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RADICAL AND RATIONAL MEANS OF DEGREE TWO

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(*communicated by P. S. Bullen*)

Abstract. In this paper, we introduce two families of means that encompass many of the classical means and we characterize the internality properties in these families. We determine the comparability relations within these two families and we study their behavior under equal increments of the variables. We also introduce a geometrical context that gives rise to one of these families.

1. Introduction

By a mean, one usually understands a function of two or more positive numbers that is *continuous*, *symmetric*, *homogeneous*, of homogeneity degree 1, and *internal*. More precisely, an n-dimensional mean on $\mathbb{R}_+ = (0, \infty)$ is a function $\mathfrak{M} : \mathbb{R}_+^n \longrightarrow \mathbb{R}_+$ with the following properties:

Continuity: $\mathfrak{M}$ is continuous in each of its n arguments;

Symmetry: $\mathfrak{M}(\mathbf{x}) = \mathfrak{M}(\sigma(\mathbf{x}))$ for every permutation σ of the positive n-tuple $\mathbf{x}$;

Homogeneity: $\mathfrak{M}(\lambda\mathbf{x}) = \lambda\mathfrak{M}(\mathbf{x})$ for all positive real numbers λ and all positive n-tuples $\mathbf{x}$;

Internality: For all positive n-tuples $\mathbf{x}$, $\min \mathbf{x} \leqslant \mathfrak{M}(\mathbf{x}) \leqslant \max \mathbf{x}$.

Although there is no universally accepted definition of means [10, page 372], [13] and [8], the above definition does seem to capture our most intuitive understanding of the term and conforms to much of the existing literature on the subject [13]. In contexts where some of the above axioms are not needed, authors drop them from their definition. However, the author holds the view expressed in [6] that internality is an essential part of any definition of means.

The (2-dimensional) *arithmetic*, *geometric* and *harmonic means* defined by

$$\frac{x+y}{2}, \qquad \sqrt{xy}, \qquad \frac{2xy}{x+y}$$

are probably the oldest and most well-known. They were known to the ancient Pythagoreans [14, page 75], [10, Chapter II], arising in their study of numbers, geometry,

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music and possibly other practical considerations. The *root-quadratic*, *contra-harmonic* and *centroidal means* defined by

$$\sqrt{\frac{x^2+y^2}{2}}, \qquad \frac{x^2+y^2}{x+y}, \qquad \frac{2}{3}\left(\frac{x^2+xy+y^2}{x+y}\right)$$

are less well-known but they appear quite frequently in the literature. Their geometric significance is described in [14, page 168]. The *Heronian mean*

$$\frac{x+\sqrt{xy}+y}{3}$$

also holds a well-known place in the history of Egyptian mathematics [15, Lecture 2]. Using the terminology of [30], the 2-modification[1] of this mean is $\sqrt{(x^2+xy+y^2)/3}$. It occurs there with similar 2-modifications of means such as

$$\sqrt{(1-t)xy+t\left(\frac{x^2+y^2}{2}\right)}.$$

The above means, or their 2-modifications, are all of the type $\sqrt{Q(x,y)}$ or of the type $Q(x,y)/(x+y)$, where Q is a symmetric quadratic form in x and y. These two families of means together with their higher dimensional analogues are the objects of study in this article. They will be given the names *quadratic radical* and *quadratic rational means* (or *radical and rational means of degree* 2). Specifically, a *quadratic radical*, respectively *rational*, *mean* is a mean of the form $\sqrt{Q}$, respectively Q/S, where Q is a symmetric quadratic form in the n variables $x_1,\dots,x_n$ and where $S=S(\mathbf{x})=x_1+\dots+x_n$.

Although both $\sqrt{Q}$ and Q/S are continuous, symmetric and homogeneous, the requirement of internality places heavy restrictions on the coefficients of Q. These resrtrictions are the subject matter of Section 2 and form the contents of Theorems 3 and 5. In Section 3, Theorem 8 describes conditions under which one of the means $\sqrt{Q_1}$ and Q_2/S dominates the other. In this connection, we note that the arithmetic mean belongs to both families since

$$\frac{x+y}{2}=\sqrt{\frac{x^2+y^2+2xy}{4}}=\frac{(x+y)^2}{2(x+y)}.$$

In fact, it is the only mean with this property; see Theorem 8(a). The behavior of these means under equal increments of their variables, the so-called *means on the move property*, is taken up in Section 4, Theorem 9. In Section 5, we give a geometrical context in which the radical means arise. Finally in the last section, there are some comments on radical and rational means of higher degree.

[1] If $\mathfrak{M}(x,y)$ is a mean, then the 2-*modification* of $\mathfrak{M}(x,y)$ is the mean $\mathfrak{M}_2(x,y)=\sqrt{\mathfrak{M}(x^2,y^2)}$.

2. Internality of quadratic radical and rational means

Let us define

$$S_n(\mathbf{x}) = \sum_{i=1}^{n} x_i,\ \ T_n(\mathbf{x}) = \sum_{i=1}^{n} x_i^2,\ \ P_n(\mathbf{x}) = \sum_{1\leqslant i<j\leqslant n} x_i x_j,\ \ \Delta_n(\mathbf{x}) = \sum_{1\leqslant i<j\leqslant n} (x_j - x_i)^2.$$

If there is no ambiguity, the subscript n and, or, the n-tuple $\mathbf{x}$ will be omitted from these notations; they then are just $S,\ T,\ P,\ \Delta$. The vector space of all symmetric quadratic forms, in n variables, has dimension 2 and is generated by the two forms T and P. Since also

$$S^2 = T + 2P, \qquad \Delta = (n-1)T - 2P,$$

we see that any two of the forms S^2, T, P, Δ generate this vector space. A further useful formula is

$$(n-1)S^2 = \Delta + 2nP. \tag{1}$$

These relations will be used, often without reference, in what follows.

The first result tells us when a symmetric quadratic form is non-negative. Although this cannot be new, we state and prove it for the reader's convenience and for ease of reference.

THEOREM 1. *If $Q = s\Delta + tP$, $s,t \in \mathbb{R}$, is a symmetric quadratic form, then*
(a) $Q(\mathbf{x}) \geqslant 0$ for all non-negative n-tuples $\mathbf{x}$ if and only if $s \geqslant 0$ and $t \geqslant 0$.
(b) $Q(\mathbf{x}) \geqslant 0$ for all n-tuples $\mathbf{x}$ if and only if $s \geqslant 0$ and $s \geqslant t/2n \geqslant 0$.

Proof. (a) If $Q \geqslant 0$ for non-negative n-tuples, then taking $x_1 = \ldots = x_n = 1$ shows that $t \geqslant 0$; then taking $x_1 = \ldots = x_{n-1} = 0, x_n = 1$ shows that $s \geqslant 0$. Conversely, if $s \geqslant 0$ and $t \geqslant 0$, then $\Delta \geqslant 0$ and $P \geqslant 0$ for all non-negative n-tuples and so $Q \geqslant 0$.

(b) If $Q \geqslant 0$ for all n-tuples, then s and t are $\geqslant 0$ by (a). Further, taking $x_1 = \ldots = x_{n-2} = 0$ and $x_n = -x_{n-1} = 1$, it follows that $s(4 + 2(n-2)) - t \geqslant 0$ and hence $2ns \geqslant t$. Conversely, suppose that $s \geqslant 0$ and that $2ns \geqslant t \geqslant 0$. If $s = 0$, then $t = 0$ and there is nothing to prove. So we may assume $s > 0$. In fact, we may then without loss in generality assume that $s = 1$ and that $2n \geqslant t \geqslant 0$. If $t = 0$, it is immediate that $Q \geqslant 0$; if $t = 2n$, then $Q = (n-1)S^2 \geqslant 0$. Since Q is linear in t, given $\mathbf{x}$, the converse is proved. □

COROLLARY 2. *If $Q = \alpha T + \beta P$, $\alpha, \beta \in \mathbb{R}$, is a symmetric quadratic form, then*
(a) $Q(\mathbf{x}) \geqslant 0$ for all non-negative n-tuples $\mathbf{x}$ if and only if

$$\alpha \geqslant 0 \quad \text{and} \quad \beta \geqslant \frac{-2\alpha}{n-1}.$$

(b) $Q(\mathbf{x}) \geqslant 0$ for all n-tuples $\mathbf{x}$ if and only if

$$\alpha \geqslant 0 \quad \text{and} \quad \alpha \geqslant \frac{\beta}{2} \geqslant \frac{-\alpha}{n-1}.$$

Proof. This follows from Theorem 1 and the identity

$$(n-1)Q = \alpha\Delta + 2\alpha P + (n-1)\beta P = \alpha\Delta + (2\alpha + (n-1)\beta)P. \qquad \square$$

It follows from Theorem 1 that for Q of the form $s\Delta + tP$ we must have that $s,t \geqslant 0$ for $\sqrt{Q}$ to be a quadratic radical mean. However for the mean to be internal s and t must satisfy further conditions. These conditions are described in the next theorem where we found it more convenient to work with another (but equivalent) form of Q.

THEOREM 3. *If*

$$Q = \frac{2}{n(n-1)}(s\Delta + tP), \qquad s,t \in \mathbb{R}$$

is a symmetric quadratic form, then $\sqrt{Q}$ *is internal,*

$$\min(\mathbf{x}) \leqslant \sqrt{Q} \leqslant \max(\mathbf{x}) \qquad \text{for all positive } n\text{−tuples } \mathbf{x} \tag{2}$$

if and only if $t = 1$ *and* $0 \leqslant s \leqslant 1$.

Proof. Clearly by Theorem 1, (2) only makes sense if $s,t \geqslant 0$. Suppose then that (2) holds. Take $x_1 = \ldots = x_n = 1$ to see that $t = 1$. Now taking $x_2 = x_3 = \ldots = x_n = 1$, $x_1 = 0$, we conclude that

$$\frac{2-n}{2} \leqslant s \leqslant 1.$$

Thus if Q satisfies (2), then, using the preliminary remark, $t = 1$ and $0 \leqslant s \leqslant 1$.

Conversely, suppose that $t = 1$ and $0 \leqslant s \leqslant 1$ and assume without loss in generality that $\mathbf{x}$ is increasing. Let $F(s)$, $0 \leqslant s \leqslant 1$ denote Q as a function of s. Since F increases with s, (2) will be established if we prove that

$$(i)\ \ F(0) \geqslant x_1^2, \qquad (ii)\ \ F(1) \leqslant x_n^2.$$

Since

$$F(0) = \frac{2}{n(n-1)}P = \frac{2}{n(n-1)} \sum_{1\leqslant i<j\leqslant n}^{n} x_i x_j \geqslant \frac{2}{n(n-1)} \sum_{1\leqslant i<j\leqslant n} x_1^2 = x_1^2,$$

(i) follows. To establish (ii), note that

$$n(n-1)(x_n^2 - F(1)) = (n-1)(n-2)x_n^2 + 2x_n S_{n-1} - 2(n-1)T_{n-1} + 2P_{n-1}. \tag{3}$$

Writing

$$A_k = (k-1)(k-2)x_k^2 + 2(n-k+1)x_k S_{k-1} - 2(n-1)T_{k-1} + 2P_{k-1}$$

for $2 \leqslant k \leqslant n$, we see that the right-hand side of (3) is just A_n. We prove that $A_n \geqslant 0$ by induction, first noting that

$$A_2 = 2(n-1)x_1x_2 - 2(n-1)x_1^2 = 2(n-1)x_1(x_2 - x_1) \geqslant 0.$$

Now assume that $A_k \geqslant 0$ for some k with $2 \leqslant k < n$. Then

$$\begin{aligned} A_{k+1} &= k(k-1)x_{k+1}^2 + 2(n-k)x_{k+1}S_k - 2\left((n-1)T_k - P_k\right) \\ &\geqslant k(k-1)x_k^2 + 2(n-k)x_kS_k - 2\left((n-1)T_k - P_k\right) \\ &= \left(k(k-1) + 2(n-k) - 2(n-1)\right)x_k^2 + \left(2(n-k)+2\right)x_kS_{k-1} \\ &\quad -2\left((n-1)T_{k-1} - P_{k-1}\right) \\ &= (k-1)(k-2)x_k^2 + 2(n-k+1)x_kS_{k-1} - 2\left((n-1)T_{k-1} - P_{k-1}\right) \\ &= A_k \geqslant 0. \end{aligned}$$

This completes the induction and the proof of the theorem. □

COROLLARY 4. *If $Q = \alpha T + \beta P$, $\alpha, \beta \in \mathbb{R}$, is a symmetric quadratic form, then $\sqrt{Q}$ is internal if and only if*

$$0 \leqslant \alpha \leqslant \frac{2}{n} \quad \text{and} \quad \beta = \frac{2(1-n\alpha)}{n(n-1)}.$$

THEOREM 5. *If*

$$Q = \frac{1}{n-1}\left(\frac{s\Delta + 2tP}{S}\right), \qquad s, t \in \mathbb{R},$$

is a symmetric quadratic form, then Q/S is internal

$$\min(\mathbf{x}) \leqslant \frac{Q}{S} \leqslant \max(\mathbf{x}) \qquad \text{for all positive } n\text{−tuples } \mathbf{x} \tag{4}$$

if and only if $t = 1$ and $0 \leqslant s \leqslant 1$.

Proof. Suppose that (4) holds. Taking $x_1 = \ldots = x_n = 1$, gives $t = 1$, and taking $x_1 = \ldots = x_{n-1} = 0$, $x_n = 1$, we conclude that $0 \leqslant s \leqslant 1$.

Conversely, suppose that $t = 1$ and $0 \leqslant s \leqslant 1$ and assume without loss of generality that $\mathbf{x}$ is a positive increasing n-tuple. Let $G(s)$, $0 \leqslant s \leqslant 1$, denote Q/S as a function of s. Since G is increasing, (4) will follow if we prove

$$(i)\ \ G(0) \geqslant x_1, \qquad (ii)\ \ G(1) \leqslant x_n.$$

Since

$$\begin{aligned} (n-1)SG(0) &= 2P = 2\sum_{1\leqslant i<j\leqslant n}^{n} x_ix_j = \sum_{i=1}^{n} x_i\Big(\sum_{j\neq i} x_j\Big) \\ &\geqslant (n-1)x_1\sum_{i=1}^{n} x_i = (n-1)Sx_1, \end{aligned}$$

(i) follows. Also,

$$(n-1)SG(1) = \Delta + 2P = (n-1)\sum_{i=1}^{n} x_i^2 \leqslant (n-1)x_n\sum_{i=1}^{n} x_i = (n-1)Sx_n.$$

This proves (ii). □

COROLLARY 6. *If* $Q = \alpha T + \beta P,\ \alpha, \beta \in \mathbb{R},$ *is a symmetric quadratic form, then* Q/S *is internal if and only if*

$$\frac{n+1}{2n} \leqslant \alpha \leqslant 1 \quad \text{and} \quad \beta = \frac{2(1-\alpha)}{n-1}.$$

Theorems 3 and 5 are summarized in Theorem 7 below, where the statement concerning strict internality follows by inspection of the proofs of Theorems 3 and 5.

THEOREM 7.
(a) Quadratic radical means are of the form

$$\mathfrak{F}_s = \sqrt{\frac{2}{n(n-1)}(s\Delta + P)}, \quad 0 \leqslant s \leqslant 1. \tag{5}$$

(b) Quadratic rational means are of the form

$$\mathfrak{G}_s = \frac{1}{n-1}\left(\frac{s\Delta + 2P}{S}\right), \quad 0 \leqslant s \leqslant 1. \tag{6}$$

Furthermore, all the means in these two families, beside being continuous, symmetric and homogeneous, are also strictly internal in the sense that $\min \mathbf{x} < \mathfrak{M}(\mathbf{x}) < \max \mathbf{x}$ *for all non-constant positive* n*-tuples* $\mathbf{x}$.

3. Comparability of quadratic radical and rational means

According to Theorem 7, the family of quadratic radical means and the family of quadratic rational means are both parametrized by a single parameter s. Further, we have the following simple comparability relations:

$$\mathfrak{F}_a \leqslant \mathfrak{F}_b \Longleftrightarrow \mathfrak{G}_a \leqslant \mathfrak{G}_b \Longleftrightarrow 0 \leqslant a \leqslant b \leqslant 1.$$

These are immediate since both $\mathfrak{F}_s$ and $\mathfrak{G}_s$ are increasing functions of s, $0 \leqslant s \leqslant 1$. The comparability of the two families is given in the following theorem.

THEOREM 8.
(a) $\mathfrak{F}_a = \mathfrak{G}_b$ *if and only if* $a = b/2 = 1/2n$, *when both means reduce to the arithmetic mean.*
(b) $\mathfrak{F}_a \leqslant \mathfrak{G}_b$ *if and only if* $b \geqslant a + 1/2n$.
(c) $\mathfrak{F}_a \geqslant \mathfrak{G}_b$ *if and only if* $a \geqslant nb^2/2$.
In all other cases, the means $\mathfrak{F}_a$ *and* $\mathfrak{G}_b$ *are not comparable.*

Proof. By (5) and (6),

$$\begin{aligned} n(n-1)^2S^2(\mathfrak{F}_a^2 - \mathfrak{G}_b^2) &= 2(n-1)S^2(a\Delta + P) - n(b\Delta + 2P)^2 \\ &= 2(\Delta + 2nP)(a\Delta + P) - n(b\Delta + 2P)^2, \qquad \text{by (1)} \\ &= \Delta\left((2a - nb^2)\Delta + 2(1 + 2an - 2bn)P\right). \end{aligned}$$

By Lemma 1, the last expression is non-negative, respectively 0, for all non-negative $\mathbf{x}$ if and only if the coefficients $\alpha = 2a - nb^2$ and $\beta = 1 + 2an - 2bn$ are non-negative, respectively 0. Noting that $\beta - n\alpha = (nb - 1)^2$, we see that α and β have the same sign. This proves (b) and (c). Finally (a) follows by solving $\alpha = \beta = 0$. □

In Theorem 8, it was pointed out that $\mathfrak{F}_{1/2n} = \mathfrak{G}_{1/n} =$ the arithmetic mean, and it is of interest to note other familiar means that belong to the above two families of means.

$$\mathfrak{F}_0 = \sqrt{\frac{2P}{n(n-1)}} = \sqrt{\frac{1}{\binom{n}{2}} \sum_{1 \leqslant i < j \leqslant n} x_i x_j}$$

is the *second elementary symmetric polynomial mean*; see [10, Chapter V];

$$\mathfrak{F}_{\frac{1}{2}} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} x_i^2}$$

is the *root-quadratic mean*; see [10, Chapter III];

$$\mathfrak{G}_1 = \frac{\sum_{i=1}^{n} x_i^2}{\sum_{i=1}^{n} x_i}$$

is an example of a *Gini mean*, the *contra-harmonic mean*; see [10, Chapter IV]. The mean $\mathfrak{F}_1$ does not seem to appear in the literature although the case $n = 2$ is the 2-modification of the generalized Heronian mean with $t = 2$. Finally, we note that the case $n = 2$ of the means $\mathfrak{G}_s$, $0 \leqslant s \leqslant 1$, are encountered in [29, p. 519] where they are called the *generalized means of harmonic type*, and that the case $n = 2$ of the means $\mathfrak{F}_s$, $0 \leqslant s \leqslant 1$, are nothing but the means $P_{2,s}(P_2, G)$ mentioned in [28, p. 512].

4. Quadratic radical and rational means on the move

The next theorem answers the question of how each of the above means behaves when all of the terms are increased by the same amount. This question was initiated in [21] and further studied in [9], [1], [2], [17].

THEOREM 9. *Let $\mathfrak{F}_s$ and $\mathfrak{G}_s$, $0 \leqslant s \leqslant 1$, be the means defined in Theorem 7, and let*

$$\phi_s(t) = \mathfrak{F}_s(x_1 + t, \ldots, x_n + t) - t, \quad \psi_s(t) = \mathfrak{G}_s(x_1 + t, \ldots, x_n + t) - t, \quad t \geqslant 0.$$

Then

(a) As a function of t, $\phi_s(t)$ increases or decreases according as s is less than or greater than $1/2n$.

(b) As a function of t, $\psi_s(t)$ increases or decreases according as s is less than or greater than $1/n$.

(c) $\lim_{t\to\infty} \phi_s(t) = \lim_{t\to\infty} \psi_s(t) = S/n$, the arithmetic mean.

Proof. (a) We may asume without loss in generality that $\mathbf{x}$ is not a constant n-tuple and that $\mathfrak{F}_s$ is not the arithmetic mean, equivalently $s \neq 1/2n$. Let

$$f_s(t) = \phi_s(t) + t = \mathfrak{F}_s(x_1 + t, \ldots, x_n + t)$$

and let

$$S^* = S(x_1 + t, \ldots, x_n + t), \ \ P^* = P(x_1 + t, \ldots, x_n + t), \ \ \Delta^* = \Delta(x_1 + t, \ldots, x_n + t).$$

Then

$$S^* = S + nt, \ \ P^* = P + \frac{n(n-1)}{2}t^2 + t(n-1)S, \ \ \Delta^* = \Delta \tag{7}$$

and

$$f_s(t) = \sqrt{\frac{2}{n(n-1)}(s\Delta^* + P^*)}.$$

So

$$\phi_s'(t) = f_s'(t) - 1 = \frac{N}{D} - 1,$$

where

$$N = n(n-1)t + (n-1)S \quad \text{and} \quad D = \sqrt{2n(n-1)(s\Delta^* + P^*)}.$$

Using (1) and (7), we get that $N^2 - D^2 = (n-1)(1-2sn)\Delta$ so that $\phi_s'(t)$ is positive or negative according as $s < 1/2n$ or $s > 1/2n$.

(b) Again, we may assume without loss in generality that $\mathbf{x}$ is not a constant n-tuple and that $\mathfrak{G}_s$ is not the arithmetic mean, equivalently $s \neq 1/n$. Let

$$g_s(t) = \psi_s(t) + t = \mathfrak{G}_s(x_1 + t, \ldots, x_n + t).$$

Then

$$g_s(t) = \frac{1}{n-1} \cdot \frac{s\Delta^* + 2P^*}{S^*},$$

and $\psi_s'(t) = g_s'(t) - 1 = N/D - 1$, where

$$N = 2(n-1)S^*(nt + S) - n(s\Delta^* + 2P^*) \quad \text{and} \quad D = (n-1)(S^*)^2.$$

Using (1) and (7), we get that $N - D = (1 - ns)\Delta$ so that $\psi_s'(t)$ is positive or negative according as $s < 1/n$ or $s > 1/n$.

(c) This follows from [1, Proposition 4] since both $\mathfrak{F}_s$ and $\mathfrak{G}_s$ are differentiable at $(1, \ldots, 1)$. $\square$

5. θ-Means: a geometric context for quadratic radical means

This section can be viewed as an application of Theorem 3 to a problem in n-dimensional geometry. It can also be viewed as a geometric context that gives rise to the study of quadratic radical means.

We define an *n-dimensional simplex* $[V] = [\mathbf{v}_0, \ldots, \mathbf{v}_n]$ to be the convex hull of a set $V = \{\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of $n+1$ affinely independent points in $\mathbb{R}^m$ where $m \geqslant n$; V is called the *set of vertices of* $[V]$. For each i, $0 \leqslant i \leqslant n$, write $V_i = V \setminus \{\mathbf{v}_i\}$. The $(n-1)$-dimensional simplex $[V_i]$ is called the *face* or *facet* of V *opposite to the vertex* $\mathbf{v}_i$. The $(n-1)$-dimensional volume or content of $[V_i]$ will be written $\mu([V_i])$.

We will always take $\mathbf{v}_0$ to be the origin and will not distinguish between a point $\mathbf{v}$ and the position vector $\vec{\mathbf{v}}$ it represents, that is the vector from $\mathbf{v}_0$ to $\mathbf{v}$. The *outward pointing unit normal to the face* $[V_i]$ will be written $\widehat{\mathbf{v}}_i$. The *angle* between $\widehat{\mathbf{v}}_i$ and $\widehat{\mathbf{v}}_j$ will be written $\widehat{v}_{i,j}$, while the *angle* between $\mathbf{v}_i$ and $\mathbf{v}_j$ will be written $v_{i,j}$.

To define θ-means, we need the following theorem.

THEOREM 10. *There exist n unit vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ in $\mathbb{R}^n$ such that the angle between any two is θ if and only if $1 \geqslant \cos\theta \geqslant -1/(n-1)$. Such vectors are linearly dependent if and only if* $\cos\theta = 1$ or $-1/(n-1)$.

Proof. If $\{\mathbf{v}_0, \ldots, \mathbf{v}_n\}$ are unit vectors in $\mathbb{R}^n$ such that the angle between any two is θ, then it follows from

$$0 \leqslant \|\mathbf{v}_1 + \ldots + \mathbf{v}_n\|^2 = n + n(n-1)\cos\theta \tag{8}$$

that $\cos\theta \geqslant -1/(n-1)$.

Conversely, let c be such that $1 \geqslant c \geqslant -1/(n-1)$. We want to construct n unit vectors $\mathbf{w}_1, \ldots, \mathbf{w}_n \in \mathbb{R}^n$ such that the cosine of the angle between any two is c. Start with a regular n-simplex in $\mathbb{R}^{n-1}$ whose centroid is at the origin and whose vertices $\mathbf{w}_1, \ldots, \mathbf{w}_n$ are unit vectors. Let α be the angle between any two of the vectors $\mathbf{w}_i$ and $\mathbf{w}_j$ and let $b = \cos\alpha$. Then it follows from $\mathbf{w}_1 + \ldots + \mathbf{w}_n = \mathbf{0}$ that $b = -1/(n-1)$. Now let $\mathbf{u}_j \in \mathbb{R}^n$ be obtained from $\mathbf{w}_j \in \mathbb{R}^{n-1}$ by adding the fixed number k as a last coordinate and let

$$\mathbf{v}_j = \frac{\mathbf{u}_j}{\|\mathbf{u}_j\|} = \frac{\mathbf{u}_j}{\sqrt{1+k^2}}.$$

If the angle between $\mathbf{v}_i$ and $\mathbf{v}_j$, $i \neq j$, is θ, then

$$\cos\theta = \frac{\mathbf{w}_i \cdot \mathbf{w}_j + k^2}{1+k^2} = \frac{b+k^2}{1+k^2}.$$

As k moves from 0 to ∞, $\cos\theta$ increases from b to 1, i.e. from $-1/(n-1)$ to 1. Thus $\cos\theta = c$ for some choice of k.

The last statement follows from [22], where it is proved that equally inclined vectors are linearly dependent if and only if they are identical (in which case $\cos\theta = 1$) or their sum is $\mathbf{0}$ (in which case $\cos\theta = -1/(n-1)$ by (8)). $\square$

Now let θ be such that $1 > c = \cos\theta > -1/(n-1)$. To define the θ-mean of n positive numbers $a_1, \ldots, a_n$, we construct unit vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ in $\mathbb{R}^n$ such that the angle between any two of them is θ and we consider the simplex $[\mathbf{v}_0, a_1\mathbf{v}_1, \ldots, a_n\mathbf{v}_n]$. Then the θ*-mean*, $\mathfrak{N}_\theta(a_1,\ldots,a_n)$, of $a_1,\ldots,a_n$ can now be defined as the number a such that the faces of the simplexes $[\mathbf{v}_0, a_1\mathbf{v}_1, \ldots, a_n\mathbf{v}_n]$ and $[\mathbf{v}_0, a\mathbf{v}_1, \ldots, a\mathbf{v}_n]$ opposite to $\mathbf{v}_0$ have the same content; that is

$$v_{i,j} = \theta,\ 1 \leqslant i < j \leqslant n \ \text{ and } \ \mu([a\mathbf{v}_1,\ldots,a\mathbf{v}_n]) = \mu([a_1\mathbf{v}_1,\ldots,a_n\mathbf{v}_n]). \tag{9}$$

This mean is well-defined in that it depends only on θ and not on the choice of the unit vectors $\mathbf{v}_1,\ldots,\mathbf{v}_n$. To see this, let $\mathbf{u}_1,\ldots,\mathbf{u}_n$ be another choice of unit vectors such that the angle between any two is θ. Then $\mathbf{v}_j - \mathbf{v}_i$ and $\mathbf{u}_j - \mathbf{u}_i$, $1 \leqslant i,j \leqslant n$, have the same length; so it follows that there is an isometry of $\mathbb{R}^n$ that fixes $\mathbf{v}_0$ and maps $\mathbf{u}_j$ to $\mathbf{v}_j$, $1 \leqslant j \leqslant n$; see [5, Proposition 9.7.1, p. 236].

Alternatively, we can define a θ-mean by starting with a simplex whose faces through $\mathbf{v}_0$ have contents $a_1,\ldots,a_n$ and requiring the smoothing simplex to have its faces through $\mathbf{v}_0$ of equal content, a say, (together of course with the requirement that the faces of the two simplexes opposite to $\mathbf{v}_0$ have equal content). Call this value $\mathfrak{M}_\theta$. It is clear that the means $\mathfrak{N}_\theta$ and $\mathfrak{M}_\theta$ are identical when $n = 2$ since the faces of a triangle are its edges. In higher dimensions, these means are different and it is the second one $\mathfrak{M}_\theta$ that we emphasize since it is the one that is quadratic radical.

The generalized law of cosines, see [18], states that

$$(\mu([V_0]))^2 = \sum_{i=1}^{n}(\mu([V_i]))^2 + 2\sum_{1\leqslant i<j\leqslant n}\mu([V_i])\,\mu([V_j])\ \cos\widehat{v}_{i,j}.$$

Thus, if $a = \mathfrak{M}_\theta$ and $\widehat{c} = \cos\widehat{v_{i,j}}$, $1 \leqslant i,j \leqslant n$, then

$$na^2 + n(n-1)a^2\widehat{c} = \sum_{i=1}^{n} a_i^2 + 2\sum_{1\leqslant i<j\leqslant n} a_ia_j\widehat{c}.$$

On simplifying, we get that $\mathfrak{M}_\theta = \mathfrak{F}_s$, where $\mathfrak{F}_s$ is given by (5) with

$$s = \frac{1}{2(1+(n-1)\widehat{c})}. \tag{10}$$

By Theorem 3, this is internal if and only if $0 \leqslant s \leqslant 1$, or equivalently if and only if

$$\widehat{c} \geqslant \frac{-1}{2(n-1)}. \tag{11}$$

It remains to express this in terms of $c = \cos\theta\ (= \mathbf{v}_i\cdot\mathbf{v}_j,\ 1 \leqslant i < j \leqslant n)$. Since the vectors $\mathbf{v}_1,\ldots,\mathbf{v}_n$ form a basis of $\mathbb{R}^n$, we have that

$$\widehat{\mathbf{v}}_i = a\mathbf{v}_i + b\sum_{j=1}^{n}\mathbf{v}_j. \tag{12}$$

The real numbers a and b do not depend on i, since every permutation of $\{\mathbf{v}_1,\ldots,\mathbf{v}_n\}$ can be extended to an isometry of $\mathbb{R}^n$. From (12) and the fact that $\widehat{\mathbf{v}}_i\cdot\mathbf{v}_j = 0$ for all $j \neq i$, it follows that $ac + b(1+(n-1)c) = 0$ and that

$$\widehat{c} = \frac{\widehat{\mathbf{v}}_i\cdot\widehat{\mathbf{v}}_j}{\sqrt{(\widehat{\mathbf{v}}_i\cdot\widehat{\mathbf{v}}_i)(\widehat{\mathbf{v}}_j\cdot\widehat{\mathbf{v}}_j)}} = \frac{-c}{1+c(n-2)}. \tag{13}$$

Using this in (11) and simplifying, we obtain the equivalent condition $c = \cos\theta \leqslant 1/n$. For ease of reference, we summarize this in the following theorem, in which (14) follows from (10) and (13).

THEOREM 11. *Let* θ *be such that* $1 \geqslant \cos\theta \geqslant -1/(n-1)$, *let* $\mathbf{v}_0 = \mathbf{0}$ *and let* $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ *be unit vectors in* $\mathbb{R}^n$ *such that the angle between any two is* θ. *For any positive numbers* $a_1, \dots, a_n$, *let* $a'_1, \dots, a'_n$ *be defined by the requirement that the content of the face of the simplex* $[\mathbf{v}_0, a'_1\mathbf{v}_1, \dots, a'_n\mathbf{v}_n]$ *opposite to* $\mathbf{v}_j$ *is* a_j, $1 \leqslant j \leqslant n$. *Let* a' *be defined by the requirement that the faces of the simplexes* $[\mathbf{v}_0, a'_1\mathbf{v}_1, \dots, a'_n\mathbf{v}_n]$ *and* $[\mathbf{v}_0, a'\mathbf{v}_1, \dots, a'\mathbf{v}_n]$ *opposite to* $\mathbf{v}_0$ *have the same content. Define* $\mathfrak{M}_\theta(a_1, \dots, a_n)$ *to be the content of any face of the simplex* $[\mathbf{v}_0, a'_1\mathbf{v}_1, \dots, a'\mathbf{v}_n]$ *through* $\mathbf{v}_0$. *Then* $\mathfrak{M}_\theta$ *is a mean (i.e. internal) if and only if* $\cos\theta \leqslant 1/n$. *In this case,* $\mathfrak{M}_\theta = \mathfrak{F}_s$, *where* $\mathfrak{F}_s$ *is given by (5) with*

$$s = \frac{1 + (n-2)\cos\theta}{2(1-\cos\theta)}. \tag{14}$$

EXAMPLE 1. When $n = 2$, the condition $c \leqslant 1/n$ says that $\theta \geqslant 60°$. Geometrically, this says that if ABC is a triangle with $\sphericalangle BAC \geqslant 60°$, then there are points X and Y on the rays AB and AC respectively such that $XY = BC$ and $AB \leqslant AX = AY \leqslant AC$. This is geometrically evident: take B' and C' on AC and AB respectively such that $AB = AB'$ and $AC = AC'$. Noting that

$$\sphericalangle AC'C = \frac{180° - \sphericalangle BAC}{2} \leqslant \frac{180° - 60°}{2} = 60°, \quad \sphericalangle CBC' \geqslant \sphericalangle CAB \geqslant 60°,$$

we conclude that $BC < CC'$ and hence $BB' < BC < CC'$. The intermediate value theorem gurantees the existence of the desired X and Y. If, however, $\sphericalangle\theta < 60°$, we can construct a triangle ABC with $3A + 2C < 180°$. Then constructing B' and C' as before, we see that $BC > CC'$. So $\mathfrak{M}_\theta(AB, AC)$ is not between AB and AC.

6. Radical and rational means of higher degree

Let H be a symmetric form of degree d in the variables $x_1, \dots, x_n$. Using the terminology of [20], we say that H is *completely positive*, respectively *co-positive*, if

$$H(\mathbf{x}) \geqslant 0 \quad \text{for all real } \mathbf{x}, \text{ respectively all positive } \mathbf{x}.$$

Similarly, we say that H is *completely internal*, respectively *co-internal* if

$$\min \mathbf{x}^d \leqslant H(\mathbf{x}) \leqslant \max \mathbf{x}^d \quad \text{for all real } \mathbf{x}, \text{ respectively all positive } \mathbf{x}.$$

In the case of quadratic H, i.e. $d = 2$, the conditions on the coefficients of H that are necessary and sufficient for H to be co-positive, completely positive and co-internal are given in Theorems 1, 3 and 5, respectively. For the sake of completeness, the next theorem gives the conditions for H to be completely internal.

THEOREM 12. *The only completely internal symmetric quadratic form is*

$$H = \frac{2}{n(n-1)}\left(\frac{1}{2}\Delta + P\right).$$

The associated quadratic radical mean is

$$\sqrt{H} = \sqrt{\frac{\sum_{i=1}^{n} x_i^2}{n}}.$$

Proof. The given form is obviously completely internal. Conversely, let a given symmetric quadratic form

$$H = \frac{2}{n(n-1)}(s\Delta + tP)$$

be completely internal. Taking $x_1 = \ldots = x_n = 1$, we obtain

$$1 = \frac{2}{n(n-1)} \cdot \frac{n(n-1)}{2} t$$

or $t = 1$. Then taking $x_1 = \ldots = x_{n-1} = 1,\ x_n = -1,$ we get that

$$1 = \frac{2}{n(n-1)}\left(4s(n-1) + t\left(\frac{(n-1)(n-2)}{2} - (n-1)\right)\right)$$

which simplifies to give $s = 1/2$. □

For symmetric forms of arbitrary degree, similar conditions may turn out to be difficult to find. For example, co-positive symmetric cubic forms in three variables were studied and fully characterized in [3] and [25] in connection with geometric inequalities. Co-positive symmetric sextics H in three variables x, y, z with some special properties – $H(1,1,1) = 0$, and not containing the terms $x^6 + y^6 + z^6$ and $x^5(y+z)+y^5(z+x)+z^5(x+y)$ – were characterized in [26]. Co-positive and completely positive symmetric quartics H in three variables satisfying $H(1,1,1) = 0$ were studied and fully characterized in [7] and [27]. Co-positive symmetric cubic forms in any number of variables were characterized in [11]; while [12] can be viewed as a first step towards the characterization of co-positive and completely positive symmetric sextics in three variables. As for co-internal and completely internal forms, as defined above, the author is unaware of any work other than [19] where co-internal symmetric cubic forms in three variables are characterized. This state of affairs hints at the magnitude of the difficulties one expects to encounter in trying to characterize co-internal and completely internal symmetric forms and of radical and rational means of higher degrees.

It is worth mentioning that the above issues are related to Hilbert's Seventeenth Problem which asks whether and when a completely positive form can be expressed as a sum of squares of forms. A detailed account can be found in [24] and [23].

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REFERENCES

[1] J. ACZÉL AND ZS. PÁLES, *The behaviour of means under equal increments of their variables*, Amer. Math. Monthly **95** (1988), 856–860.
[2] J. ACZÉL, S. LOSONCZI AND ZS. PÁLES, *The behaviour of comprehensive classes of means under equal increments of their variables*, Numerical Mathematics **80** (1987), 459–461.
[3] P. J VAN ALBADA, *Geometric inequalities and their geometry*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. **338–352** (1971), 41–45.
[4] E. F. BECKENBACH, *A class of mean value functions*, Amer. Math. Monthly **57** (1950), 1–6.
[5] M. BERGER, *Geometry I*, Springer–Verlag, Berlin, 1987.
[6] J. M. BORWEIN AND P. B. BORWEIN, *The way of all means*, Amer. Math. Monthly **94** (1987), 519–522.
[7] O. BOTTEMA AND J. T. GROENMAN, *On some triangle inequalities*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. **577–598** (1977), 11–20.
[8] J. L. BRENNER AND M. E. MAYS, *Some reproducing identities for families of mean values*, Aequationes Math. **33** (1987), 106–113.
[9] P. S. BULLEN, *Averages still on the move*, Math. Mag. **63** (1990), 250–255.
[10] P. S. BULLEN, D. S. MITRINOVIĆ AND P. M. VASIĆ, *Means and Their Inequalities*, D. Reidel Publishing Company, Dordecht, Holland, 1988.
[11] M. D. CHOI, T. Y. LAM AND B. REZNICK, *Even symmetric sextics*, Math. Z. **195** (1987), 559–580.
[12] M. D. CHOI, T. Y. LAM AND B. REZNICK, *Positive sextics and Schur's inequalities*, J. Algebra **141** (1991), 36–77.
[13] E. L. DODD, *The substitutive mean and certain subclasses of this general mean*, Ann. Math. Stat. **11** (1940), 163–176.
[14] H. EVES, *An Introduction to the History of Mathematics*, 5th edition, Saunders, Philadelphia, 1983.
[15] H. EVES, *Great Moments in Mathematics (before 1650)*, Doliciani Math. Exp., No. 5, MAA, Washington, D.C., 1980.
[16] D. FARNSWORTH AND R. ORR, *Gini means*, Amer. Math. Monthly **93** (1986), 603–607.
[17] D. FARNSWORTH AND R. ORR, *Transformation of power means and a new class of means*, J. Math. Anal. Appl. **129** (1988), 394–400.
[18] R. J. GREGORAC, *A law of cosines in* $\mathbb{R}^n$, Nieuw Arch. Wisk. **3** (1991), 267–269.
[19] M. HAJJA, *Internal cubic forms in three variables*, preprint.
[20] M. HALL AND M. NEWMAN, *Copositive and completely positive quadratic forms*, Proc. Camb. Phil. Soc. **59** (1963), 329–339.
[21] L. HOEHN AND I. NIVEN, *Averages on the move*, Math. Mag. **58** (1985), 151–156.
[22] M. S. KLAMKIN, *On some symmetric sets of unit vectors*, Math. Mag. **64** (1991), 271–273.
[23] A. PRESTEL AND C. N. DELZELL, *Positive Polynomials*, Springer, New York, 2001.
[24] RAJWADE, *Squares*, London Math. Soc. Lecture Note Ser. 171, Cambridge Univ. Press, 1993.
[25] J. F. RIGBY, *A method of obtaining related triangle inequalities, with applications*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. **412–460** (1973), 217–226.
[26] J. F. RIGBY, *Sextic inequalities for the sides of a triangle*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. **498–541** (1975), 51–58.
[27] J. F. RIGBY, *Quartic and sextic inequalities for the sides of triangles, and best possible inequalities*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. **602–633** (1978), 195–202.
[28] G. TOADER, *Integral generalized means*, Math. Inequal. Appl. **5** (2002), 511–516.
[29] S. TOADER, *Derivatives of generalized means*, Math. Inequal. Appl. **5** (2002), 517–523.
[30] M. K. VAMANAMURTHY AND M. VUORINEN, *Inequalities for means*, J. Math. Anal. Appl. **183** (1994), 155–166.

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