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A NOTE ON COMPLEMENTARITY PROBLEM IN BANACH SPACE

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Abstract. The purpose of this note is to establish an existence theorem for nonlinear complementarity problem for strictly pseudo-monotone operator over convex cone in a Banach space.

1. Introduction and statement of the theorem

Variational inequality and complementarity problem are related subjects having applications in various areas. These problems have been studied by several authors including Browder [1], Hartman and Stampacchia [2], Karamardian [3], Lions and Stampacchia [4], Mosco [5], Nanda [6,7], Thera [8] and Yao [9].

The purpose of this note is to establish an existence theorem for nonlinear complementarity problem under most general assumptions on the operator and the cone.

In order to state our theorem we need the following definitions.

Let B be a reflexive real Banach space, and let B^* be its dual. Let the value of $u \in B^*$ at $x \in B$ be denoted by (u, x). Let K be a closed convex cone in B with vertex at 0.

The polar of K is the cone K^* defined by

$$K^* = \{u \in B^* : (u, x) \geqslant 0 \text{ for each } x \in K\}.$$

For any $e \in K^*$ and $r > 0$ we define

$$D_r(e) = \{x \in K : 0 \leqslant (e, x) \leqslant r\}$$
$$D_r^0(e) = \{x \in K : 0 < (e, x) < r\}$$
$$S_r(e) = \{x \in K : (e, x) = r\}$$

and also

$$D_r = \{x \in K : \|x\| \leqslant r\}$$
$$D_r^0 = \{x \in K : \|x\| < r\}$$
$$S_r = \{x \in K : \|x\| = r\}.$$

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$T : K \mapsto B^*$ is said to be pseudo-monotone if $(Ty, x-y) \geqslant 0$ implies $(Tx, x-y) \geqslant 0$ for all $x,\ y \in K$ and strictly pseudo-monotone if $(Ty, x-y) > 0$ implies $(Tx, x-y) > 0$ for $x \neq y$.

In this note we prove

THEOREM 1. *Let K be a closed convex cone in a real Banach space B. Let K^* denote the polar cone of K. Suppose that T is a strictly pseudo-monotone operator from K into B^* satisfying the following conditions:*

(i) the function $x \mapsto (Tx, y)$ is sequentially weakly upper semicontinuous on K for each $y \in B$;

(ii) for any sequence $\{x_n\} \subset K$ converging weakly to x, $\limsup_{n\to\infty} (Tx_n, x_n) \geqslant (Tx, x)$.

Then the complementarity problem (CP)

$$x \in K,\ Tx \in K^* \ \text{ such that } \ (Tx, x) = 0 \tag{1}$$

has a solution, if any one of the following conditions holds:

(iii) $\exists$ an $x \in K$ with $Tx \in \operatorname{int} K^*$.

(iii)' for at least one $r > 0$, $\exists$ an $u \in D_r^0$ with $(Tx, x-u) \geqslant 0\ \forall\, x \in S_r$.

To prove this theorem we need the following result of Jen-Chih Yao [9].

THEOREM. [Yao] *Let K be a nonempty weakly compact convex subset of the real Banach space B. Suppose that T is an operator from K into B^* satisfying the following assumptions:*

(i) the function $x \mapsto (Tx, y)$ is sequentially weakly upper semicontinuous on K for each $y \in B$.

(ii) for any sequence $\{x_n\} \subset K$ converging weakly to x, $\limsup_{n\to\infty} (Tx_n, x_n) \geqslant (Tx, x)$. Then the variational inequality $x \in K : \ (Tx, u-x) \geqslant 0,\ \forall\, u \in K$, has a solution.

2. Proof of the theorem

Proof for condition (iii).

For any $e \in \operatorname{int} K^*$ and each $r > 0$, $D_r(e)$ is a weakly compact convex subset of B (see Nanda [6]). Hence by Theorem [Yao], see Yao [9], for each $r > 0$ there is an $x_r \in D_r(e)$ such that

$$(Tx_r, y - x_r) \geqslant 0, \quad \forall\, y \in D_r(e). \tag{2}$$

Since $0 \in D_r(e)$, we have $(Tx_r, x_r) \leqslant 0$. For $r > 0$, $e \in K^*$ if $\exists\ x_r \in D_r^0(e)$ then there is some $\lambda > 1$ such that $\lambda x_r \in S_r(e) \subset D_r(e)$. Then from (2) we have $(Tx_r, x_r) \leqslant (Tx_r, \lambda x_r) = \lambda (Tx_r, x_r)$. Since $(Tx_r, x_r) \leqslant 0$, it is impossible unless $(Tx_r, x_r) = 0$, thus x_r is a solution to CP.

Let $x_r \in S_r(e)\ \forall\ e \in$ int K^* and $r > 0$. By our assumption there is an $x \in K$ with $Tx \in$ int K^*. Set $e = Tx$. Choose $r > (Tx, x) \geqslant 0$, now $x \in D_r^0(Tx)$. For $z \in S_r(Tx)$, $(Tx, z) = r$ and hence $(Tx, z - x) = (Tx, z) - (Tx, x) > 0$. Since T is strictly pseudo-monotone and $(Tx, z-x) > 0$ for $z \in S_r(Tx)$, we have $(Tz, z-x) > 0$ for $z \in S_r(Tx)$.

But $x_r \in S_r(Tx)$, hence

$$(Tx_r, x_r - x) > 0. \tag{3}$$

Since $x \in D_r^0(Tx) \subset D_r(Tx)$, it follows from (2) that $(Tx_r, x - x_r) \geqslant 0$ i.e. $(Tx_r, x_r - x) \leqslant 0$, this contradicts (3) and hence the assumption that $x_r \in S_r(e)$ for all r has thus been shown not to hold when $e = Tx$. Thus the proof is reduced to the previous case, the case where $\exists\, e \in \text{int } K^*$ and $r > 0$ such that $x \in D_r^0(e)$. This completes the proof under condition (iii).

Proof for condition (iii)'.

Since for each $r > 0$, D_r is weakly compact convex subset of B, it follows from theorem [Yao] that for each $r > 0$ there exists an $x_r \in D_r$ such that

$$(Tx_r, y - x_r) \geqslant 0 \quad \text{for all} \quad y \in D_r. \tag{4}$$

Since $0 \in D_r$, $(Tx_r, x_r) \leqslant 0$. If there is some $r > 0$ such that $S_r \in D_r^0$, then there is some $\lambda > 0$ such that $\lambda x_r \in S_r \subset D_r$. So from (4), $(Tx_r, x_r) \leqslant \lambda (Tx_r, x_r)$. Since $(Tx_r, x_r) \leqslant 0$ this is impossible unless $(Tx_r, x_r) = 0$. Thus (x_r) is the solution to CP (1).

If $x_r \in S_r$ for all $r > 0$, then by the hypothesis, for at least one $r > 0$, there is a $u \in D_r^0$ such that

$$(Tx_r, x_r - u) \geqslant 0 \quad \text{for all} \quad x_r \in S_r \tag{5}$$

For that $r > 0$, we have from (4) and (5) $(Tx_r, y - u) \geqslant 0,\ \forall\, y \in D_r$. Let $z \in K$ and $w = \lambda z + (1-\lambda)u$, $0 < \lambda < 1$. We can choose λ sufficiently small so that w lies in D_r, then

$$0 \leqslant (Tx_r, w - u) = \lambda (Tx_r, z - u) \Rightarrow (Tx_r, z - u) \geqslant 0 \quad \text{for all} \quad z \in K. \tag{6}$$

Since $u \in D_r^0$ it follows from (4) that

$$(Tx_r, u - x_r) \geqslant 0. \tag{7}$$

From (6) and (7)

$$(Tx_r, z - x_r) \geqslant 0 \quad \text{for all} \quad z \in K. \tag{8}$$

Taking $z = \lambda x_r$, $\lambda > 1$ in (8) we get $(Tx_r, x_r) \geqslant 0$ Since $(Tx_r, x_r) \leqslant 0$ we obtain $(Tx_r, x_r) = 0$, $Tx_r \in K^*$ thus x_r is a solution of CP (1), and this completes the proof.

3. Another result

In [2] Hartman and Stampacchia proved the following theorem

THEOREM.

(i) If $T : K \mapsto B^*$ is hemicontinuous and monotone, where K is closed convex and bounded, then variational inequality has a solution.

(ii) If $B = R^n$, $T : K \mapsto R^n$ is a continuous operator, where K is compact, convex, then variational inequality has a solution.

We have

THEOREM 2. *If K is a closed convex cone, $T : K \mapsto R^n$ is continuous and strictly pseudo-monotone and $\exists\, x \in K$ with $Tx \in$ int K^* then complementarity problem has a solution.*

This can be proved by using the theorem of Hartman and Stampacchia [2] and by similar arguments used to prove the previous theorem under the conditions (i), (ii) and (iii).

In fact the following example shows that if $\exists\, x \in K$ with $Tx \in K^*$ but $Tx \notin$ int K^*, then the complementarity problem does not have a solution.

Take $B = R^3$ and $K = \{(x,y,z) \in R^3 : x,z \geqslant 0,\, xz \geqslant y^2\}$. Define T by $T(x,y,z) = (0, y+2, z+2)$. Then T is monotone and continuous. The point $(2,-2,2) \in K$ and $T(2,-2,2) = (0,0,4) \in K^*$. If $u = (x,y,z) \in K$ with $Tu \in K^*$, then $y = -2$, and hence $z > 0$. Hence for any such u, $(Tu, u) = z(z+2) > 0$.

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