant applications in variational inequalities and optimization.

LEMMA 2.1. *For a given* $u \in \mathbf{R}^n$, $z \in \mathbf{R}^n$ *satisfies the inequality*

$$\langle u - z, v - u\rangle + \rho\varphi(v,u) - \rho\varphi(u,u) \geqslant 0, \quad \forall v \in \mathbf{R}^n, \tag{6}$$

*if and only if*

$$u = J_{\varphi(u)}z,$$

*where* $J_{\varphi(u)}$ *is resolvent operator defined by (2.5).*

Note that for $\varphi(v,u) = \varphi(v)$, $\forall u \in \mathbf{R}^n$, Lemma 2.1 is well known, see, for example [18].

Using Lemma 2.1, one can easily show that the mixed quasi variational inequalities are equivalent to the fixed-point problem, which is the motivation of our next result.

LEMMA 2.2. [19]. *The function $u \in \mathbf{R}^n$ is a solution of problem (2.1) if and only if $u \in \mathbf{R}^n$ satisfies the relation*

$$u = J_{\varphi(u)}[u - \rho A(u)], \tag{7}$$

*where $\rho > 0$ is a constant and $J_{\varphi(u)} = (I + \rho\partial\varphi(.\,,u))^{-1}$ is the resolvent operator.*

Lemma 2.2 implies that the quasi variational inequality (2.1) is equivalent to the fixed-point formulation. This alternative formulation has been used to discuss the existence of the solution and to suggest iterative methods for the quasi variational inequalities and related optimization problems, see [19].

We now define the residual vector $R(u)$ by the relation

$$R(u) = u - J_{\varphi(u)}[u - \rho A(u)]. \tag{8}$$

Invoking Lemma 2.2, we see that $u \in \mathbf{R}^n$ is a solution of the mixed quasi variational inequality (2.1) if and only if $u \in \mathbf{R}^n$ is a zero of the equation

$$R(u) = 0. \tag{9}$$

We now use the fixed-point formulation (2.7) to consider the following dynamical system

$$\frac{du}{dt} = \lambda\{J_{\varphi(u)}[u - \rho A(u)] - u\}, \quad u(t_0) = u_0 \in \mathbf{R}^n, \tag{10}$$

associated with the mixed quasi variational inequality (2.1), where $\lambda$ is a constant. Here the right hand side is related to the resolvent and is discontinuous on the boundary. It is clear from the definition that the solution to (2.10) always stays in the constraint set. This implies that the qualitative results such as the existence, uniqueness and continuous dependence of the solution on the given data of (2.10) can be studied.

For $\varphi(.\,,.) = \varphi(v), \forall u \in H,$ the dynamical system (2.10) ie equivalent to:

$$\frac{du}{dt} = \lambda\{J_{\varphi}[u - \rho A(u)] - u\}, \quad u(t_0) = u_0 \in \mathbf{R}^n. \tag{11}$$

This dynamical system has been studied by Noor [21]. These dynamical systems describe the disequilibrium adjustment processes, which may produce important transient phenomena prior to the achievement of a steady state. It has been shown that such type of the dynamical systems are useful for computational schemes. From the view point of neural computation, the structure of these systems are simple and can be easily implemented in a parallel circuit, see, for example, [7, 8, 25, 26].

DEFINITION 2.2. The dynamical system is said to converge to the solution set $K^*$ of (2.1) if, irrespective of the initial point, the trajectory of the dynamical system satisfies

$$\lim_{t\longrightarrow\infty} \operatorname{dist}(u(t), K^*) = 0, \tag{12}$$

where

$$\operatorname{dist}(u, K^*) = \inf_{v \in K^*} \|u - v\|.$$

It is easy to see that, if the set $K^*$ has a unique point $u^*$, then (2.12) implies that

$$\lim_{t \longrightarrow \infty} u(t) = u^*.$$

If the dynamical system is still stable at $u^*$ in the Lyapunov sense, then the dynamical system is globally asymptotically stable at $u^*$.

DEFINITION 2.3. The dynamical system is said to be globally exponentially stable with degree $\eta$ at $u^*$ if, irrespective of the initial point, the trajectory of the system $u(t)$ satisfies

$$\|u(t) - u^*\| \leqslant \mu_1 \|u(t_0) - u^*\| \exp(-\eta(t - t_0)), \quad \forall\, t \geqslant t_0,$$

where $\mu_1$ and $\eta$ are positive constants independent of the initial point. It is clear that globally exponential stability is necessarily globally asymptotical stability and the dynamical system converges arbitrarily fast.

LEMMA 2.4. (Gronwall [7]). *Let $\hat{u}$ and $\hat{v}$ be real-valued nonnegative conitnuous functions with domain $\{t : t \geqslant t_0\}$ and let $\alpha(t) = \alpha_0(|t - t_0|)$, where $\alpha_0$ is a monotone increasing function. If, for $t \geqslant t_0$,*

$$\hat{u}(t) \leqslant \alpha(t) + \int_{t_0}^{t} \hat{u}(s)\hat{v}(s)ds,$$

*then*

$$\hat{u}(t) \leqslant \alpha(t) \exp\left\{ \int_{t_0}^{t} \hat{v}(s)ds \right\}.$$

*We also need the following condition.*

ASSUMPTION 2.1. *$\forall\, u, v, w \in \mathbf{R}^n$, the operator $J_{\varphi(u)}$ satisfies the condition*

$$\|J_{\varphi(u)} w - J_{\varphi(v)} w\| \leqslant \nu \|u - v\|,$$

*where $\nu > 0$ is a constant.*

## 3. Main results

In this section we study the main properties of the implicit dynamical systems and analyze the global stability of the systems. First of all, we discuss the existence and uniqueness of the dynamical system (2.10) and this is the main motivation of our next result.

THEOREM 3.1. *Let the operator $A$ be a Lipschitz continuous operator and let Assumption 2.1 hold. Then, for each $u_0 \in \mathbf{R}^n$, there exists a unique continuous solution $u(t)$ of dynamical system (2.10) with $u(t_0) = u_0$ over $[t_0, \infty)$.*

*Proof.* Let

$$G(u) = \lambda\{J_{\varphi(u)}[u - \rho A(u)] - u\},$$

where $\lambda > 0$ is a constant. $\forall u, v \in \mathbf{R}^n$, and using Assumption 2.1, we have

$$\begin{aligned}
\|G(u) - G(v)\| &\leqslant \lambda\{\|J_{\varphi(u)}[u - \rho A(u)] - J_{\varphi(v)}[v - \rho A(v)]\| + \|u - v\|\} \\
&\leqslant \lambda\|u - v\| + \lambda\|J_{\varphi(u)}[u - \rho A(u)] - J_{\varphi(u)}[v - \rho A(v)]\| \\
&\quad + \lambda\|J_{\varphi(u)}[v - \rho A(v)] - J_{\varphi(v)}[v - \rho A(v)]\| \\
&\leqslant \lambda\{\|u - v\| + \|u - v + \rho(A(u) - A(v))\| + \mu\|u - v\|\} \\
&\leqslant \lambda\{2 + \mu + \rho\beta\}\|u - v\|,
\end{aligned}$$

where $\beta > 0$ is a Lipschitz constant of the operator $A$. This implies that the operator $G(u)$ is a Lipschitz continuous in $\mathbf{R}^n$. So, for each $u_0 \in \mathbf{R}^n$, there exists a unique and continuous solution $u(t)$ of the implicit dynamical system of (2.10), defined in a interval $t_0 \leqslant t < T$ with the initial condition $u(t_0) = u_0$. Let $[t_0, T)$ be its maximal interval of existence; we show that $T = \infty$. Consider

$$\begin{aligned}
\|G(u)\| &= \lambda\|J_{\varphi(u)}[u - \rho A(u)] - u\| \\
&\leqslant \lambda\{\|J_{\varphi(u)}[u - \rho A(u)] - J_{\varphi(u)}[u]\| + \|J_{\varphi(u)}[u] - J_{\varphi(u^*)}[u]\| \\
&+ \|J_{\varphi(u)}[u^*] - J_{\varphi(u^*)}[u^*]\| + \|J_{\varphi(u^*)[u^*]} - u\|\} \\
&\leqslant \lambda\rho\|A(u)\| + \lambda\mu\|u - u^*\| + \lambda\mu\|u - u^*\| + \lambda\|J_{\varphi(u^*)}[u^*]\| + \lambda\|u\| \\
&= \lambda(1 + \beta_1 + 2\mu)\|u\| + \lambda\{2\mu\|u^*\| + \|J_{\varphi(u^*)}[u^*]\|\}
\end{aligned}$$

for any $u \in \mathbf{R}^n$, then

$$\begin{aligned}
\|u(t)\| &\leqslant \|u_0\| + \int_{t_0}^{t} \|Tu(s)\| ds \\
&\leqslant (\|u_0\| + k_1(t - t_0)) + k_2 \int_{t_0}^{t} \|u(s)\| ds,
\end{aligned}$$

where $k_1 = \lambda(2\mu)\|u^*\| + \lambda\|J_{\varphi(u^*)}[u^*]\|$ and $k_2 = \lambda(1 + \beta_1 + 2\mu)$. Hence, by invoking Lemma 2.4, we have

$$\|u(t)\| \leqslant \{\|u_0\| + k_1(t - t_0)\}e^{k_2(t - t_0)}, \quad t \in [t_0, T).$$

This shows that the solution $u(t)$ is bounded on $[t_0, T)$. So $T = \infty$. □

We now study the stability of the implicit dynamical system (2.10). The analysis is in the spirit of Noor [20, 21] and Xia and Wang [26].

THEOREM 3.2. *Let $A$ be a pseudomonotone Lipschitz continuous operator and let Assumption 2.1 hold. If the bifunction $\varphi(.,.)$ is skew-symmetric, then the dynamical system (2.10) is stable in the sense of Lyapunov and globally converges to the solution subset of (2.1).*

*Proof.* Since the operator $A$ is a Lipschitz continuous operator, it follows from Theorem 3.1, that the dynamical system (2.12) has a unique continuous solution $u(t)$

over $[t_0, T)$ for any fixed $u_0 \in \mathbf{R}^n$. Let $u(t) = u(t, t_0; u_0)$ be the solution of the initial value problem (2.12). For a given $u^* \in \mathbf{R}^n$, consider the following Lyapunov function

$$L(u) = \lambda \|u - u^*\|^2, \quad u \in \mathbf{R}^n. \tag{1}$$

It is clear that $\lim_{n\to\infty} L(u_n) = +\infty$ whenever the sequence $\{u_n\} \subset \mathbf{R}^n$ and $\lim_{n\to\infty} u_n = +\infty$. Consequently, we conclude that the level sets of $L$ are bounded. Let $u^* \in \mathbf{R}^n$ be a solution of (2.1). Then

$$\langle A(u^*), v - u^* \rangle + \varphi(v, u^*) - \varphi(u^*, u^*) \geqslant 0, \quad \forall v \in H$$

which implies that

$$\langle A(v), v - u^* \rangle + \varphi(v, u^*) - \varphi(u^*, u^*) \geqslant 0, \tag{2}$$

since the operator $A$ is monotone.

Taking $v = J_{\varphi(u)}[u - \rho A(u)]$ in (3.2), we have

$$\begin{aligned} &\langle AJ_{\varphi(u)}[u - \rho A(u)], J_{\varphi(u)}[u - \rho A(u)] - u^* \rangle \\ &\qquad + \varphi(J_{\varphi(u)}[u - \rho A(u)], u^*) - \varphi(u^*, u^*) \geqslant 0. \end{aligned} \tag{3}$$

Setting $v = u^*$, $u = J_{\varphi(u)}[u - \rho A(u)]$, and $z = u - \rho A(u)$ in (2.6), we have

$$\begin{aligned} &\langle J_{\varphi(u)}[u - \rho A(u)] - u + \rho A(u), u^* - J_{\varphi(u)}[u - \rho A(u)] \rangle \\ &+ \rho\varphi(u^*, J_{\varphi(u)}[u - \rho A(u)]) - \rho\varphi(J_{\varphi(u)}[u - \rho A(u)], J_{\varphi(u)}[u - \rho A(u)]) \geqslant 0. \end{aligned} \tag{4}$$

Adding (3.3), (3.4), using the skew-symmetricity of $\varphi(.,.)$, and from (2.8), we obtain

$$\langle -R(u), u^* - u + R(u) \rangle \geqslant 0,$$

which implies that

$$\langle u - u^*, R(u) \rangle \geqslant \|R(u)\|^2. \tag{5}$$

Thus, from (2.8), (2.10), (3.1) and (3.5), we have

$$\begin{aligned} \frac{d}{dt} L(u) &= \frac{dL}{du}\frac{du}{dt} \\ &= 2\lambda \langle u - u^*, J_{\varphi(u)}[u - \rho A(u)] - u \rangle \\ &= 2\lambda \langle u - u^*, -R(u) \rangle \\ &\leqslant -2\lambda \|R(u)\|^2 \leqslant 0. \end{aligned}$$

This implies that $L(u)$ is a global Lyapunov function for the implicit dynamical system in (2.10) and the implicit dynamical system (2.10) is stable in the sense of Lyapunov. Since $\{u(t) : t \geqslant t_0\} \subset K_0$, where $K_0 = \{u \in \mathbf{R}^n : L(u) \leqslant L(u_0)\}$ and the function $L(u)$ is continuously differentiable on the bounded and closed set $\mathbf{R}^n$, it follows from LaSalle's invariance principle that the trajectory will converge to $\Omega$, the largest invariant subset of the following subset:

$$E = \left\{ u \in \mathbf{R}^n : \frac{dL}{dt} = 0 \right\}.$$

Note that, if $\frac{dL}{dt} = 0$, then

$$\|u - J_{\varphi(u)}[u - \rho A(u)]\|^2 = 0,$$

and hence $u$ is an equilibrium point of the implicit dynamical system (2.10), that is,

$$\frac{du}{dt} = 0.$$

Conversely, if $\frac{du}{dt} = 0$, then it follows that $\frac{dL}{dt} = 0$. Thus, we conclude that

$$E = \left\{ u \in \mathbf{R}^n : \frac{du}{dt} = 0 \right\} = K_0 \cap K^*,$$

which is nonempty, convex and invariant set containing the solution set $K^*$. So

$$\lim_{t \longrightarrow \infty} \text{dist}\,(u(t), E) = 0.$$

Therefore, the implicit dynamical system (2.10) converges globally to the solution set of (2.1). In particular, if the set $E = \{u^*\}$, then

$$\lim_{t \longrightarrow \infty} u(t) = u^*.$$

Hence the system (2.10) is globally asymptotically stable. □

THEOREM 3.3. *Let the operator $A$ be Lipschitz continuous with a constant $\beta > 0$ and let Assumption 2.1 hold. If $\lambda < 0$, then the implicit dynamical system (2.10) converges globally exponentially to the unique solution of the quasi variational inequalities (2.1).*

*Proof.* From Theorem 3.1, we see that there exists a unique continuously differentiable solution of the implicit dynamical system (2.10) over $[t_0, \infty)$. Then, (2.10) and (3.1), we have

$$\begin{aligned} \frac{dL}{dt} &= 2\lambda \langle u(t) - u^*, J_{\varphi(u(t))}[u(t) - \rho A(u(t))] - u(t) \rangle \\ &= -2\lambda \|u(t) - u^*\|^2 + 2\lambda \langle u(t) - u^*, J_{\varphi(u(t))}[u(t) - \rho A(u(t))] - u^* \rangle, \end{aligned} \tag{6}$$

where $u^* \in \mathbf{R}^n$ is the solution of the quasi variational inequality (2.1), that is,

$$u^* = J_{\varphi(u^*)}[u^* - \rho A(u^*)].$$

Now, using the Assumption (2.1) and Lipschitz continuity of the operator $A$, we have

$$\begin{aligned} \|J_{\varphi(u)}[u - \rho A(u)] &- J_{\varphi(u^*)}[u^* - \rho A(u^*)]\| \\ &\leqslant \|J_{\varphi(u)}[u - \rho A(u)] - J_{\varphi(u^*)}[u - \rho A(u)]\| \\ &+ \|J_{\varphi(u^*)}[u - \rho A(u)] - J_{\varphi(u^*)}[u^* - \rho A(u^*)]\| \\ &\leqslant \mu\|u - u^*\| + \|u - u^* - \rho(Au - Au^*)\| \\ &\leqslant \mu\|u - u^*\| + \|u - u^*\| + \rho\beta\|u - u^*\| \\ &\leqslant (1 + \mu + \rho\beta)\|u - u^*\|. \end{aligned} \tag{7}$$

From (3.6) and (3.7), we have

$$\frac{d}{dt}\|u(t)-u^*\|^2 \leqslant 2\alpha\lambda\|u(t)-u^*\|^2,$$

where

$$\alpha = \mu + \rho\beta.$$

Thus, for $\lambda = -\lambda_1$, where $\lambda_1$ is a positive constant, we have

$$\|u(t)-u^*\| \leqslant \|u(t_0)-u^*\|e^{-\alpha\lambda_1(t-t_0)},$$

which shows that the trajectory of the solution of the implicit dynamical system (2.10) will globally exponentially converge to the unique solution of the quasi variational inequalities (2.1). □

## Conclusions

We have suggested some implicit resolvent dynamical systems associated with the quasi variational inequalities. We have proved the global asymptotical stability of the systems for monotone operators. The suggested dynamical systems can be used in designing recurrent neural networks for solving solving quasi variational inequalities and optimization problems. Thus, these implicit dynamical systems have wide applicability than the existing optimization neural networks. We hope that the theoretical results obtained in this paper may provide a different approach for stability analysis, computation, analysis and design of new neural networks.

## REFERENCES


[1] C. Baiocchi and A. Capelo, *Variational and Quasi Variational Inequalities*, J. Wiley and Sons, New York, 1984.

[2] T. R. Bielecki and S. R. Pliska, *Risk sensitive asset management with transaction costs*, Finance and Stochastics, **4** (2000), 1–33.

[3] R. W. Cottle, F. Giannessi and J. L. Lions, *Variational Inequalities and Complementarity Problems: Theory and Applications*, J. Wiley and Sons, New York, 1980.

[4] J. Crank, *Free and Moving Boundary Problems*, Clarendon Press, Oxford, U. K., 1984.

[5] J. Dong, D. Zhang and A. Nagurney, *A projected dynamical systems model of general financial equilibrium with stability analysis*, Math. Computer Modelling, **24** (2) (1996), 35–44.

[6] P. Dupuis and A. Nagurney, *Dynamical systems and variational inequalities*, Annals Opers. Research, **44** (1993), 19–42.

[7] T. L. Friesz, D. H. Bernstein, N. J. Mehta and S. Ganjliazadeh, *Day-to-day dynamic network disequilibria and idealized traveler information systems*, Operations Research, **42** (1994), 1120–1136.

[8] T. L. Friesz, D. H. Bernstein and R. Stough, *Dynamic systems, variational inequalities and control theoretic models for predicting time-varying urban network flows*, Trans. Science, **30** (1996), 14–31.

[9] F. Giannessi and A. Maugeri, *Variational Inequalities and Network Equilibrium Problems*, Plenum Press, New York, 1995.

[10] F. Giannessi, A. Magueri and P. M. Pardalos, *Equilibrium Problems: Nonsmooth Optimization and Variational inequality Models*, Kluwer Academic Publishers, Dordrecht, Holland, 2001.

[11] R. Glowinski, J. L. Lions and R. Tremolieres, *Numerical Analysis of Variational inequalities*, North-Holland, Amsterdam, Holland, 1981.

[12] N. Kikuchi and J. T. Oden, *Contact Problems in Elasticity*, SIAM Publishing Co., Philadelphia, 1988.

[13] D. KINDERLEHRER AND G. STAMPACCHIA, *An Introduction to Variational Inequalities and Their Applications*, SIAM Publishing Co., Philadelphia, 2000.
[14] A. NAGURNEY AND D. ZHANG, *Projected Dynamical Systems and Variational Inequalities with Applications*, Kluwer Academic Publishers, Dordrecht, Holland, 1995.
[15] M. ASLAM NOOR, *On a class of variational inequalities*, J. Math. Anal. Appl. **128** (1987), 135–155.
[16] M. ASLAM NOOR, *Nonlinear variational inequalities in elastostatics*, Int. J. Engg. Sci. **26** (1988), 1043–1053.
[17] M. ASLAM NOOR, *Some recent advances in variational inequalities*, Part I, basic concepts, New Zealand J. Math. **26** (1997), 53–80.
[18] M. ASLAM NOOR, *Some recent advances in variational inequalities*, Part II, other concepts, New Zealand J. Math. **26** (1997), 229–255.
[19] M. ASLAM NOOR, *Set-valued mixed quasi variational inequalities and implicit resolvent equations*, Math. Computer Modelling, **29** (1999), 1–11.
[20] M. ASLAM NOOR, *Implicit dynamical systems for quasi variational inequalities*, Appl. Math. Comput. **134** (2002), 69–81.
[21] M. ASLAM NOOR, *Resolvent dynamical systems for mixed variational inequalities*, Korean J. Comput. Appl. Math. **9** (2002), 15–26.
[22] M. ASLAM NOOR, *A Wiener-Hopf dynamical system for variational inequalities*, New Zealand J. Math, **31** (2002), 173–182.
[23] M. ASLAM NOOR, K. INAYAT NOOR AND TH. M. RASSIAS, *Some aspects of variational inequalities*, J. Comput. Appl. Math. **47** (1993), 285–312.
[24] G. STAMPACCHIA, *Formes bilineaires coercitives sur les ensembles convexes*, C. R. Acad. Sci. Paris, **258** (1964), 4413–4416.
[25] Y. S. XIA AND J. WANG, *A recurrent neural network for solving linear projection equations*, Neural Network, **13** (2000), 337–350.
[26] Y. S. XIA AND J. WANG, *On the stability of globally projected dynamical systems*, J. Optim. Theory Appl. **106** (2000), 129–150.
[27] D. ZHANG AND A. NAGURNEY, *On the stability of the projected dynamical systems*, J. Optim. Theory and Appl. **85** (1995), 97–124.

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