Mathematical
Inequalities
& Applications
Volume 6, Number 4 (2003), 625–658

ON THE WAY OF WEIGHT COEFFICIENT AND RESEARCH FOR THE HILBERT–TYPE INEQUALITIES

BICHENG YANG AND THEMISTOCLES M. RASSIAS

(*communicated by S. Saitoh*)

Abstract. The Hilbert-type inequalities are certain significant weight inequalities, which play an important role in mathematical analysis and its applications. In this paper, we introduce the way of weight coefficient and consider its applications to the Hilbert-type inequalities. We will summarize how to use the way of weight coefficient to obtain some new improvements and generalizations of the Hilbert-type inequalities.

1. Introduction

1.1. Hilbert's inequality and equivalent form

If $\{a_n\}, \{b_n\}$ are sequences of real numbers such that $0 < \sum_{n=1}^{\infty} a_n^2 < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^2 < \infty$, then (see Hardy et al. [1, Ch. 9])

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \pi\left(\sum_{n=1}^{\infty} a_n^2 \sum_{n=1}^{\infty} b_n^2\right)^{1/2}, \tag{1.1}$$

where the constant factor π is the best possible. Its equivalent form is

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{a_m}{m+n}\right)^2 < \pi^2 \sum_{n=1}^{\infty} a_n^2, \tag{1.2}$$

where the constant factor π^2 is also the best possible. Inequality (1.1) is well known as Hilbert's inequality, which is important in mathematical analysis and its applications (see Mintrinović et al. [2, Ch. 5]).

The associated equivalent integral forms of (1.1) and (1.2) are the following:

If f, g are real functions such that $\int_0^{\infty} f^2(t)dt < \infty$ and $\int_0^{\infty} g^2(t)dt < \infty$, then

---

Mathematics subject classification (2000): 26D15.

Key words and phrases: Hilbert's inequality, weight coefficient, Cauchy's inequality, β function, Hölder's inequality.

Research supported by Natural Science Foundation of Guangdong Institutions of Higher Learning, College and University (China, No. 0177).

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \pi \left(\int_0^\infty f^2(t)dt \int_0^\infty g^2(t)dt \right)^{1/2}; \tag{1.1a}$$

$$\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y} dx \right)^2 dy < \pi^2 \int_0^\infty f^2(t)dt, \tag{1.2a}$$

where the constant factors π and π^2 are both the best possible.

More accurate equivalent forms of (1.1) and (1.2) are (see Mintrinović et al. [3]):

$$\sum_{n=0}^\infty \sum_{m=0}^\infty \frac{a_m b_n}{m+n+1} < \pi \left(\sum_{n=0}^\infty a_n^2 \sum_{n=0}^\infty b_n^2 \right)^{1/2}; \tag{1.1b}$$

$$\sum_{n=0}^\infty \left(\sum_{m=0}^\infty \frac{a_m}{m+n+1} \right)^2 < \pi^2 \sum_{n=0}^\infty a_n^2, \tag{1.2b}$$

where the constant factors π and π^2 are both the best possible.

Hardy [1, Th. 324] pointed out an important role of (1.1b) in building the following Hardy-Littlewood's inequality:

If $f(x)$ is a real function in $L^2(0,1)$ such that $\int_0^1 f^2(t)dt < \infty$, we define $a_n := \int_0^1 x^n f(x)dx$ $(n = 0, 1, \ldots)$. Then

$$\sum_{n=0}^\infty a_n^2 < \pi \int_0^1 f^2(x)dx, \tag{1.3}$$

where the constant π is the best possible.

Since Ingham [4] gave the following result:

$$\sum_{n=0}^\infty \sum_{m=0}^\infty \frac{a_m b_n}{m+n+2\alpha} < \pi \Big(\sum_{n=0}^\infty a_n^2 \sum_{n=0}^\infty b_n^2 \Big)^{1/2} \qquad \Big(\alpha \geqslant \frac{1}{4}\Big), \tag{1.1c}$$

we can obtain an equivalent form as follows:

$$\sum_{n=0}^\infty \Big(\sum_{m=0}^\infty \frac{a_m}{m+n+2\alpha} \Big)^2 < \pi^2 \sum_{n=0}^\infty a_n^2 \qquad \Big(\alpha \geqslant \frac{1}{4}\Big). \tag{1.2c}$$

It is obvious that (1.1) and (1.2) are particular results of (1.1c) and (1.2c) for $\alpha = 1$, and (1.1b) and (1.2b) are particular results of (1.1c) and (1.2c) for $\alpha = 1/2$. Hence, (1.1c) provides a unification of Hilbert's inequalities and (1.2c) unifies their equivalent form.

We can write the general equivalent integral forms of (1.1c) and (1.2c) as:

If $\alpha \in R, f, g$ are real functions, such that

$$\int_{-\alpha}^\infty f^2(x)dx < \infty \quad \text{and} \quad \int_{-\alpha}^\infty g^2(x)dx < \infty,$$

then one has

$$\int_{-\alpha}^{\infty}\int_{-\alpha}^{\infty}\frac{f(x)g(y)}{x+y+2\alpha}dxdy<\pi\left(\int_{-\alpha}^{\infty}f^2(x)dx\int_{-\alpha}^{\infty}g^2(x)dx\right)^{1/2}; \tag{1.1d}$$

$$\int_{-\alpha}^{\infty}\left(\int_{-\alpha}^{\infty}\frac{f(x)}{x+y+2\alpha}dx\right)^2dy<\pi^2\int_{-\alpha}^{\infty}f^2(x)dx, \tag{1.2d}$$

where the constants π and π^2 are the best possible.

REMARK 1.1. For $\alpha=0$, (1.1d) reduces to (1.1a). Inequality (1.1d) provides a generalization of (1.1a). On the other hand, setting $X=x-\alpha$, $Y=y-\alpha$, $F(X)=f(x)$ and $G(Y)=g(y)$ in (1.1a), we can obtain

$$\int_{-\alpha}^{\infty}\int_{-\alpha}^{\infty}\frac{F(X)G(Y)}{x+y+2\alpha}dXdY<\pi\left\{\int_{-\alpha}^{\infty}F^2(X)dX\int_{-\alpha}^{\infty}G^2(Y)dY\right\}^{1/2}.$$

It follows that (1.1a) and (1.1d) are equivalent; so are (1.2a) and (1.2d).

1.2. Some research results on Hilbert's inequality

(1) In the year 1991, Xu et al. [5] initialed the way of weight coefficient and gave a strengthened version of (1.1) as follows:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n}<\left\{\sum_{n=1}^{\infty}\left(\pi-\frac{\theta}{n^{1/2}}\right)a_n^2\sum_{n=1}^{\infty}\left(\pi-\frac{\theta}{n^{1/2}}\right)b_n^2\right\}^{1/2}, \tag{1.4}$$

where $\theta=1.1213^+$. In [5] it was asked to determine the best constant θ, that keeps (1.4) valid. In 1992, by using the improved Euler-Maclaurin's formula (see [6, (2.6)]), Gao [6] found that the best constant is $\max\theta=1.281669^+$.

Xu et al. [7] also gave a strengthened version of Hardy-Hilbert's inequality applying the same method. In the last ten years, by using the way of the weight coefficient, several new weight inequalities are being given.

(2) In the year 1992, by introducing the function as

$$k(x)=\frac{2}{\pi}\int_0^{\infty}\frac{e(xt^2)}{1+t^2}dt-e(x), \qquad (1-e(x)+e(y)\geqslant 0)$$

and using Cauchy's inequality, Hu [8] gave an improvement of (1.1a) in the form:

$$\left\{\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{x+y}dxdy\right\}^2\leqslant\pi^2\left\{\left[\left(\int_0^{\infty}f^2(x)dx\right)^2-\left(\int_0^{\infty}f^2(x)k(x)dx\right)^2\right]\right.$$
$$\left.\times\left[\left(\int_0^{\infty}g^2(x)dx\right)^2-\left(\int_0^{\infty}g^2(x)k(x)dx\right)^2\right]\right\}^{1/2}. \tag{1.5}$$

Similarly, Hu [9] gave an improvement of (1.1b) and Hu [10] provided a survey of his reseach results in inequalities.

(3) In the year 1998, by using the way of matrix and inner product, Gao [11] gave an improvement of (1.1a) as follows:

$$\left\{\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}dxdy\right\}^2 \leqslant \pi^2\int_0^\infty f^2(x)dx\int_0^\infty g^2(x)dx - G(\xi,\eta,\delta), \quad (1.6)$$

where $G(\xi,\eta,\delta) = \big(\|\xi\|(\eta,\delta)\big)^2 - 2(\xi,\delta)(\eta,\delta)(\xi,\delta) + \big(\|\eta\|(\xi,\eta)\big)^2 > 0$ and

$$\xi = \frac{1}{(x+y)^{1/2}}\Big(\frac{x}{y}\Big)^{1/4}f(x), \qquad \eta = \frac{1}{(x+y)^{1/2}}\Big(\frac{y}{x}\Big)^{1/4}g(y).$$

Furthermore by the same method, Gao [12] gave another improvement of (1.1a) as:

$$\left\{\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}dxdy\right\}^2 < \pi^2(1-R)\int_0^\infty f^2(x)dx\int_0^\infty g^2(x)dx, \quad (1.7)$$

where,

$$R = \frac{1}{\pi}\Big(\frac{x}{\|g\|} - \frac{y}{\|f\|}\Big)^2 > 0,$$

with $x = \big(\frac{2}{\pi}\big)^{1/2}(g,e), y = \big(2\pi\big)^{1/2}(f,e^{-s})$, and $e(t) = \int_0^\infty \frac{e^{-s}}{s+t}ds$.

(4) In the year 1998, B. G. Pachpatte [13] builded a new inequality similar to (1.2) as follows

$$\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{\sum\limits_{s=1}^{m}\sum\limits_{i=1}^{n}a_sb_i}{m+n} \leqslant \frac{1}{2}\sqrt{kr}\Big\{\sum_{n=1}^{k}(k-m+1)a_m^2\sum_{n=1}^{r}(r-n+1)b_n^2\Big\}^{1/2}. \quad (1.8)$$

Zhao [14, 15] considered some further generalizations of (1.8). Recently, Yang [16, 17] considered some new Hardy-Hilbert's type inequalities.

(5) In the year 2001, by introducing the Γ function, Hong [18] provided a generalization of (1.1a) in the form:

If $a>0$, $b\geqslant 0$, $p_i>1$ satisfy $\sum\limits_{i=1}^{n}\frac{1}{p_i}=1$ and $f_i\geqslant 0$, $r_i>b$, $\lambda>\frac{1}{a}(n-1-\frac{b}{r_i})$ $(r_i=[\prod\limits_{j=1}^{n}p_j]/p_i,\ i=1,2,\dots,n)$, then

$$\int_0^\infty\int_0^\infty\cdots\int_0^\infty\frac{1}{(\sum\limits_{i=1}^{n}x_i^a)^\lambda}\prod_{i=1}^{n}f_i(x_i)dx_1dx_2\dots dx_n$$

$$\leqslant \frac{\Gamma^{n-2}(\frac{1}{a})}{a^{n-1}\Gamma(\lambda)}\prod_{i=1}^{n}\Big\{\Gamma\Big(\frac{1}{a}\Big(1-\frac{b}{r_i}\Big)\Big)\Gamma\Big(\lambda-\frac{1}{a}\Big(n-1-\frac{b}{r_i}\Big)\Big)\int_0^\infty x^{n-1-a\lambda}f_i^{p_i}(x)dx\Big\}^{\frac{1}{p_i}}. \quad (1.9)$$

In particular for $a=b=1$, $\lambda=n-1$, $p_i=n$, $n\in\mathbf{N}\setminus\{1\}$, and $r_i=n^{n-1}$, we obtain

$$\int_0^\infty\int_0^\infty\cdots\int_0^\infty\frac{1}{(\sum\limits_{i=1}^{n}x_i)^{n-1}}\prod_{i=1}^{n}f_i(x_i)dx_1dx_2\dots dx_n$$

$$\leqslant \frac{1}{(n-2)!}\Big[\Gamma\Big(1-\frac{1}{n^{n-1}}\Big)\Gamma\Big(\frac{1}{n^{n-1}}\Big)\Big]\prod_{i=1}^{n}\Big\{\int_0^{\infty} f_i^n(x)dx\Big\}^{1/n}. \qquad (1.10)$$

REMARK 1.2. For $n=2$, inequality (1.10) reduces to (1.1a). Hence (1.10) is a generalization of (1.1a); so is (1.9). In the following we will obtain an improvement of inequality (1.9).

2. The way of weight coefficient and research for Hilbert's inequality

2.1. Simple introduction to Xu Lizhi's way of weight coefficient

In the year 1991, Xu et al. [5] in order to improve (1.1) initiated the way of weight coefficient, that is as follows:

For the left-hand side of (1.1), uses Cauchy's inequality as follows:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\Big[\frac{1}{(m+n)^{1/2}}\Big(\frac{m}{n}\Big)^{1/4}a_m\Big]\Big[\frac{1}{(m+n)^{1/2}}\Big(\frac{n}{m}\Big)^{1/4}b_n\Big]$$

$$\leqslant \Big\{\sum_{m=1}^{\infty}\Big[\sum_{n=1}^{\infty}\frac{1}{(m+n)}\Big(\frac{m}{n}\Big)^{1/2}\Big]a_m^2\sum_{n=1}^{\infty}\Big[\sum_{m=1}^{\infty}\frac{1}{(m+n)}\Big(\frac{n}{m}\Big)^{1/2}\Big]b_n^2\Big\}^{1/2}. \qquad (2.1)$$

Then, it is defined the weight coefficient $\omega(n)$ as follows

$$\omega(n) := \sum_{m=1}^{\infty}\frac{1}{(m+n)}\Big(\frac{n}{m}\Big)^{1/2} \qquad (n\in\mathbf{N}). \qquad (2.2)$$

By (2.1), it follows that

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} \leqslant \Big\{\sum_{m=1}^{\infty}\omega(m)a_m^2\sum_{n=1}^{\infty}\omega(n)b_n^2\Big\}^{1/2}. \qquad (2.3)$$

A decomposition of the weight coefficient $\omega(n)$ is written in the form

$$\omega(n) = \pi - \frac{\theta(n)}{n^{1/2}} \qquad (n\in\mathbf{N}), \qquad (2.4)$$

where $\theta(n) := (\pi-\omega(n))n^{1/2}$. We obtain

$$\theta(n) = \Big[\pi - \sum_{m=1}^{\infty}\frac{1}{(m+n)}\Big(\frac{n}{m}\Big)^{1/2}\Big]n^{1/2} > \theta := 1.1213^{+} \qquad (n\in\mathbf{N}).$$

Using (2.4), we have

$$\omega(n) < \pi - \frac{\theta}{n^{1/2}} \qquad (n\in\mathbf{N},\ \theta = 1.1213^{+}). \qquad (2.5)$$

Hence by (2.3), it follows that (1.4) is valid.

The open problem that was posed in [5] is asking for the maximal value of θ, that keeps (2.5) true. Since Gao [6] proved that $\{\theta(n)\}$ is a strictly increasing sequence, it follows that

$$\theta_0 := \min_{n\in\mathbf{N}} \theta(n) = \theta(1) = 1.281669^+,$$

and $\theta(n) \geqslant \theta_0$, $(n \in \mathbf{N})$. The equality holds when $n = 1$. Hence (2.5) reduces to

$$\omega(n) \leqslant \pi - \frac{\theta_0}{n^{1/2}} \qquad (n \in \mathbf{N},\ \theta_0 = 1.281669^+),$$

where the equality holds for $n = 1$.

The answer of Xu's open problem is that $\theta_0 = 1.281669^+$.

2.2. Some strengthened inequalities of (1.1b)

First, we introduce the improved Euler-Maclaurin's formula (see [19, 20]) in the form:

If $f \in C^7[0,\infty)$ satisfies $(-1)^i f^{(i)}(x) > 0$, $f^{(i)}(\infty) = 0$, $(i = 1,2,\ldots,7)$ and $\int_n^\infty f(x)dx < \infty$, $(n \in \mathbf{N}_0)$, then

$$\sum_{k=n}^{\infty} f(k) = \int_n^\infty f(x)dx + \frac{1}{2}f(n) + \int_n^\infty \rho_1(x)f'(x)dx,$$

$$-\frac{1}{12}f'(n) + \frac{1}{720}f^{(3)}(n) < \int_n^\infty \rho_1(x)f'(x)dx < -\frac{1}{12}f'(n), \tag{2.6}$$

where $\rho_1(x) = x - [x] - 1/2$ is the first-order Bernoulli's function. One applies here a method similar to the process of building (2.1). We have

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{a_m b_n}{m+n+1} \leqslant \left\{\sum_{n=0}^{\infty} \tilde{\omega}(n)a_n^2 \sum_{n=0}^{\infty} \tilde{\omega}(n)b_n^2\right\}^{1/2}, \tag{2.7}$$

where the weight coefficient $\tilde{\omega}(n)$ is defined by

$$\tilde{\omega}(n) := \sum_{m=0}^{\infty} \frac{1}{(m+n+1)}\left(\frac{n+1}{m+1}\right)^{1/2} \qquad (n \in \mathbf{N}_0 = \mathbf{N} \cup \{0\}). \tag{2.8}$$

Applying the following decomposition:

$$\tilde{\omega}(n) = \pi - \frac{\vartheta(n)}{(n+1)^{1/2}},$$

$$\vartheta(n) = \left[\pi - \sum_{m=0}^{\infty} \frac{1}{(m+n+1)}\left(\frac{n+1}{m+1}\right)^{1/2}\right](n+1)^{1/2},$$

by (2.6) we obtain (see [20])

$$\vartheta(n) \geqslant \vartheta_0 := \min_{n\in\mathbf{N}_0} \vartheta(n) = \vartheta(0) = 0.5292496^+.$$

Then we have

$$\tilde{\omega}(n) \leqslant \pi - \frac{\vartheta_0}{(n+1)^{1/2}} \qquad (n \in \mathbf{N}_0, \vartheta_0 = 0.5292496^+), \tag{2.9}$$

where the equality holds for $n = 0$. Substituing (2.9) into (2.7), we have a strengthened version of (1.1b) in the form

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{a_m b_n}{m+n+1} < \left\{ \sum_{n=0}^{\infty} \left[\pi - \frac{\vartheta_0}{\sqrt{n+1}}\right] a_n^2 \sum_{n=0}^{\infty} \left[\pi - \frac{\vartheta_0}{\sqrt{n+1}}\right] b_n^2 \right\}^{1/2}, \tag{2.10}$$

where $\vartheta_0 = 0.5292496^+$.

REMARK 2.1. (i) Yang et al. [21] gave a new inequality of (2.8) as follows:

$$\tilde{\omega}(n) < \pi - \frac{7}{5(\sqrt{n}+3)} \qquad (n \in \mathbf{N}_0), \tag{2.11}$$

and proved that the right-hand side of (2.11) and (2.9) were not comparable. Hence a new strengthened version of (1.1b) is given by:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{a_m b_n}{m+n+1} < \left\{ \sum_{n=0}^{\infty} \left[\pi - \frac{7}{5(\sqrt{n}+3)}\right] a_n^2 \sum_{n=0}^{\infty} \left[\pi - \frac{7}{5(\sqrt{n}+3)}\right] b_n^2 \right\}^{1/2}. \tag{2.10a}$$

(ii) Yang et al. [22] set a new weight coefficient of (1.1b) by considering

$$\tilde{\omega}_1(n) := \sum_{m=0}^{\infty} \frac{1}{(m+n+1)} \left(\frac{2n+1}{2m+1}\right)^{1/2} \qquad (n \in \mathbf{N}_0),$$

and obtained the following inequality

$$\tilde{\omega}_1(n) < \pi - \frac{5}{6\sqrt{2n+1}} \qquad (n \in \mathbf{N}_0).$$

Then a new strengthened version of (1.1b) is obtained by

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{a_m b_n}{m+n+1} < \left\{ \sum_{n=0}^{\infty} \left[\pi - \frac{5}{6\sqrt{2n+1}}\right] a_n^2 \sum_{n=0}^{\infty} \left[\pi - \frac{5}{6\sqrt{2n+1}}\right] b_n^2 \right\}^{1/2}, \tag{2.10b}$$

which is not comparable to (2.10) and (2.10a).

(iii) Yang [23] also obtained

$$\tilde{\omega}_1(n) < \pi - \frac{\theta}{(2n+1)^{3/2}} \qquad (\theta = 0.92955^+,\ n \in \mathbf{N}_0).$$

Then a new strengthened version of (1.1b)was built as:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{a_m b_n}{m+n+1} < \left\{ \sum_{n=0}^{\infty} \left[\pi - \frac{\theta}{\sqrt{(2n+1)^3}}\right] a_n^2 \sum_{n=0}^{\infty} \left[\pi - \frac{\theta}{\sqrt{(2n+1)^3}}\right] b_n^2 \right\}^{1/2}. \tag{2.10c}$$

2.3. Applications to the equivalent form and Hardy-Littlewood's inequality

(1) *Application to an improvement of (1.2b).*

Setting $b_n := \sum_{m=0}^{\infty} \frac{a_m}{m+n+1}$, by (1.2b), we have

$$0 < \sum_{n=0}^{\infty} b_n^2 = \sum_{n=0}^{\infty} \Big(\sum_{m=0}^{\infty} \frac{a_m}{m+n+1} \Big)^2 < \infty.$$

In view of (2.10), we obtain

$$\begin{aligned} 0 < \Big(\sum_{n=0}^{\infty} b_n^2 \Big)^2 &= \Big[\sum_{n=0}^{\infty} \Big(\sum_{m=0}^{\infty} \frac{a_m}{m+n+1} \Big)^2 \Big]^2 = \Big(\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_m b_n}{m+n+1} \Big)^2 \\ &< \sum_{n=0}^{\infty} \Big[\pi - \frac{\vartheta_0}{\sqrt{n+1}} \Big] a_n^2 \sum_{n=0}^{\infty} \Big[\pi - \frac{\vartheta_0}{\sqrt{n+1}} \Big] b_n^2 \\ &< \pi \sum_{n=0}^{\infty} \Big[\pi - \frac{\vartheta_0}{\sqrt{n+1}} \Big] a_n^2 \Big(\sum_{n=0}^{\infty} b_n^2 \Big). \end{aligned}$$

Hence we have a strengthened version of (1.2b) in the form:

$$\sum_{n=0}^{\infty} \Big(\sum_{m=0}^{\infty} \frac{a_m}{m+n+1} \Big)^2 < \pi \sum_{n=0}^{\infty} \Big[\pi - \frac{\vartheta_0}{\sqrt{n+1}} \Big] a_n^2 \qquad (\vartheta_0 = 0.5292496^+). \quad (2.12)$$

(2) *Application to an improvement of (1.3).*

Since we have

$$a_n = \int_0^1 x^n f(x)dx \qquad (n = 0, 1, \ldots),$$

suppose $\{b_n\}$ is a sequence of real numbers, such that $0 < \sum_{n=0}^{\infty} b_n^2 < \infty$. Then by Cauchy's inequality,we have

$$\begin{aligned} \Big(\sum_{n=0}^{\infty} a_n b_n \Big)^2 &= \Big(\sum_{n=0}^{\infty} \int_0^1 b_n x^n f(x)dx \Big)^2 = \Big[\int_0^1 \Big(\sum_{n=0}^{\infty} b_n x^n \Big) f(x)dx \Big]^2 \\ &\leqslant \int_0^1 \Big(\sum_{n=0}^{\infty} b_n x^n \Big)^2 dx \int_0^1 f^2(x)dx \\ &= \int_0^1 \Big(\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} b_m b_n x^{m+n} \Big) dx \int_0^1 f^2(x)dx \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} b_m b_n \int_0^1 x^{m+n} dx \int_0^1 f^2(x)dx \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{b_m b_n}{m+n+1} \int_0^1 f^2(x)dx. \end{aligned}$$

By (2.10), it follows that

$$\Big(\sum_{n=0}^{\infty} a_n b_n\Big)^2 < \sum_{n=0}^{\infty}\Big[\pi - \frac{\vartheta_0}{\sqrt{n+1}}\Big] b_n^2 \int_0^1 f^2(x)dx.$$

Hence, for $b_n = a_n$ in the above inequality, we have an improvement of (1.3) in the form:

$$\Big(\sum_{n=0}^{\infty} a_n^2\Big)^2 < \sum_{n=0}^{\infty}\Big[\pi - \frac{\vartheta_0}{\sqrt{n+1}}\Big] a_n^2 \int_0^1 f^2(x)dx \qquad (\vartheta_0 = 0.5292496^+). \tag{2.13}$$

2.4. Some generalizations of Hilbert's integral inequality (1.1a)

In the year 1998, Yang [24] first introduced the β function and considered some generalizations of (1.1a). By introducing some parameters, Yang [25, 26, 27, 28] obtained the following two theorems.

THEOREM 2.1. *If $\lambda > 0, f, g$ are real functions, such that*

$$0 < \int_0^\infty t^{1-\lambda} f^2(t)dt < \infty \quad and \quad 0 < \int_0^\infty t^{1-\lambda} g^2(t)dt < \infty,$$

then

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}dxdy < B\Big(\frac{\lambda}{2},\frac{\lambda}{2}\Big)\Big\{\int_0^\infty t^{1-\lambda} f^2(t)dt \int_0^\infty t^{1-\lambda} g^2(t)dt\Big\}^{1/2}, \tag{2.14}$$

where the constant factor

$$B\Big(\frac{\lambda}{2},\frac{\lambda}{2}\Big) = \int_0^\infty \frac{1}{(1+u)^\lambda} u^{-1+\lambda/2}du$$

is the best possible ($B(u,v)$, $(u,v>0)$ is the β function). Its equivalent form is given by

$$\int_0^\infty y^{\lambda-1}\Big[\int_0^\infty \frac{f(x)}{(x+y)^\lambda}dx\Big]^2 dy < \Big[B\Big(\frac{\lambda}{2},\frac{\lambda}{2}\Big)\Big]^2 \int_0^\infty t^{1-\lambda} f^2(t)dt, \tag{2.15}$$

where the constant factor $\big[B\big(\frac{\lambda}{2},\frac{\lambda}{2}\big)\big]^2$ is still the best possible. In particular,

(i) for $\lambda = 2n$ $(n \in \mathbf{N})$, we have

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^{2n}}dxdy < \frac{[(n-1)!]^2}{(2n-1)!}\Big\{\int_0^\infty \frac{1}{t^{2n-1}} f^2(t)dt \int_0^\infty \frac{1}{t^{2n-1}} g^2(t)dt\Big\}^{1/2}; \tag{2.14a}$$

$$\int_0^\infty y^{2n-1}\Big[\int_0^\infty \frac{f(x)}{(x+y)^{2n}}dx\Big]^2 dy < \frac{[(n-1)!]^4}{[(2n-1)!]^2}\int_0^\infty \frac{1}{t^{2n-1}} f^2(t)dt, \tag{2.15a}$$

(ii) for $\lambda = 2n+1$ $(n \in \mathbf{N})$, we have

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^{2n+1}}dxdy < \frac{[(2n-1)!!]^2}{(4n)!}\pi\Big\{\int_0^\infty \frac{1}{t^{2n}} f^2(t)dt \int_0^\infty \frac{1}{t^{2n}} g^2(t)dt\Big\}^{1/2}; \tag{2.14b}$$

$$\int_0^\infty y^{2n}\Big[\int_0^\infty \frac{f(x)}{(x+y)^{2n+1}}dx\Big]^2 dy < \frac{[(2n-1)!!]^4}{[(4n)!]^2}\pi^2 \int_0^\infty \frac{1}{t^{2n}} f^2(t)dt, \qquad (2.15b)$$

where the constant factors in the above inequalities are all the best possible.

Proof. Define the weight function $\omega(x)$ as

$$\omega(x) := \int_0^\infty \frac{1}{(x+y)^\lambda}\Big(\frac{x}{y}\Big)^{1-\lambda/2} dy, \qquad x \in (0,\infty).$$

Setting $u = y/x$ in the above integral, we find $\omega(x) = B\big(\frac{\lambda}{2},\frac{\lambda}{2}\big)x^{1-\lambda}$. By using Cauchy's inequality and the method similar to building (2.1), we have

$$\begin{aligned}
\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}dxdy &= \int_0^\infty\int_0^\infty \Big[\frac{f(x)}{(x+y)^{\frac{\lambda}{2}}}\Big(\frac{x}{y}\Big)^{\frac{1-\lambda/2}{2}}\Big]\Big[\frac{g(y)}{(x+y)^{\frac{\lambda}{2}}}\Big(\frac{y}{x}\Big)^{\frac{1-\lambda/2}{2}}\Big]dxdy\\
&\leqslant \Big\{\int_0^\infty\int_0^\infty \frac{f^2(x)}{(x+y)^\lambda}\Big(\frac{x}{y}\Big)^{(1-\lambda/2)}dxdy\int_0^\infty\int_0^\infty \frac{g^2(y)}{(x+y)^\lambda}\Big(\frac{y}{x}\Big)^{(1-\lambda/2)}dxdy\Big\}^2\\
&= \Big\{\int_0^\infty \omega(x)f^2(x)dx\int_0^\infty \omega(y)g^2(y)dy\Big\}^{1/2}\\
&= B\Big(\frac{\lambda}{2},\frac{\lambda}{2}\Big)\Big\{\int_0^\infty x^{1-\lambda}f^2(x)dx\int_0^\infty y^{1-\lambda}g^2(y)dy\Big\}^{1/2}. \qquad (2.16)
\end{aligned}$$

If (2.16) takes the form of equality, then there exists constants a and b, which are not all zero, such that (see [29, p. 29])

$$a\frac{f^2(x)}{(x+y)^\lambda}\Big(\frac{x}{y}\Big)^{(1-\lambda/2)} = b\frac{g^2(y)}{(x+y)^\lambda}\Big(\frac{y}{x}\Big)^{(1-\lambda/2)}, \quad \text{a.e. in } (0,\infty)\times(0,\infty).$$

It follows that $ax^{2-\lambda}f^2(x) = by^{2-\lambda}g^2(y)$, a.e. in $(0,\infty)\times(0,\infty)$. Hence there exist a real constant C and

$$ax^{2-\lambda}f^2(x) = by^{2-\lambda}g^2(y) = C, \quad \text{a.e. in } (0,\infty).$$

Suppose $a \neq 0$ such that $x^{1-\lambda}f^2(x) = (C/a)x^{-1}$, a.e. *in* $(0,\infty)$. It would contradict the fact that $0 < \int_0^\infty x^{1-\lambda}f^2(x)dx < \infty$. Hence, (2.16) takes the form of strict inequality and (2.14) holds.

If the constant factor $B\big(\frac{\lambda}{2},\frac{\lambda}{2}\big)$ in (2.14) is not the best possible, then there exists a positive constant K (with $K < B\big(\frac{\lambda}{2},\frac{\lambda}{2}\big)$), such that (2.14) is still valid if we replace $B\big(\frac{\lambda}{2},\frac{\lambda}{2}\big)$ by K. For $0<\varepsilon<\lambda/2$, setting

$$f_\varepsilon(t) = t^{(\lambda-2-\varepsilon)/2}, \quad t \in [1,\infty); \qquad f_\varepsilon(t) = 0, \quad t \in (0,1),$$

one has

$$\begin{aligned}
\frac{1}{\varepsilon}\Big(B\Big(\frac{\lambda}{2},\frac{\lambda}{2}\Big)+o(1)\Big) &= \int_0^\infty\int_0^\infty \frac{f_\varepsilon(x)g_\varepsilon(y)}{(x+y)^\lambda}dxdy\\
&< K\Big\{\int_0^\infty t^{1-\lambda}f_\varepsilon^2(t)dt\int_0^\infty t^{1-\lambda}g_\varepsilon^2(t)dt\Big\}^{1/2} = \frac{K}{\varepsilon}.
\end{aligned}$$

It follows that $B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right) \leqslant K$, which contradicts the fact $K < B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)$. Hence the constant factor $B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)$ in (2.14) is the best possible.

Setting $g(y)$ as:

$$g(y) := y^{\lambda-1}\int_0^\infty \frac{|f(x)|}{(x+y)^\lambda}dx \qquad (y \in (0,\infty)),$$

one obtains

$$\begin{aligned}\left\{\int_0^\infty y^{1-\lambda}g^2(y)dy\right\}^2 &= \left\{\int_0^\infty y^{\lambda-1}\left[\int_0^\infty \frac{|f(x)|}{(x+y)^\lambda}dx\right]^2 dy\right\}^2\\ &= \left\{\int_0^\infty\int_0^\infty \frac{|f(x)|g(y)}{(x+y)^\lambda}dxdy\right\}^2\\ &\leqslant \left[B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\right]^2\int_0^\infty x^{1-\lambda}f^2(x)dx\int_0^\infty y^{1-\lambda}g^2(y)dy;\\ 0 < \int_0^\infty y^{\lambda-1}&\left[\int_0^\infty \frac{f(x)}{(x+y)^\lambda}dx\right]^2 dy \int_0^\infty y^{1-\lambda}g^2(y)dy\\ &\leqslant \left[B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\right]^2\int_0^\infty x^{1-\lambda}f^2(x)dx < \infty.\end{aligned}$$

If follows that the above two inequalities take the form of strict inequalities by the fact $0 < \int_0^\infty y^{1-\lambda}g^2(y)dy < \infty$ and using (2.14). Hence, we have (2.15).

On the other hand, if (2.15) is valid, by Cauchy's inequality, we have

$$\begin{aligned}\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}dxdy &= \int_0^\infty \left[y^{(\lambda-1)/2}\int_0^\infty \frac{f(x)}{(x+y)^\lambda}dx\right]\left[y^{(1-\lambda)/2}g(y)\right]dy\\ &\leqslant \left\{\int_0^\infty y^{\lambda-1}\left[\int_0^\infty \frac{f(x)}{(x+y)^\lambda}dx\right]^2 dy\int_0^\infty y^{1-\lambda}g^2(y)dy\right\}^{1/2}.\end{aligned}$$

By (2.15), we have (2.14). Hence, (2.14) and (2.15) are equivalent. We can conclude that the constant factor in (2.15) is the best possible by the equivalence of (2.14) and (2.15).

The theorem is proved.

THEOREM 2.2. *If $0 \leqslant a < b \leqslant \infty, \lambda > 0$, f and g are real functions, such that*

$$\int_0^\infty t^{1-\lambda}f^2(t)dt < \infty \quad \text{and} \quad \int_0^\infty t^{1-\lambda}g^2(t)dt < \infty,$$

then, (i) for $0 < a < b < \infty$, we have

$$\int_a^b\int_a^b \frac{f(x)g(y)}{(x+y)^\lambda}dxdy < B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left[1-\left(\frac{a}{b}\right)^{\frac{\lambda}{4}}\right]\left\{\int_a^b t^{1-\lambda}f^2(t)dt\int_a^b t^{1-\lambda}g^2(t)dt\right\}^{\frac{1}{2}}; \tag{2.17}$$

$$\int_a^b y^{\lambda-1}\left[\int_a^b \frac{f(x)}{(x+y)^\lambda}dx\right]^2 dy < \left\{B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left[1-\left(\frac{a}{b}\right)^{\lambda/4}\right]\right\}^2\int_a^b t^{1-\lambda}f^2(t)dt, \tag{2.18}$$

(ii) for $0 = a < b < \infty$, *we have*

$$\int_0^b \int_0^b \frac{f(x)g(y)}{(x+y)^\lambda} dxdy < B\Big(\frac{\lambda}{2}, \frac{\lambda}{2}\Big)$$

$$\times\Big\{ \int_0^b \Big[1 - \frac{1}{2}\Big(\frac{t}{b}\Big)^{\lambda/2}\Big] t^{1-\lambda} f^2(t)dt \int_0^b \Big[1 - \frac{1}{2}\Big(\frac{t}{b}\Big)^{\lambda/2}\Big] t^{1-\lambda} g^2(t)dt \Big\}^{1/2}; \quad (2.19)$$

$$\int_0^b y^{\lambda-1} \Big[\int_0^b \frac{f(x)}{(x+y)^\lambda} dx\Big]^2 dy < \Big[B\Big(\frac{\lambda}{2}, \frac{\lambda}{2}\Big)\Big]^2 \int_0^b \Big[1 - \frac{1}{2}\Big(\frac{t}{b}\Big)^{\lambda/2}\Big] t^{1-\lambda} f^2(t)dt, \quad (2.20)$$

(iii) for $0 < a < b = \infty$, *we have*

$$\int_a^\infty \int_a^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dxdy < B\Big(\frac{\lambda}{2}, \frac{\lambda}{2}\Big)$$

$$\times\Big\{ \int_a^\infty \Big[1 - \frac{1}{2}\Big(\frac{a}{t}\Big)^{\lambda/2}\Big] t^{1-\lambda} f^2(t)dt \int_a^\infty \Big[1 - \frac{1}{2}\Big(\frac{a}{t}\Big)^{\lambda/2}\Big] t^{1-\lambda} g^2(t)dt \Big\}^{1/2}; \quad (2.21)$$

$$\int_a^\infty y^{\lambda-1} \Big[\int_a^\infty \frac{f(x)}{(x+y)^\lambda} dx\Big]^2 dy < \Big[B\Big(\frac{\lambda}{2}, \frac{\lambda}{2}\Big)\Big]^2 \int_a^\infty \Big[1 - \frac{1}{2}\Big(\frac{a}{t}\Big)^{\lambda/2}\Big] t^{1-\lambda} f^2(t)dt. \quad (2.22)$$

REMARK 2.2. (i) For $\lambda = 1$, since $B(1/2, 1/2) = \pi$, (2.14) reduces to (1.1a). Hence (2.14) is a generalization of (1.1a) with a single parameter. It is obvious that (2.17), (2.19) and (2.21) are generalizations of (1.1a) with a single parameter, and (2.18), (2.20) and (2.22) are generalizations of (1.2a) and (2.15) with some parameters.

(ii) For $\lambda = 1$, some new improvement of (2.17), (2.19) and (2.21) are given as follows (see Xie et al. [30]):

$$\int_a^b \int_a^b \frac{f(x)g(y)}{x+y} dxdy < \Big(\pi - 4 \arctan\sqrt{\frac{a}{b}}\Big)\Big\{ \int_a^b f^2(t)dt \int_a^b g^2(t)dt \Big\}^{1/2}; \quad (2.17a)$$

$$\int_0^b \int_0^b \frac{f(x)g(y)}{x+y} dxdy < \Big\{ \int_0^b \Big(\pi - 2 \arctan\sqrt{\frac{t}{b}}\Big) f^2(t)dt \times \int_0^b \Big(\pi - 2 \arctan\sqrt{\frac{t}{b}}\Big) g^2(t)dt \Big\}^{1/2}; \quad (2.19a)$$

$$\int_a^\infty \int_a^\infty \frac{f(x)g(y)}{x+y} dxdy < \Big\{ \int_a^\infty \Big(\pi - 2 \arctan\sqrt{\frac{a}{t}}\Big) f^2(t)dt \times \int_a^\infty \Big(\pi - 2 \arctan\sqrt{\frac{a}{t}}\Big) g^2(t)dt \Big\}^{1/2}. \quad (2.21a)$$

2.5. A generalization of Hilbert's inequality (1.1) with a single parameter

THEOREM 2.3. *If* $0 < \lambda \leqslant 4, \{a_n\}, \{b_n\}$ *are real sequences, such that*

$$0 < \sum_{n=1}^{\infty} n^{1-\lambda} a_n^2 < \infty \quad \text{and} \quad 0 < \sum_{n=1}^{\infty} n^{1-\lambda} b_n^2 < \infty,$$

then (see Yang [31]*)*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{(m+n)^{\lambda}} < B\Big(\frac{\lambda}{2}, \frac{\lambda}{2}\Big)\Big\{\sum_{n=1}^{\infty} n^{1-\lambda} a_n^2 \sum_{n=1}^{\infty} n^{1-\lambda} b_n^2\Big\}^{1/2}, \tag{2.23}$$

where the constant factor $B\big(\frac{\lambda}{2}, \frac{\lambda}{2}\big)$ *is the best possible. In particular, for* $\lambda = 1/2, 2, 3, 4,$ *we have*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{\sqrt{m+n}} < \frac{\Gamma^2(1/4)}{\sqrt{\pi}}\Big\{\sum_{n=1}^{\infty} \sqrt{n} a_n^2 \sum_{n=1}^{\infty} \sqrt{n} b_n^2\Big\}^{1/2}; \tag{2.24}$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{(m+n)^2} < \Big\{\sum_{n=1}^{\infty} \frac{1}{n} a_n^2 \sum_{n=1}^{\infty} \frac{1}{n} b_n^2\Big\}^{1/2}; \tag{2.25}$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{(m+n)^3} < \frac{\pi}{8}\Big\{\sum_{n=1}^{\infty} \Big(\frac{a_n}{n}\Big)^2 \sum_{n=1}^{\infty} \Big(\frac{a_n}{n}\Big)^2\Big\}^{1/2}; \tag{2.26}$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{(m+n)^4} < \frac{1}{6}\Big\{\sum_{n=1}^{\infty} \frac{1}{n^3} a_n^2 \sum_{n=1}^{\infty} \frac{1}{n^3} b_n^2\Big\}^{1/2}. \tag{2.27}$$

The equivalent form of (2.23) *is given by*

$$\sum_{n=1}^{\infty} n^{\lambda-1}\Big[\sum_{m=1}^{\infty} \frac{a_m}{(m+n)^{\lambda}}\Big]^2 < \Big[B\Big(\frac{\lambda}{2}, \frac{\lambda}{2}\Big)\Big]^2 \sum_{n=1}^{\infty} n^{1-\lambda} a_n^2, \tag{2.28}$$

where the constant factor $\big[B\big(\frac{\lambda}{2}, \frac{\lambda}{2}\big)\big]^2$ *is still the best possible. In particular, for* $\lambda = 1/2, 2, 3, 4,$ *we have*

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}\Big[\sum_{m=1}^{\infty} \frac{a_m}{\sqrt{m+n}}\Big]^2 < \frac{\Gamma^4(1/4)}{\pi} \sum_{n=1}^{\infty} \sqrt{n} a_n^2; \tag{2.29}$$

$$\sum_{n=1}^{\infty} n\Big[\sum_{m=1}^{\infty} \frac{a_m}{(m+n)^2}\Big]^2 < \sum_{n=1}^{\infty} \frac{1}{n} a_n^2; \tag{2.30}$$

$$\sum_{n=1}^{\infty} \Big[n\sum_{m=1}^{\infty} \frac{a_m}{(m+n)^3}\Big]^2 < \frac{\pi^2}{64} \sum_{n=1}^{\infty} \Big(\frac{a_n}{n}\Big)^2; \tag{2.31}$$

$$\sum_{n=1}^{\infty} n^3\Big[\sum_{m=1}^{\infty} \frac{a_m}{(m+n)^4}\Big]^2 < \frac{1}{36} \sum_{n=1}^{\infty} \frac{1}{n^3} a_n^2. \tag{2.32}$$

The idea for the proof of Theorem 2.3. Similarly as we builded (2.1), we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}} \leqslant \left\{\sum_{m=1}^{\infty}\tilde{\omega}_{\lambda}(m)a_m^2\sum_{n=1}^{\infty}\tilde{\omega}_{\lambda}(n)b_n^2\right\}^{1/2}, \tag{2.33}$$

where the weight coefficient $\tilde{\omega}_{\lambda}(n)$ is defined by

$$\tilde{\omega}_{\lambda}(n) := \sum_{m=1}^{\infty}\frac{1}{(m+n)^{\lambda}}\left(\frac{n}{m}\right)^{1-\lambda/2} \qquad (n \in \mathbf{N}). \tag{2.34}$$

Setting $f_n(t) = \frac{1}{(t+n)t^{1-\lambda/2}}$ $(t \in (0,\infty))$, we find

$$\tilde{\omega}_{\lambda}(n) = n^{1-\lambda/2}\sum_{m=1}^{\infty}f_n(m) = n^{1-\lambda/2}\left[\int_0^{\infty}f_n(t)dt - Q(n)\right]. \tag{2.35}$$

By using (2.6) and integration by parts, we obtain

$$Q(n) := \int_0^{\infty}f_n(t)dt - \sum_{m=1}^{\infty}f_n(m) > 0 \qquad (0 < \lambda \leqslant 4, n \in \mathbf{N}),$$

and $\int_0^{\infty}f_n(t)dt = n^{-\lambda/2}B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)$. By (2.35), we find

$$\tilde{\omega}_{\lambda}(n) < B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)n^{1-\lambda} \qquad (0 < \lambda \leqslant 4, n \in \mathbf{N}). \tag{2.36}$$

In view of (2.33), we have (2.23).

For $0 < \varepsilon < \lambda/2$, setting $\tilde{a}_n := n^{-1+(\lambda-\varepsilon)/2}$, then by (2.6) we find

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\tilde{a}_m\tilde{a}_n}{(m+n)^{\lambda}} \geqslant \left(B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)+o(1)\right)\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}; \tag{2.37}$$

$$\left\{\sum_{n=1}^{\infty}n^{1-\lambda}\tilde{a}_n^2\sum_{n=1}^{\infty}n^{1-\lambda}\tilde{a}_n^2\right\}^{1/2} = \sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}. \tag{2.38}$$

If the constant factor $B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)$ in (2.23) is not the best possible, then by (2.37) and (2.38), we can obtain a contradiction.

REMARK 2.3. When $\lambda > 4$, we can not verify that (2.23) and (2.28) are valid.

3. Some strengthened versions of Hardy-Hilbert's inequalities

In the year 1925, Hardy-Riesz extended Hilbert's inequality (1.1) as (see Hardy [32]):

If $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $\{a_n\}$, $\{b_n\}$ are non-negative sequences of real numbers, such that

$$0 < \sum_{n=1}^{\infty}a_n^p < \infty \quad \text{and} \quad 0 < \sum_{n=1}^{\infty}b_n^q < \infty,$$

then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\Big\{\sum_{n=1}^{\infty} a_n^p\Big\}^{1/p}\Big\{\sum_{n=1}^{\infty} b_n^q\Big\}^{1/q}, \tag{3.1}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. Inequality (3.1) is well known as Hardy-Hilbert's inequality, and the equivalent form is

$$\sum_{n=1}^{\infty}\Big(\sum_{m=1}^{\infty}\frac{a_m}{m+n}\Big)^p < \Big[\frac{\pi}{\sin(\pi/p)}\Big]^p\sum_{n=1}^{\infty} a_n^p, \tag{3.2}$$

where the constant factor $\left[\frac{\pi}{\sin(\pi/p)}\right]^p$ is still the best possible.

More exact the following Hardy-Hilbert's inequalities and equivalent forms are

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{m+n+1} < \frac{\pi}{\sin(\pi/p)}\Big\{\sum_{n=0}^{\infty} a_n^p\Big\}^{1/p}\Big\{\sum_{n=0}^{\infty} b_n^q\Big\}^{1/q}; \tag{3.3}$$

$$\sum_{n=0}^{\infty}\Big(\sum_{m=0}^{\infty}\frac{a_m}{m+n+1}\Big)^p < \Big[\frac{\pi}{\sin(\pi/p)}\Big]^p\sum_{n=0}^{\infty} a_n^p, \tag{3.4}$$

where the constant factors $\frac{\pi}{\sin(\pi/p)}$ and $\left[\frac{\pi}{\sin(\pi/p)}\right]^p$ are all the best possible.

In the year 1991, Xu et al. [7] used the way of weight coefficient, and built a strengthened version of (3.1) as follows:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \Big\{\sum_{n=1}^{\infty}\Big[\frac{\pi}{\sin(\pi/p)} - \frac{p-1}{n^{1/p}+n^{-1/q}}\Big]a_n^p\Big\}^{1/p}$$
$$\times\Big\{\sum_{n=1}^{\infty}\Big[\frac{\pi}{\sin(\pi/p)} - \frac{q-1}{n^{1/q}+n^{-1/p}}\Big]b_n^q\Big\}^{1/q}. \tag{3.5}$$

For $p=q=2$, (3.5) reduces to

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \Big\{\sum_{n=1}^{\infty}\Big[\pi - \frac{1}{n^{1/2}+n^{-1/2}}\Big]a_n^2\sum_{n=1}^{\infty}\Big[\pi - \frac{1}{n^{1/2}+n^{-1/2}}\Big]b_n^2\Big\}^{1/2}, \tag{3.6}$$

which is different from (1.4).

About the meaning of the word "strengthened", in 1986, Mikhlin [33] gave the following Karlson's inequality and its improvement for illustration:

If $\{a_n\}$ is a sequence of real numbers, such that $0 < \sum_{n=1}^{\infty} n^2 a_n^2 < \infty$, then

$$\Big(\sum_{n=1}^{\infty} a_n^2\Big)^4 < \pi^2\sum_{n=1}^{\infty} a_n^2\sum_{n=1}^{\infty} n^2 a_n^2, \tag{3.7}$$

where the constant factor π^2 is the best possible. Mikhlin said, the constant π^2 in (3.7) could not be made smaller, although itself might be strengthened as:

$$\Big(\sum_{n=1}^{\infty} a_n^2\Big)^4 < \pi^2\sum_{n=1}^{\infty} a_n^2\sum_{n=1}^{\infty}\Big(n-\frac{1}{2}\Big)^2 a_n^2. \tag{3.8}$$

For getting a version similar to (1.4), we can rewrite (3.8) in the form:

$$\Big(\sum_{n=1}^{\infty} a_n^2\Big)^4 < \sum_{n=1}^{\infty} a_n^2 \sum_{n=1}^{\infty} \Big(\pi - \frac{\pi}{2n}\Big)^2 n^2 a_n^2. \tag{3.8a}$$

3.1. Two strengthened versions of (3.1)

(1) *A strengthened version of (3.1) similar to (1.4).*

Let γ be the Euler constant, $1-\gamma = 0.42278433^+$. We established a strengthened version of (3.1) as

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{m+n} < \Big\{\sum_{n=1}^{\infty}\Big[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)} - \frac{1-\gamma}{n^{\frac{1}{p}}}\Big] a_n^p\Big\}^{\frac{1}{p}} \Big\{\sum_{n=1}^{\infty}\Big[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)} - \frac{1-\gamma}{n^{\frac{1}{q}}}\Big] b_n^q\Big\}^{\frac{1}{q}}. \tag{3.9}$$

In the year 1997, Yang and Gao [34] derived (3.9). In 1998, Gao and Yang [35] considered some of its applications. In the following we discuss the Key idea of the proof:

First, by Hölder's inequality, we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{m+n} \leqslant \Big\{\sum_{n=1}^{\infty} \omega_q(n) a_n^p\Big\}^{1/p} \Big\{\sum_{n=1}^{\infty} \omega_p(n) b_n^q\Big\}^{1/q}, \tag{3.10}$$

where the weight coefficient $\omega_r(n)$ is defined by:

$$\omega_r(n) := \sum_{m=1}^{\infty} \frac{1}{m+n}\Big(\frac{n}{m}\Big)^{1/r} \qquad (n \in \mathbf{N},\ r = p, q). \tag{3.11}$$

Setting the decomposition of $\omega_r(n)$ in the form

$$\omega_r(n) = \frac{\pi}{\sin(\pi/r)} - \frac{\theta_r(n)}{n^{1-1/r}}, \tag{3.12}$$

by (3.11) and (3.12), we have

$$\theta_r(n) := \Big[\frac{\pi}{\sin(\pi/r)} - \sum_{m=1}^{\infty} \frac{1}{m+n}\Big(\frac{n}{m}\Big)^{1/r}\Big] n^{1-1/r}. \tag{3.13}$$

By using (2.6) and some computational analysis, we show that $\{\theta_r(n)\}$ is a strictly increasing sequence. Therefore

$$\theta_r(n) \geqslant \min_{n\in\mathbf{N}}\{\theta_r(n)\} = \theta_r(1) = \frac{\pi}{\sin(\pi/r)} - \sum_{m=1}^{\infty} \frac{1}{m+1}\Big(\frac{1}{m}\Big)^{1/r}. \tag{3.14}$$

It follows that

$$\theta_r(1) > \inf_{r>1}\{\theta_r(1)\} = \lim_{r\to\infty} \theta_r(1) = 1-\gamma.$$

Hence we find

$$\theta_r(n) > \inf_{r>1}\min_{n\in\mathbf{N}}\{\theta_r(n)\} = 1-\gamma \qquad (n \in \mathbf{N}).$$

In view of (3.12), we build the following inequality of the weight coefficient:

$$\omega_r(n) < \frac{\pi}{\sin(\pi/p)} - \frac{1-\gamma}{n^{1-1/r}} \qquad (r = p, q, n \in \mathbf{N}). \tag{3.15}$$

Hence, by (3.10) we have (3.9).

(2) *A strengthened version of (3.1) similar to (3.5).*

In 1998, Yang and Debnath [36] obtained the following interesting inequality:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{m+n} < \Big\{ \sum_{n=1}^{\infty} \Big[\frac{\pi}{\sin(\pi/p)} - \frac{1}{2n^{1/p} + n^{-1/q}}\Big] a_n^p \Big\}^{1/p}$$
$$\times \Big\{ \sum_{n=1}^{\infty} \Big[\frac{\pi}{\sin(\pi/p)} - \frac{1}{2n^{1/q} + n^{-1/p}}\Big] b_n^q \Big\}^{1/q}. \tag{3.16}$$

By using (3.9) and (2.6), the following inequality of (3.11) was obtained

$$\omega_r(n) < \frac{\pi}{\sin(\pi/p)} - \frac{1}{2n^{1-1/r} + n^{-1/r}} \qquad (r = p, q, n \in \mathbf{N}). \tag{3.17}$$

Then by (3.10) we get (3.16).

REMARK 3.1. Inequality (3.16) is an improvement of (3.5). We compare the right-hand side of (3.15) and (3.17) as follows.

For $n = 1$, 2, we obtain

$$\frac{\pi}{\sin(\pi/p)} - \frac{1-\gamma}{n^{1-1/r}} < \frac{\pi}{\sin(\pi/p)} - \frac{1}{2n^{1-1/r} + n^{-1/r}},$$

and for $n \geqslant 3$, we deduce

$$\frac{\pi}{\sin(\pi/p)} - \frac{1-\gamma}{n^{1-1/r}} > \frac{\pi}{\sin(\pi/p)} - \frac{1}{2n^{1-1/r} + n^{-1/r}}.$$

It follows that (3.9) and (3.16) are not related and these are two distinct strengthened versions of (3.5).

(3) *Two distinct strengthened versions of (3.2).*

We have (see [36])

$$\sum_{n=1}^{\infty}\Big(\sum_{m=1}^{\infty} \frac{a_m}{m+n}\Big)^p < \Big[\frac{\pi}{\sin(\pi/p)}\Big]^{p-1} \sum_{n=1}^{\infty} \Big[\frac{\pi}{\sin(\pi/p)} - \frac{1-\gamma}{n^{1/p}}\Big] a_n^p; \tag{3.18}$$

$$\sum_{n=1}^{\infty}\Big(\sum_{m=1}^{\infty} \frac{a_m}{m+n}\Big)^p < \Big[\frac{\pi}{\sin(\frac{\pi}{p})}\Big]^{p-1} \sum_{n=1}^{\infty} \Big[\frac{\pi}{\sin(\frac{\pi}{p})} - \frac{1-\gamma}{2n^{\frac{1}{p}} + n^{\frac{-1}{q}}}\Big] a_n^p. \tag{3.19}$$

3.2. Two strengthened versions of (3.3)

(1) *A strengthened version of (3.3) similar to (3.9).*

In the year 1999, Yang [37] built the following strengthened version of (3.3):

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{m+n+1} < \left\{\sum_{n=0}^{\infty}\left[\frac{\pi}{\sin(\pi/p)}-\frac{\ln 2-\gamma}{(2n+1)^{1+1/p}}\right]a_n^p\right\}^{1/p}$$
$$\times\left\{\sum_{n=0}^{\infty}\left[\frac{\pi}{\sin(\pi/p)}-\frac{\ln 2-\gamma}{(2n+1)^{1+1/q}}\right]b_n^q\right\}^{1/q}, \tag{3.20}$$

where, $\ln 2-\gamma = 0.1159315^{+}$ (γ is Euler constant).

The idea of proof. By Hölder's inequality, we have

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{m+n+1} \leqslant \left\{\sum_{n=0}^{\infty}\tilde{\omega}_q(n)a_n^p\right\}^{1/p}\left\{\sum_{n=0}^{\infty}\tilde{\omega}_p(n)b_n^q\right\}^{1/q}, \tag{3.21}$$

where the weight coefficient $\tilde{\omega}_r(n)$ is defined by

$$\tilde{\omega}_r(n) := \sum_{m=0}^{\infty}\frac{1}{m+n+1}\left(\frac{2n+1}{2m+1}\right)^{1/r} \qquad (n\in\mathbf{N}_0,\ r=p,q). \tag{3.22}$$

Consider the decomposition of $\tilde{\omega}_r(n)$ in the form:

$$\tilde{\omega}_r(n) = \frac{\pi}{\sin(\pi/r)}-\frac{\tilde{\theta}(n,r)}{(2n+1)^{2-1/r}}. \tag{3.23}$$

By (3.22) and (3.23), we have

$$\tilde{\theta}(n,r) := \left[\frac{\pi}{\sin(\pi/r)}-\sum_{m=0}^{\infty}\frac{1}{m+n+1}\left(\frac{2n+1}{2m+1}\right)^{1/r}\right](2n+1)^{2-1/r}. \tag{3.24}$$

By using (2.6), we may show that $\{\tilde{\theta}(n,r)\}$ is strictly decreasing for r, and $\{\tilde{\theta}(n,\infty)\}$ is strictly increasing. Hence, we obtain

$$\tilde{\theta}(n,r) > \min_{n\in\mathbf{N}_0}\inf_{r>1}\{\tilde{\theta}(n,r)\} = \tilde{\theta}(n0,r) = \ln 2-\gamma. \tag{3.25}$$

By (3.23), we find the following inequality of the weight coefficient as:

$$\tilde{\omega}_r(n) < \frac{\pi}{\sin(\pi/p)}-\frac{\ln 2-\gamma}{(2n+1)^{2-1/r}} \qquad (r=p,q,\ n\in\mathbf{N}_0). \tag{3.26}$$

By (3.21) we derive (3.20).

(2) *Another strengthened version of (3.3) similar to (3.16).*

In the year 2000, Yang [38] obtained another strengthened version of (3.3) as:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{m+n+1} < \left\{\sum_{n=0}^{\infty}\left[\frac{\pi}{\sin(\pi/p)}-\frac{1}{13(n+1)(2n+1)^{1/p}}\right]a_n^p\right\}^{1/p}$$

$$\times\Big\{\sum_{n=0}^{\infty}\Big[\frac{\pi}{\sin(\pi/p)}-\frac{1}{13(n+1)(2n+1)^{1/q}}\Big]b_n^q\Big\}^{1/q}. \tag{3.27}$$

Furthermore using (3.20) and (2.6), we find the following inequality of the weight coefficient:

$$\tilde{\omega}_r(n)<\frac{\pi}{\sin(\pi/p)}-\frac{1}{13(n+1)(2n+1)^{1-1/r}} \qquad (r=p,q,\ n\in\mathbf{N}_0). \tag{3.28}$$

Hence, by (3.21) we have (3.27). Obviously, we can show that (3.20) and (3.27) are not comparable.

(3) *Two distinct strengthened versions of (3.4).*

By the proof of the equivalence in Theorem 2.1 and (3.28), we have

$$\sum_{n=0}^{\infty}\Big(\sum_{m=0}^{\infty}\frac{a_m}{m+n+1}\Big)^p<\Big[\frac{\pi}{\sin(\frac{\pi}{p})}\Big]^{p-1}\sum_{n=0}^{\infty}\Big[\frac{\pi}{\sin(\frac{\pi}{p})}-\frac{\ln 2-\gamma}{(2n+1)^{1+\frac{1}{p}}}\Big]a_n^p; \tag{3.29}$$

$$\sum_{n=0}^{\infty}\Big(\sum_{m=0}^{\infty}\frac{a_m}{m+n+1}\Big)^p<\Big[\frac{\pi}{\sin(\frac{\pi}{p})}\Big]^{p-1}\sum_{n=0}^{\infty}\Big[\frac{\pi}{\sin(\frac{\pi}{p})}-\frac{1}{13(n+1)(2n+1)^{\frac{1}{p}}}\Big]a_n^p; \tag{3.30}$$

3.3. Generalization and improvement of Hardy-Littlewood's inequality (1.3)

In the year 2000, by using (3.27), Yang [39] gave a generalization and an improvement of (1.3) as follows:

THEOREM 3.1. *If* $p\geqslant 2,\frac{1}{p}+\frac{1}{q}=1, f\in L^2(0,1),$ *such that*

$$0<\int_0^1 f^2(x)dx<\infty \quad and \quad a_n=\int_0^1 x^n f(x)dx \qquad (n\in\mathbf{N}_0),$$

then

$$\Big(\sum_{n=0}^{\infty}a_n^p\Big)^{1+\frac{1}{p}}<\Big[\frac{\pi}{\sin(\frac{\pi}{p})}\Big]^{\frac{1}{q}}\Big\{\sum_{n=0}^{\infty}\Big[\frac{\pi}{\sin(\frac{\pi}{p})}-\frac{1}{13(n+1)(2n+1)^{\frac{1}{p}}}\Big]a_n^{p(p-1)}\Big\}^{\frac{1}{p}}\int_0^1 f^2(x)dx; \tag{3.31}$$

$$\Big(\sum_{n=0}^{\infty}a_n^p\Big)^{1+\frac{1}{p}}<\frac{\pi}{\sin(\frac{\pi}{p})}\Big\{\sum_{n=0}^{\infty}a_n^{p(p-1)}\Big\}^{\frac{1}{p}}\int_0^1 f^2(x)dx. \tag{3.32}$$

Proof. Since $l^q\subseteq l^p$, $(0<q\leqslant p)$ (see [29,p. 24]), and for $p\geqslant 2$, $\{a_n\}\in l^2$, $(\subseteq l^{p(p-1)})$, then we have $\sum_{n=0}^{\infty}a_n^{p(p-1)}<\infty$. By Cauchy's inequality we obtain

$$\begin{aligned}\Big(\sum_{n=0}^{\infty}a_n^p\Big)^2&=\Big\{\sum_{n=0}^{\infty}a_n^{p-1}\int_0^1 x^n f(x)dx\Big\}^2\\&=\Big\{\int_0^1\Big(\sum_{n=0}^{\infty}a_n^{p-1}x^n\Big)f(x)dx\Big\}^2\leqslant\int_0^1\Big(\sum_{n=0}^{\infty}a_n^{p-1}x^n\Big)^2dx\int_0^1 f^2(x)dx\end{aligned}$$

$$= \Big(\int_0^1 \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_m^{p-1} a_n^{p-1} x^{m+n} dx\Big)\int_0^1 f^2(x)dx$$
$$= \Big(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_m^{p-1} a_n^{p-1}\int_0^1 x^{m+n} dx\Big)\int_0^1 f^2(x)dx$$
$$= \Big[\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{a_m^{p-1} b_n^{p-1}}{m+n+1}\Big]\int_0^1 f^2(x)dx.$$

In view of $p = q(p-1)$, by (3.27) we have

$$\Big(\sum_{n=0}^{\infty} a_n^p\Big)^2 \leqslant \Big\{\sum_{n=0}^{\infty}\Big[\frac{\pi}{\sin(\pi/p)} - \frac{1}{13(n+1)(2n+1)^{1/p}}\Big] a_n^{p(p-1)}\Big\}^{1/p}$$
$$\times\Big\{\sum_{n=0}^{\infty}\Big[\frac{\pi}{\sin(\pi/p)} - \frac{1}{13(n+1)(2n+1)^{1/q}}\Big] a_n^{p}\Big\}^{1/q}\int_0^1 f^2(x)dx. \quad (3.33)$$

Hence we find

$$\Big(\sum_{n=0}^{\infty} a_n^p\Big)^{1+\frac{1}{p}} \leqslant \Big[\frac{\pi}{\sin(\frac{\pi}{p})}\Big]^{\frac{1}{q}}\Big\{\sum_{n=0}^{\infty}\Big[\frac{\pi}{\sin(\frac{\pi}{p})} - \frac{1}{13(n+1)(2n+1)^{\frac{1}{p}}}\Big] a_n^{p(p-1)}\Big\}^{\frac{1}{p}}\int_0^1 f^2(x)dx. \quad (3.34)$$

It is obvious that $\{a_n\} \in l^p$ $(p \geqslant 2)$. Then by (3.27), inequality (3.33) takes the form of strict inequality; so does (3.34). Hence we have (3.31), and then we have (3.32).

REMARK 3.4. For $p = 2$, inequality (3.32) reduces to (1.3). It follows that (3.32) is a generalization of (1.3). In (3.33), for $p = q = 2$, since it does not take the form of equality, we have

$$\Big(\sum_{n=0}^{\infty} a_n^2\Big)^2 < \sum_{n=0}^{\infty}\Big[\pi - \frac{1}{13(n+1)(2n+1)^{1/2}}\Big] a_n^2\int_0^1 f^2(x)dx, \quad (3.35)$$

which is an improvement of (1.3). Obviously, (3.35) and (2.13) are not comparable.

4. Some generalizations of Hardy-Hilbert's inequality

Hardy et al [1, Ch. 9] pointed out that the integral form relating (3.1) and (3.2) can be expressed in the form:

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, f, g are real functions such that

$$0 < \int_0^{\infty} f^p(t)dt < \infty \quad \text{and} \quad 0 < \int_0^{\infty} g^q(t)dt < \infty,$$

then

$$\int_0^{\infty}\int_0^{\infty} \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin(\pi/p)}\Big\{\int_0^{\infty} f^p(t)dt\Big\}^{1/p}\Big\{\int_0^{\infty} g^q(t)dt\Big\}^{1/q}; \quad (4.1)$$
$$\int_0^{\infty}\Big(\int_0^{\infty} \frac{f(x)}{x+y} dx\Big)^p dy < \Big[\frac{\pi}{\sin(\pi/p)}\Big]^p\int_0^{\infty} f^p(t)dt, \quad (4.2)$$

where the constants $\frac{\pi}{\sin(\pi/p)}$ and $\left[\frac{\pi}{\sin(\pi/p)}\right]^p$ are all the best possible. Inequalities (4.1) and (4.2) are equivalent.

Inequality (4.1) is well known as Hardy-Hilbert's integral inequality. For $p = q = 2$, (4.1) reduces to (1.1a), and (4.2) reduces to (1.2a). On generalizing (4.1) with single parameter, Hardy et al [1, Th. 340] claimed:

If $p > 1, q > 1, \frac{1}{p} + \frac{1}{q} \geqslant 1$ and $0 < \lambda = 2 - \frac{1}{p} - \frac{1}{q} \leqslant 1$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dxdy \leqslant K\left\{\int_0^\infty f^p(x)dx\right\}^{1/p}\left\{\int_0^\infty g^q(y)dy\right\}^{1/q}, \tag{4.3}$$

where the constant $K = K(p,q)$ is related p and q.

For $\frac{1}{p} + \frac{1}{q} = 1, \lambda = 2 - \frac{1}{p} - \frac{1}{q} = 1$, since $K(p,q) = \frac{\pi}{\sin(\pi/p)}$, it follows that (4.3) is a generalization of (1.1a). Since the parameter λ in (4.3) relates p, q, it may be improved. In the year 1998, by introducing the β function and some parameters, Yang [24] gave some generalizations of (1.1a) and (1.2a), and Yang [40, 41] gave some further generalizations of (4.1). In the year 2000, Yang [42,43] proved that the constant factors in the extended inequalities are all the best possible.

4.1. On generalizations of (4.1) and (3.1) with single parameter

THEOREM 4.1. *If $p > 1, \frac{1}{p} + \frac{1}{q} = 1$, $\lambda > 2 - \min\{p,q\}$, f, g are non-negative real functions, such that*

$$0 < \int_0^\infty t^{1-\lambda} f^p(t)dt < \infty \quad and \quad 0 < \int_0^\infty t^{1-\lambda} g^q(t)dt < \infty,$$

then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dxdy < B\Big(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\Big) \times\left\{\int_0^\infty t^{1-\lambda} f^p(t)dt\right\}^{\frac{1}{p}}\left\{\int_0^\infty t^{1-\lambda} g^q(t)dt\right\}^{\frac{1}{q}}; \tag{4.4}$$

$$\int_0^\infty y^{(\lambda-1)(p-1)}\left[\int_0^\infty \frac{f(x)}{(x+y)^\lambda}dx\right]^p dy < \left[B\Big(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\Big)\right]^p \int_0^\infty t^{1-\lambda} f^p(t)dt, \tag{4.5}$$

where the constant factors $B\big(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\big)$ and $\left[B\big(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\big)\right]^p$ are all the best possible. Inequalities (4.4) and (4.5) are equivalent. In particular, for $\lambda = 2$, we have

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^2} dxdy < \left\{\int_0^\infty \frac{1}{t} f^p(t)dt\right\}^{1/p}\left\{\int_0^\infty \frac{1}{t} g^q(t)dt\right\}^{1/q}; \tag{4.6}$$

$$\int_0^\infty y^{p-1}\left[\int_0^\infty \frac{f(x)}{(x+y)^2}dx\right]^p dy < \int_0^\infty \frac{1}{t} f^p(t)dt. \tag{4.7}$$

REMARK 4.1. For $\lambda = 1$, (4.4) reduces to (4.1), and (4.5) reduces to (4.2). It follows that (4.4) and (4.5) are generalizations of (4.1) and (4.2) with the best constant factors. It is obvious that (4.4) is more elegant than (4.3).

The method of proving Theorem 4.1. Similarly to the proof of Theorem 2.1, by Hölder's inequality, we have

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dxdy < \Big\{ \int_0^\infty \omega_\lambda(q,x) f^p(x) dx \Big\}^{\frac{1}{p}} \Big\{ \int_0^\infty \omega_\lambda(p,y) g^q(y) dty \Big\}^{\frac{1}{q}}, \tag{4.8}$$

where the weight function $\omega_\lambda(r,t)$ is defined by

$$\omega_\lambda(r,t) := \int_0^\infty \frac{1}{(t+s)^\lambda} \Big(\frac{t}{s}\Big)^{1-\lambda/r} ds \qquad (r = p, q, t \in (0,\infty)). \tag{4.9}$$

Setting $u = s/t$ in the integral of (4.9), we have

$$\omega_\lambda(r,t) = t^{1-\lambda} \int_0^\infty \frac{1}{(1+u)^\lambda} u^{-1+(r+\lambda-2)/r} du \qquad (r = p, q).$$

In view of the following formula related the β function and the Γ function (see [44, p. 117]):

$$B(u,v) = \int_0^\infty \frac{t^{-1+v}}{(1+t)^{u+v}} dt = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)} = B(v,u) \qquad (u, v > 0), \tag{4.10}$$

and $\frac{p+\lambda-2}{p} + \frac{q+\lambda-2}{q} = \lambda$, we have

$$\omega_\lambda(q,x) = \omega_\lambda(p,x) = B\Big(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\Big) x^{1-\lambda}. \tag{4.11}$$

Hence, by (4.8), we have (4.4).

For $\varepsilon > 0$ small enough, define the functions $f_\varepsilon(t)$ and $g_\varepsilon(t)$ as:

$$f_\varepsilon(t) = g_\varepsilon(t) = 0, \quad \text{for} \quad t \in (0,1);$$

$$f_\varepsilon(t) = t^{(\lambda-2-\varepsilon)/p}, \qquad g_\varepsilon(t) = t^{(\lambda-2-\varepsilon)/q}, \quad \text{for} \quad t \in [1,\infty);$$

we find that

$$\Big\{ \int_0^\infty t^{1-\lambda} f_\varepsilon^p(t) dt \Big\}^{1/p} \Big\{ \int_0^\infty t^{1-\lambda} g_\varepsilon^q(t) dt \Big\}^{1/q} = \frac{1}{\varepsilon};$$

$$\int_0^\infty \int_0^\infty \frac{f_\varepsilon(x) g_\varepsilon(y)}{(x+y)^\lambda} dxdy = \frac{1}{\varepsilon} B\Big(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\Big) + o(1) \qquad (\varepsilon \to 0^+).$$

We may show that the constant factor $B\big(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\big)$ in (4.4) is the best possible, by using the method of Theorem 2.1.

Setting $g(y)$ as:

$$g(y) := y^{(\lambda-1)(p-1)} \Big[\int_0^\infty \frac{f(x)}{(x+y)^\lambda} dx \Big]^{p-1} \qquad (y \in (0,\infty)),$$

then we have

$$\int_0^\infty y^{1-\lambda} g^q(y) dy = \int_0^\infty y^{(\lambda-1)(p-1)} \Big[\int_0^\infty \frac{f(x)}{(x+y)^\lambda} dx \Big]^p dy$$
$$= \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dxdy.$$

By (4.4), we have (4.5). Furthermore using the method of Theorem 2.1, by (4.5) and Hölder's inequality, we have (4.4). Hence (4.4) and (4.5) are equivalent. We can show that the constant factor in (4.5) is the best possible by the equivalence of (4.4) and (4.5).

THEOREM 4.2. *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $2 - \min\{p,q\} < \lambda \leqslant 2$, $\{a_n\}$, $\{b_n\}$ *are non-negative real sequences such that*

$$0 < \sum_{n=1}^{\infty} n^{1-\lambda} a_n^p < \infty \quad \textit{and} \quad 0 < \sum_{n=1}^{\infty} n^{1-\lambda} b_n^q < \infty,$$

then (see [45]*)*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{(m+n)^{\lambda}} < B\Big(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\Big)\Big\{\sum_{n=1}^{\infty} n^{1-\lambda} a_n^p\Big\}^{\frac{1}{p}}\Big\{\sum_{n=1}^{\infty} n^{1-\lambda} b_n^q\Big\}^{\frac{1}{q}}; \tag{4.12}$$

$$\sum_{n=1}^{\infty} n^{(\lambda-1)(p-1)}\Big[\sum_{m=1}^{\infty} \frac{a_m}{(m+n)^{\lambda}}\Big]^p < \Big[B\Big(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\Big)\Big]^p \sum_{n=1}^{\infty} n^{1-\lambda} a_n^p, \tag{4.13}$$

where the constant factors $B\big(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\big)$ *and* $\big[B\big(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\big)\big]^p$ *are the best possible. Inequalities* (4.12) *and* (4.13) *are equivalent. In particular, for* $\lambda = 2$, *we have*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{(m+n)^2} < \Big\{\sum_{n=1}^{\infty} \frac{1}{n} a_n^p\Big\}^{1/p}\Big\{\sum_{n=1}^{\infty} \frac{1}{n} b_n^q\Big\}^{1/q}; \tag{4.14}$$

$$\sum_{n=1}^{\infty} n^{(p-1)}\Big[\sum_{m=1}^{\infty} \frac{a_m}{(m+n)^2}\Big]^p < \sum_{n=1}^{\infty} \frac{1}{n} a_n^p. \tag{4.15}$$

REMARK 4.2. For $\lambda = 1$, (4.12) reduces to (3.1), and (4.13) reduces to (3.2); for $p = q = 2$, (4.12) reduces to (2.23), and (4.13) reduces to (2.28). It follows that (4.12) is a generalization of (3.1) and (2.23), and (4.13) is a generalization of (3.2) and (2.28). Their associated integral inequalities are (4.4) and (4.5).

The method of proving Theorem 4.2. By Hölder's inequality, we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{(m+n)^{\lambda}} \leqslant \Big\{\sum_{n=1}^{\infty} \tilde{\omega}_{\lambda}(q,n) a_n^p\Big\}^{1/p}\Big\{\sum_{n=1}^{\infty} \tilde{\omega}_{\lambda}(p,n) b_n^q\Big\}^{1/q}, \tag{4.16}$$

where the weight coefficient $\tilde{\omega}_{\lambda}(r,n)$ is defined by:

$$\tilde{\omega}_{\lambda}(r,n) := \sum_{m=1}^{\infty} \frac{1}{(m+n)^{\lambda}}\Big(\frac{n}{m}\Big)^{(2-\lambda)/r} \qquad (n \in \mathbf{N}, r = p,q). \tag{4.17}$$

For $0 \leqslant 2 - \min\{p,q\} < \lambda \leqslant 2$, we have

$$\tilde{\omega}_{\lambda}(r,n) < \omega_{\lambda}(r,n) = \int_0^{\infty} \frac{1}{(y+n)^{\lambda}}\Big(\frac{n}{y}\Big)^{(2-\lambda)/r} dy.$$

Then by (4.11), we have

$$\tilde{\omega}_\lambda(r,n) < B\Big(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\Big)n^{1-\lambda} \qquad (n \in \mathbf{N}, r = p, q).$$

In view of (4.16), we have (4.12).

For $\varepsilon > 0$ small enough, setting

$$\tilde{a}_n = n^{(\lambda-2-\varepsilon)/p}, \qquad \tilde{b}_n = n^{(\lambda-2-\varepsilon)/q} \qquad (n \in \mathbf{N}),$$

we find

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\tilde{a}_m\tilde{b}_n}{(m+n)^\lambda} &> \int_1^\infty x^{(\lambda-2-\varepsilon)/p}\int_1^\infty \frac{1}{(x+y)^\lambda}y^{(\lambda-2-\varepsilon)/q}dydx \\ &= \frac{1}{\varepsilon}B\Big(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\Big) + o(1);\end{aligned}$$

$$\Big\{\sum_{n=1}^{\infty}\tilde{a}_n^p\Big\}^{1/p}\Big\{\sum_{n=1}^{\infty}\tilde{b}_n^q\Big\}^{1/q} = 1 + \sum_{n=2}^{\infty}\frac{1}{n^{1+\varepsilon}} < 1 + \int_1^\infty \frac{1}{t^{1+\varepsilon}}dt = \frac{1}{\varepsilon}(1+\varepsilon).$$

If the constant factor in (4.12) is not the best possible, we may get a contradiction following the above results. The rest of the proof is similar to the method of proving Theorem 4.1.

4.2. Further generalizations of (4.1) and (3.1) with a single parameter

In the year 1998, Kuang [46] provided the following generalization of (4.1):
If $\max\{1/p, 1/q\} < \lambda \leqslant 1$, then

$$\begin{aligned}\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x^\lambda + y^\lambda}dxdy &< \frac{\pi}{\lambda(\sin(\pi/p\lambda))^{1/p}(\sin(\pi/q\lambda))^{1/q}} \\ &\times\Big\{\int_0^\infty t^{1-\lambda}f^p(t)dt\Big\}^{1/p}\Big\{\int_0^\infty t^{1-\lambda}g^q(t)dt\Big\}^{1/q}. \qquad (4.18)\end{aligned}$$

For $\lambda = 1$, inequality (4.18) reduces to (4.1). In the year 2000, Hu [47] gave a generalization of Hardy-Littlewood-Polya's inequality with a single parameter λ, and then, Yang [48] improved (4.18) in the following:

Setting $x = X^\lambda, y = Y^\lambda$ $(\lambda > 0)$ in (4.1), one has $dx = \lambda X^{\lambda-1}dX, dy = \lambda Y^{\lambda-1}dY$ and

$$\begin{aligned}\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}dxdy &= \int_0^\infty\int_0^\infty \frac{[X^{\lambda-1}f(X^\lambda)][Y^{\lambda-1}g(Y^\lambda)]}{X^\lambda+Y^\lambda}\lambda^2 dXdY \\ &< \frac{\pi}{\sin(\pi/p)}\Big\{\int_0^\infty \lambda X^{\lambda-1}f^p(X^\lambda)dX\Big\}^{1/p}\Big\{\int_0^\infty \lambda Y^{\lambda-1}g^q(Y^\lambda)dY\Big\}^{1/q} \\ &= \frac{\pi}{\sin(\pi/p)}\Big\{\int_0^\infty f^p(x)dx\Big\}^{1/p}\Big\{\int_0^\infty g^q(y)dy\Big\}^{1/q}.\end{aligned}$$

Setting $F(X) = X^{\lambda-1}f(X^\lambda), G(Y) = Y^{\lambda-1}g(Y^\lambda)$ in the above inequality, we find

$$\int_0^\infty \int_0^\infty \frac{F(X)G(Y)}{X^\lambda + Y^\lambda} dXdY < \frac{\pi}{\lambda \sin(\pi/p)}$$

$$\times \Big\{ \int_0^\infty X^{(p-1)(1-\lambda)} F^p(X)dX \Big\}^{1/p} \Big\{ \int_0^\infty Y^{(q-1)(1-\lambda)} G^q(Y)dY \Big\}^{1/q}. \quad (4.19)$$

It is obvious that (4.19) and (4.1) are equivalent, and the constant factor in (4.19) is the best possible. Applying the same way, we can build a generalization of (4.2), that is equivalent to (4.19).

THEOREM 4.3. *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda > 0$, f, g *are non-negative real functions such that*

$$\int_0^\infty t^{(p-1)(1-\lambda)} f^p(t)dt < \infty, \quad \textit{and} \quad \int_0^\infty t^{(q-1)(1-\lambda)} g^q(t)dt < \infty,$$

then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\lambda + y^\lambda} dxdy < \frac{\pi}{\lambda \sin(\pi/p)}$$

$$\times \Big\{ \int_0^\infty t^{(p-1)(1-\lambda)} f^p(t)dt \Big\}^{1/p} \Big\{ \int_0^\infty t^{(q-1)(1-\lambda)} g^q(t)dt \Big\}^{1/q}; \quad (4.20)$$

$$\int_0^\infty y^{\lambda-1} \Big(\int_0^\infty \frac{f(x)}{x^\lambda + y^\lambda} dx \Big)^p dy < \Big[\frac{\pi}{\lambda \sin(\pi/p)} \Big]^p \int_0^\infty t^{(p-1)(1-\lambda)} f^p(t)dt, \quad (4.21)$$

where the constant $\frac{\pi}{\lambda \sin(\pi/p)}$ *and* $\big[\frac{\pi}{\lambda \sin(\pi/p)}\big]^p$ *are all the best possible. Inequalities (4.20) and (4.21) are equivalent.*

REMARK 4.3. Inequality (4.20) is obviously more elegant than (4.18), which is a generalization of (4.1) with a single parameter.

THEOREM 4.4. *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \lambda \leqslant \min\{p, q\}$, $\{a_n\}$, $\{b_n\}$ *are non-negative real sequences, such that*

$$\sum_{n=1}^\infty n^{(p-1)(1-\lambda)} a_n^p < \infty, \textit{ and } \sum_{n=1}^\infty n^{(q-1)(1-\lambda)} b_n^q < \infty,$$

then

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{m^\lambda + n^\lambda} < \frac{\pi}{\lambda \sin(\frac{\pi}{p})} \Big\{ \sum_{n=1}^\infty n^{(p-1)(1-\lambda)} a_n^p \Big\}^{\frac{1}{p}} \Big\{ \sum_{n=1}^\infty n^{(q-1)(1-\lambda)} b_n^q \Big\}^{\frac{1}{q}}, \quad (4.22)$$

$$\sum_{n=1}^\infty n^{\lambda-1} \Big(\sum_{m=1}^\infty \frac{a_m}{m^\lambda + n^\lambda} \Big)^p < \Big[\frac{\pi}{\lambda \sin(\pi/p)} \Big]^p \sum_{n=1}^\infty n^{(p-1)(1-\lambda)} a_n^p, \quad (4.23)$$

where the constant factors $\frac{\pi}{\lambda\sin(\pi/p)}$ *and* $\left[\frac{\pi}{\lambda\sin(\pi/p)}\right]^p$ *are all the best possible. Inequalities (4.22) and (4.23) are equivalent.*

Idea of the proof. By Hölder's inequality, we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m^\lambda+n^\lambda}=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\Big[\frac{a_m}{(m^\lambda+n^\lambda)^{\frac{1}{p}}}\Big(\frac{m^{1/q}}{n^{1/p}}\Big)^{1-\lambda}\Big(\frac{m}{n}\Big)^{\frac{\lambda}{qp}}\Big]$$
$$\times\Big[\frac{b_n}{(m^\lambda+n^\lambda)^{\frac{1}{q}}}\Big(\frac{n^{1/p}}{m^{1/q}}\Big)^{1-\lambda}\Big(\frac{n}{m}\Big)^{\frac{\lambda}{pq}}\Big]$$
$$\leqslant\Big\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m^p}{m^\lambda+n^\lambda}\Big(\frac{m^{p/q}}{n}\Big)^{1-\lambda}\Big(\frac{m}{n}\Big)^{\frac{\lambda}{q}}\Big\}^{\frac{1}{p}}\Big\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{b_n^q}{m^\lambda+n^\lambda}\Big(\frac{n^{q/p}}{m}\Big)^{1-\lambda}\Big(\frac{n}{m}\Big)^{\frac{\lambda}{p}}\Big\}^{\frac{1}{q}}$$
$$=\Big\{\sum_{m=1}^{\infty}\tilde{\omega}_\lambda(p,m)m^{(p-1)(1-\lambda)}a_m^p\Big\}^{1/p}\Big\{\sum_{n=1}^{\infty}\tilde{\omega}_\lambda(q,n)n^{(q-1)(1-\lambda)}b_n^q\Big\}^{1/q},\tag{4.24}$$

where the weight coefficient $\tilde{\omega}_\lambda(r,n)$ is defined by

$$\tilde{\omega}_\lambda(r,n):=\sum_{m=1}^{\infty}\frac{1}{m^\lambda+n^\lambda}\cdot\frac{n^{\lambda(1-1/r)}}{m^{1-\lambda/r}}\qquad(r=p,q,n\in\mathbf{N}).\tag{4.25}$$

For $0<\lambda\leqslant\min\{p,q\}$, we have $1-\lambda/r\geqslant 0$ $(r=p,q)$, and then

$$\tilde{\omega}_\lambda(r,n)<\int_0^\infty\frac{1}{t^\lambda+n^\lambda}\cdot\frac{n^{\lambda(1-1/r)}}{t^{1-\lambda/r}}dt.$$

Setting $y=(t/n)^\lambda$ in the above integral, we obtain $\tilde{\omega}_\lambda(r,n)<\pi/[\lambda\sin(\pi/r)]$. By (4.24), we have (4.22).

The second step. For $\varepsilon>0$ small enough, setting

$$\tilde{a}_n=n^{-[1+\varepsilon+(p-1)(1-\lambda)]/p},\qquad \tilde{b}_n=n^{-[1+\varepsilon+(q-1)(1-\lambda)]/q}\qquad(n\in\mathbf{N}),$$

then we obtain

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\tilde{a}_m\tilde{b}_n}{m^\lambda+n^\lambda}>\int_1^\infty x^{-[1+\varepsilon+(p-1)(1-\lambda)]/p}\int_1^\infty\frac{y^{-[1+\varepsilon+(q-1)(1-\lambda)]/q}}{x^\lambda+y^\lambda}dydx$$
$$=\frac{1}{\varepsilon}\Big(\frac{\pi}{\lambda\sin(\pi/p)}+o(1)\Big)\ (\varepsilon\to 0^+);$$
$$\Big\{\sum_{n=1}^{\infty}n^{(p-1)(1-\lambda)}\tilde{a}_n^p\Big\}^{1/p}\Big\{\sum_{n=1}^{\infty}n^{(q-1)(1-\lambda)}\tilde{b}_n^q\Big\}^{1/q}$$
$$=1+\sum_{n=2}^{\infty}\frac{1}{n^{1+\varepsilon}}<1+\int_1^\infty\frac{1}{t^{1+\varepsilon}}dt=\frac{1}{\varepsilon}(1+\varepsilon).$$

If the constant factor in (4.22) is not the best possible, we may get a contradiction by using the above results.

The third step. Similar to the way of proof in Theorem 4.1, we can show that (4.22) and (4.23) are equivalent. Then by the equivalence of (4.22) and (4.23), we may show that the constant factor in (4.23) is the best possible.

4.3. Two new generalizations of (3.3) with single parameter

(1) In the year 1999, by introducing a parameter λ and the β function, Yang and Debnath [49] gave generalizations of (3.3) and (3.4) as follows:

THEOREM 4.5. *If* $p > 1, \frac{1}{p} + \frac{1}{q} = 1, 2 - \min p, q < \lambda \leqslant 2, \{a_n\}, \{b_n\}$ *are non-negative real sequences, such that*

$$\sum_{n=0}^{\infty}(n+1/2)^{1-\lambda}a_n^p < \infty, \text{ and } \sum_{n=0}^{\infty}(n+1/2)^{1-\lambda}b_n^q < \infty,$$

then

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{(m+n+1)^{\lambda}} < B\Big(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\Big) \times \Big\{\sum_{n=0}^{\infty}(n+\frac{1}{2})^{1-\lambda}a_n^p\Big\}^{1/p}\Big\{\sum_{n=0}^{\infty}(n+\frac{1}{2})^{1-\lambda}b_n^q\Big\}^{1/q}; \tag{4.26}$$

$$\sum_{n=0}^{\infty}(n+\frac{1}{2})^{(p-1)(\lambda-1)}\Big[\sum_{m=0}^{\infty}\frac{a_m}{(m+n+1)^{\lambda}}\Big]^p < \Big[B\Big(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\Big)\Big]^p\sum_{n=0}^{\infty}(n+\frac{1}{2})^{1-\lambda}a_n^p, \tag{4.27}$$

where the constant factors $B\big(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\big)$ *and* $\big[B\big(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\big)\big]^p$ *are all the best possible. Inequalities (4.26) and (4.27) are equivalent. In particular,*

(i) for $\lambda = 2$*, we have two equivalent inequalities as:*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{(m+n+1)^2} < 2\Big\{\sum_{n=0}^{\infty}\frac{1}{2n+1}a_n^p\Big\}^{1/p}\Big\{\sum_{n=0}^{\infty}\frac{1}{2n+1}b_n^q\Big\}^{1/q}; \tag{4.28}$$

$$\sum_{n=0}^{\infty}(2n+1)^{p-1}\Big[\sum_{m=0}^{\infty}\frac{a_m}{(m+n+1)^2}\Big]^p < 2^p\sum_{n=0}^{\infty}\frac{1}{2n+1}a_n^p, \tag{4.29}$$

(ii) for $p = q = 2$*, we have* $0 < \lambda \leqslant 2$*, and*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{(m+n+1)^{\lambda}} < B\Big(\frac{\lambda}{2}, \frac{\lambda}{2}\Big)\Big\{\sum_{n=0}^{\infty}(n+\frac{1}{2})^{1-\lambda}a_n^2\sum_{n=0}^{\infty}(n+\frac{1}{2})^{1-\lambda}b_n^2\Big\}^{1/2}; \tag{4.30}$$

$$\sum_{n=0}^{\infty}(n+\frac{1}{2})^{\lambda-1}\Big[\sum_{m=0}^{\infty}\frac{a_m}{(m+n+1)^{\lambda}}\Big]^2 < \Big[B\Big(\frac{\lambda}{2}, \frac{\lambda}{2}\Big)\Big]^2\sum_{n=0}^{\infty}(n+\frac{1}{2})^{1-\lambda}a_n^2. \tag{4.31}$$

REMARK 4.4. For $\lambda = 1$, (4.26) reduces to (3.3), and (4.27) reduces to (3.4). Hence inequalities (4.26) and (4.27) are generalizations of (3.3) and (3.4) with the best possible constant factors.

The idea of proof of Theorem 4.5. By using Hölder's inequality, we have

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{(m+n+1)^{\lambda}} \leqslant \Big\{\sum_{n=0}^{\infty}\omega_{\lambda}(q,n)a_n^p\Big\}^{1/p}\Big\{\sum_{n=0}^{\infty}\omega_{\lambda}(p,n)b_n^q\Big\}^{1/q}, \qquad (4.32)$$

where the weight coefficient $\omega_{\lambda}(r,n)$ is defined by

$$\omega_{\lambda}(r,n) := \sum_{m=0}^{\infty}\frac{1}{(m+n+1)^{\lambda}}\Big(\frac{n+1/2}{m+1/2}\Big)^{(2-\lambda)/r} \qquad (n\in\mathbf{N}_0, r=p,q). \qquad (4.33)$$

By using (2.6), we obtain

$$\omega_{\lambda}(r,n) \leqslant \Big(n+\frac{1}{2}\Big)^{1-\lambda}\int_0^{\infty}\frac{1}{(1+u)^{\lambda}}\Big(\frac{1}{u}\Big)^{\frac{2-\lambda}{r}}du - \Big(n+\frac{1}{2}\Big)^{\frac{2-\lambda}{r}}R_{\lambda}(r,n), \qquad (4.34)$$

where,

$$R_{\lambda}(r,n) : = \Big(n+\frac{1}{2}\Big)^{1-\lambda-\frac{2-\lambda}{r}}\int_0^{1/(2n+1)}\frac{1}{(1+u)^{\lambda}}\Big(\frac{1}{u}\Big)^{(2-\lambda)/r}du - \Big[\frac{3r+2-\lambda}{6r}+\frac{\lambda}{12(n+1)}\Big]\frac{2^{(2-\lambda)/r}}{(n+1)^{\lambda}}.$$

Integration by parts, implies $R_{\lambda}(r,n) > 0$. Hence by (4.34), we have

$$\omega_{\lambda}(r,n) < \Big(n+\frac{1}{2}\Big)^{1-\lambda}\int_0^{\infty}\frac{1}{(1+u)^{\lambda}}\Big(\frac{1}{u}\Big)^{(2-\lambda)/r}du = \Big(n+\frac{1}{2}\Big)^{1-\lambda}B\Big(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}\Big) \qquad (r=p,q). \qquad (4.35)$$

By (4.32), we have (4.30). Similarly to the proof of Theorem 4.4, one can finish the proof.

(2) In the year 2003, Yang [50] gave further generalizations of (3.3) and (3.4) as follows:

THEOREM 4.5. *If* $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $0<\lambda\leqslant \min p,q$, $\{a_n\}$, $\{b_n\}$ *are non-negative real sequences, such that*

$$\sum_{n=0}^{\infty}(n+1/2)^{(p-1)(1-\lambda)}a_n^p<\infty \quad \textit{and} \quad \sum_{n=0}^{\infty}(n+1/2)^{(q-1)(1-\lambda)}b_n^q<\infty,$$

then

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{(m+n+1)^{\lambda}} < B\Big(\frac{\lambda}{p},\frac{\lambda}{q}\Big) \times\Big\{\sum_{n=0}^{\infty}(n+\frac{1}{2})^{(p-1)(1-\lambda)}a_n^p\Big\}^{1/p}\Big\{\sum_{n=0}^{\infty}(n+\frac{1}{2})^{(q-1)(1-\lambda)}b_n^q\Big\}^{1/q}; \qquad (4.36)$$

$$\sum_{n=0}^{\infty}(n+\frac{1}{2})^{\lambda-1}\Big[\sum_{m=0}^{\infty}\frac{a_m}{(m+n+1)^{\lambda}}\Big]^p < \Big[B\Big(\frac{\lambda}{p},\frac{\lambda}{q}\Big)\Big]^p\sum_{n=0}^{\infty}(n+\frac{1}{2})^{(p-1)(1-\lambda)}a_n^p, \quad (4.37)$$

where the constant factors $B\big(\frac{\lambda}{p},\frac{\lambda}{q}\big)$ *and* $\big[B\big(\frac{\lambda}{p},\frac{\lambda}{q}\big)\big]^p$ *are all the best possible. Inequalities (4.36) and (4.37) are equivalent.*

REMARK 4.6. By (4.26) and (4.36), it follows that there are two more distinct generalizations of (3.3) with the same double series form and the best constant factors. The method of proof of Theorem 4.5 is following the same way as in Theorem 4.4.

5. On an extended multiple Hardy-Hilbert's integral inequality

On multiple Hardy-Hilbert's integral inequality, Hardy et al. [1, Th. 322] stated:

If $p,q,\dots,r$ are n numbers, such that $p>1, q>1,\dots,r>1, \frac{1}{p}+\frac{1}{q}+\dots+\frac{1}{r}=1$, $K(x,y,\dots,z)$ is a positive function of n variables $x,y,\dots,z,$ and of a homogenous form of degree $-n+1$, such that

$$\int_0^{\infty}\cdots\int_0^{\infty}K(1,y,\dots,z)y^{-1/q}\dots z^{-1/r}dy\dots dz=k,$$

then

$$\int_0^{\infty}\int_0^{\infty}\cdots\int_0^{\infty}K(x,y,\dots,z)f(x)g(y)\dots h(z)dxdy\dots dz$$
$$\leqslant k\Big(\int_0^{\infty}f^p(x)dx\Big)^{1/p}\Big(\int_0^{\infty}g^q(y)dy\Big)^{1/q}\cdots\Big(\int_0^{\infty}h^r(z)dz\Big)^{1/r}. \quad (5.1)$$

We give an improvement and an extension of (5.1)in the following (see Yang [51]).

THEOREM 5.1. *If* $n\in\mathbf{N}\setminus\{1\}, p_i>1, \lambda>n-\min\{p_i; 1\leqslant i\leqslant n\}$, *satisfy* $\sum_{i=1}^{n}\frac{1}{p_i}=1$, *and*

$$f_i\geqslant 0,\quad 0<\int_0^{\infty}t^{n-1-\lambda}f_i^{p_i}(t)dt<\infty\quad (i=1,2,\dots,n),$$

then

$$\int_0^{\infty}\cdots\int_0^{\infty}\frac{\prod_{i=1}^{n}f_i(x_i)}{(\sum_{i=1}^{n}x_i)^{\lambda}}dx_1\dots dx_n<\frac{1}{\Gamma(\lambda)}\prod_{i=1}^{n}\Gamma\Big(\frac{p_i+\lambda-n}{p_i}\Big)\Big\{\int_0^{\infty}t^{n-1-\lambda}f_i^{p_i}(t)dt\Big\}^{\frac{1}{p_i}}, \quad (5.2)$$

where the constant factor $\frac{1}{\Gamma(\lambda)}\prod_{i=1}^{n}\Gamma\big(\frac{p_i+\lambda-n}{p_i}\big)$ *is the best possible. In particular,*

(i) for $\lambda=n-1$, *we have*

$$\int_0^{\infty}\cdots\int_0^{\infty}\frac{\prod_{i=1}^{n}f_i(x_i)}{(\sum_{i=1}^{n}x_i)^{n-1}}dx_1\dots dx_n<\frac{1}{(n-1)!}\prod_{i=1}^{n}\Gamma\Big(\frac{p_i-1}{p_i}\Big)\Big\{\int_0^{\infty}f_i^{p_i}(t)dt\Big\}^{\frac{1}{p_i}}; \quad (5.3)$$

(ii) for $p_i = n$, $(i = 1, 2, \dots, n)$, we have $\lambda > 0$, and

$$\int_0^\infty \cdots \int_0^\infty \frac{\prod_{i=1}^n f_i(x_i)}{(\sum_{i=1}^n x_i)^\lambda} dx_1 \dots dx_n < \frac{1}{\Gamma(\lambda)} \Big[\Gamma\Big(\frac{\lambda}{n}\Big)\Big]^n \Big\{ \prod_{i=1}^n \int_0^\infty t^{n-1-\lambda} f_i^n(t) dt \Big\}^{\frac{1}{n}}. \tag{5.4}$$

REMARK 5.1. For $n = 2$, (5.2), (5.3) and (5.4) reduce respectively to (4.4), (4.1) and (1.1a). Inequality (5.2) is a generalization of (5.3) with a single parameter, which is more elegant than (5.1) and (1.9) (for $a = b = 1$). The main result of [52] is (5.4).

We prove some lemmas before we give a proof of the theorem.

LEMMA 5.1. *If $r_i > 0$ $(i \in \mathbf{N})$, setting $\lambda(k) = \sum_{i=1}^{k+1} r_i$ $(k \in \mathbf{N})$, then*

$$\int_0^\infty \cdots \int_0^\infty \frac{1}{(1 + \sum_{j=1}^k u_j)^{\lambda(k)}} \prod_{i=1}^k u_i^{r_i - 1} du_1 \dots du_k = \frac{1}{\Gamma(\lambda(k))} \prod_{i=1}^{k+1} \Gamma(r_i). \tag{5.5}$$

Proof. We establish (5.5) by mathematical induction. For $k = 1$, we have (5.5) by (4.10). Suppose for $k \in \mathbf{N}$, that (5.5) is valid. Then for $k + 1$, since $\lambda(k+1) = \sum_{i=2}^{k+2} r_i + r_1$, by setting $v = u_1/(1 + \sum_{i=2}^{k+1})$, we obtain

$$\int_0^\infty \cdots \int_0^\infty \frac{1}{(1 + \sum_{j=1}^{k+1} u_j)^{\lambda(k+1)}} \prod_{i=1}^{k+1} u_i^{r_i - 1} du_1 \dots du_{k+1}$$

$$= \int_0^\infty \cdots \int_0^\infty \frac{\prod_{i=2}^{k+1} u_i^{r_i - 1}}{(1 + \sum_{j=2}^{k+1} u_j)^{\lambda(k+1) - r_1}} du_2 \dots du_{k+1} \Big[\int_0^\infty \frac{v^{r_1 - 1}}{(1 + v)^{\lambda(k+1)}} dv \Big]. \tag{5.6}$$

Since we have

$$\int_0^\infty \frac{v^{r_1 - 1}}{(1 + v)^{\lambda(k+1)}} dv = \frac{1}{\Gamma(\lambda(k+1))} \Gamma\Big(\sum_{i=2}^{k+1} r_i\Big) \Gamma(r_1).$$

By the assumption of Mathematical induction for k, we have

$$\int_0^\infty \cdots \int_0^\infty \frac{\prod_{i=2}^{k+1} u_i^{r_i - 1}}{(1 + \sum_{j=2}^{k+1} u_j)^{\lambda(k+1) - r_1}} du_2 \dots du_{k+1} = \frac{1}{\Gamma(\sum_{i=2}^{k+2} r_i)} \prod_{i=2}^{k+2} \Gamma(r_i).$$

By (5.6), we have (5.5) for $k + 1$. By induction, it follows that (5.5) is valid for $k \in \mathbf{N}$.

LEMMA 5.2. *If* $n \in \mathbf{N} \setminus \{1\}$, $p_i > 1$, $\lambda > n - \min\{p_i; 1 \leqslant i \leqslant n\}$, *satisfy* $\sum_{i=1}^{n} \frac{1}{p_i} = 1$, *and for* $j \in \{1, 2, \ldots, n\}$, *set* $\omega_j(x_j)$ *as*

$$\omega_j(x_j) := x_j^{1+\frac{\lambda-n}{p_j}} \int_0^\infty \ldots \int_0^\infty \frac{1}{(\sum_{j=1}^{n} x_j)^\lambda} \prod_{1 \leqslant i \neq j \leqslant n} x_i^{\frac{\lambda-n}{p_i}} dx_1 \ldots dx_{j-1} dx_{j+1} \ldots dx_n. \quad (5.7)$$

Then each $\omega_j(x_j)$ *is constant, that is*

$$\omega_j(x_j) = \frac{1}{\Gamma(\lambda)} \prod_{i=1}^{n} \Gamma\Big(\frac{p_i + \lambda - n}{p_i}\Big) \qquad (j = 1, 2, \ldots, n). \quad (5.8)$$

Proof. Fix j. Setting $\tilde{p}_i$ and u_i as $\tilde{p}_n = p_j$, and

$$\tilde{p}_i = p_i, u_i = \frac{x_i}{x_j}, \quad \text{for} \quad i = 1, 2, \ldots, j-1;$$

$$\tilde{p}_i = p_{i+1}, u_i = \frac{x_{i+1}}{x_j}, \quad \text{for} \quad i = j, \ldots, n-1$$

in (5.7), by simplification we have

$$\omega_j(x_j) = \int_0^\infty \ldots \int_0^\infty \frac{1}{(1 + \sum_{i=1}^{n-1} u_i)^\lambda} \prod_{i=1}^{n-1} u_i^{\frac{\lambda-n}{\tilde{p}_i}} du_1 \ldots du_{n-1}. \quad (5.9)$$

Substitution of $n-1$ for k, λ for $\lambda(k)$, and $1 - (n-\lambda)/\tilde{p}_i$ for r_i $(i = 1, 2, \ldots, n)$ into (5.5), by (5.9) we have

$$\omega_j(x_j) = \frac{1}{\Gamma(\lambda)} \prod_{i=1}^{n} \Gamma\Big(\frac{\tilde{p}_i + \lambda - n}{\tilde{p}_i}\Big) = \frac{1}{\Gamma(\lambda)} \prod_{i=1}^{n} \Gamma\Big(\frac{p_i + \lambda - n}{p_i}\Big) \quad (j = 1, 2, \ldots, n). \quad (5.10)$$

Equality (5.8) is valid.

LEMMA 5.3. *Following the assumption of Lemma 5.2, if* $0 < \varepsilon < \lambda - n + \min\{p_i; 1 \leqslant i \leqslant n\}$, *then*

$$\varepsilon \int_0^\infty \ldots \int_0^\infty \frac{\prod_{i=1}^{n} x_i^{\frac{\lambda-n-\varepsilon}{p_i}}}{(\sum_{j=1}^{n} x_j)^\lambda} dx_1 \ldots dx_n \geqslant \frac{1}{\Gamma(\lambda)} \prod_{i=1}^{n} \Gamma\Big(\frac{p_i + \lambda - n}{p_i}\Big) + o(1) \quad (\varepsilon \to 0^+). \quad (5.11)$$

Proof. Setting $u_i = x_i/x_n$ $(i = 1, 2, \ldots, n-1)$, we have

$$\varepsilon \int_0^\infty \ldots \int_0^\infty \frac{1}{(\sum_{j=1}^{n} x_j)^\lambda} \prod_{1}^{n} x_i^{\frac{\lambda-n-\varepsilon}{p_i}} dx_1 \ldots dx_n \geqslant [\omega_n(x_n) + o(1)] - \varepsilon A, \quad (5.12)$$

where, A is defined by

$$A := \int_1^\infty x_n^{-1} \sum_{j=1}^{n-1} \Big[\int \cdots \int_{D_j} \frac{1}{(1+\sum_{i=1}^{n-1} u_i)^\lambda} u_i^{\frac{\lambda-n-\varepsilon}{\tilde{p}_i}} du_1 \ldots du_{n-1} \Big] dx_n,$$

with

$$D_j = \Big\{ (u_1, u_2, \ldots, u_{n-1}) : 0, \quad u_j \leqslant 1/x_n, \quad 0 < u_k < \infty \quad (k \neq j) \Big\}.$$

We can obtain that $A = O(1)$, and by (5.12) and (5.8) (for $j = n$), we have (5.11).

Proof of Theorem 5.1. By Hölder's inequality and (5.9), we have

$$\begin{aligned}
&\int_0^\infty \cdots \int_0^\infty \frac{1}{(\sum_{i=1}^n x_i)^\lambda} \prod_{i=1}^n f_i(x_i) dx_1 \ldots dx_n \\
&= \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^n \Big\{ \frac{f_j(x_j)}{(\sum_{i=1}^n x_i)^{\lambda/p_j}} \Big[x_j^{(n-\lambda)(1-\frac{1}{p_j})} \prod_{1 \leqslant i \neq j \leqslant n} x_i^{\frac{\lambda-n}{p_i}} \Big] \Big\} dx_1 \ldots dx_n \\
&\leqslant \prod_{j=1}^n \Big\{ \int_0^\infty \omega_j(x_j) x^{n-1-\lambda} f_j^{p_j}(x_j) dx_j \Big\}^{\frac{1}{p_j}}.
\end{aligned}$$

Since the above equality does not hold (see [29, p. 29]), by (5.8), we have (5.2).

For $0 < \varepsilon < \lambda - n + \min\{p - i; 1 \leqslant i \leqslant n\}$, setting

$$\tilde{f_j}(x_j) = 0,\ x \in (0,1); \quad \tilde{f_j}(x_j) = x_j^{\frac{\lambda-n-\varepsilon}{p_j}},\ x_j \in [1,\infty) \quad (j = 1, 2, \ldots, n),$$

we obtain $\varepsilon \prod_{i=1}^n \Big\{ \int_0^\infty t^{n-1-\lambda} \tilde{f}_i^{p_i}(t) dt \Big\}^{\frac{1}{p_i}} = 1$, and by (5.11), we have

$$\varepsilon \int_0^\infty \cdots \int_0^\infty \frac{1}{(\sum_{i=1}^n x_i)^\lambda} \prod_{i=1}^n \tilde{f_i}(x_i) dx_1 \ldots dx_n \geqslant \frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \Gamma\Big(\frac{p_i+\lambda-n}{p_i}\Big) + o(1).$$

If the constant factor in (5.2) is not the best possible, then by the above results, we may get a contradiction.

REMARK 5.2. We do not consider the equivalent form and the series form of (5.2).

REFERENCES

[1] G. H. HARDY, J. E. LITTLEWOOD AND G. POLYA, *Inequalities*, Cambridge Univ. Press, London, 1952.

[2] D. S. MITRINOVIC, J. E. PECARIC AND A. M. FINK, *Inequalities Involving Functions and their Integrals and Derivatives*, Kluwer Academic Publishers, Boston, 1991.

[3] D. S. MITRINOVIC, *Analytic inequalities*, Springer-Verlag, New York, 1970.

[4] A. E. INGHAM, *A note on Hilbert's inequality*, J. London Math. Soc. 11 (1936), 237–240.

[5] L. C. HSU AND Y. J. WANG, *A refinement of Hilbert's double series theorem*, J. Math. Res. Exp. 11:1 (1991), 143–144.

[6] GAO MINGZHE, *A note on Hilbert double series theorem*, Hunan Annals of Math., 12:1–2 (1992), 143–147.

[7] XU LIZHI AND GUO YONGKANG, *Not on Hardy-Riesz's extension of Hilbert's inequality*, Chin. Quart. J. Math., 6:1 (1991), 75–77.

[8] HU KE, *On Hilbert's inequality*, Chin. Ann. of Math., 13B:1 (1992), 37–39.2.

[9] HU KE, *On Hilbert inequality and its application*, Advances in Math., 22:2 (1993), 160–163.

[10] HU KE, *Establishments and applications of the base inequalities*, Jianxi College and University Press, Nanchang, 1998.

[11] GAO MINGZHE AND TAN LI, *Some improvements on Hilbert's integral inequality*, J. Math. Anal. Appl., 229 (1999), 682–689. 237–240.

[12] GAO MINGZHE, *On the Hilbert inequality*, J. for Anal. Appl., 18:4 (1999), 1117–1122.

[13] B. G. PACHPATTE, *On some new inequalities similar to Hilbert's inequality*, J. Math. Anal. Appl., 226 (1998), 166–179.

[14] ZHAO CHANGJIAN, *Generalization on two new Hilbert type inequalities*, J. Math., 20:4 (2000), 413–416.

[15] ZHAO CHANGJIAN AND L. DEBNATH, *Some new type Hilbert integral inequalities*, J. Math. Anal. Appl., 262 (2001), 411–418.

[16] YANG BICHENG, *On a new inequality similar to Hardy-Hilbert's inequality*, Math. Ineq. Appl., 6:1 (2003), 37–44.

[17] YANG BICHENG, *On a new Hardy-Hilbert's type inequality and applications*, Acta Mathematica Sinica, 46:5 (2003) (to appear).

[18] HONG YONG, *All-sided generalization about Hardy-Hilbert integral inequalities*, Acta Mathematica Sinica, 44:4 (2001), 619–626.

[19] XU LIZHI AND WANG XINGHUA, *The way of mathematical analysis and some examples*, High Education Press, Beijing, 1985.

[20] YANG BICHENG, *A refinement of Hilbert's inequality*, Huanghuai J., 13:2 (1997), 47–51. (1993), 160–163.

[21] YANG BICHENG AND L. DEBNATH, *Some inequalities involving* π *and an application to Hilbert's inequality*, Appl. Math. Lett., 12 (1999), 101–105.11.

[22] YANG BICHENG AND L. DEBNATH, *On a refinement of Hilbert's inequality*, Classroom Notes, 2000, 773–779.

[23] YANG BICHENG, *A strengthened Hilbert's inequality and its application*, Journal of Xinyang Teachers College, 14:1 (2001), 40–43.

[24] YANG BICHENG, *On Hilbert's integral inequality*, J. Math. Anal. Appl., 220 (1998), 778–785.

[25] YANG BICHENG, *A note on Hilbert's integral inequalities*, Chin. Quart. J. Math., 13:4 (1998), 83–86.

[26] YANG BICHENG, *On generalization of Hilbert's theorem*, J. Nanjing Univ. Math. Biquarterly, 17:1 (2000), 152–156.

[27] YANG BICHENG AND L. DEBNATH, *On a generalization of Hilbert's integral inequality and its applications*, Research Report Collection, 2:5 (1999), 731–741.

[28] YANG BICHENG, *On a generalization of Hilbert's double series theorem*, Math. Ineq. Appl., 5:2 (2002), 197–204.

[29] KUANG JICHANG, *Applied Inequalities*, Hunan Education Press, Changsha, 1993.

[30] XIE HONGGZHENG, LU ZHONGXUE AND XIN YUMEI, *New Majorized results on Hilbert's integral inequality*, Chin. Quart. J. Math., 16:4 (2001), 67–75.

[31] YANG BICHENG, *On a generalization of Hilbert double series theorem*, J. Nanjing Univ. Math. Biquarterly, 18:1 (2001), 145–152.

[32] G. H. HARDY, *Note on a theorem of Hilbert concerning series of positive terms*, Proc. London Math. Soc., 23:2 (1925), Records of Proc. xlv–xlvi.

[33] S. G. MIKHLIN, *Constants in some inequalities of analysis*, John Wiley and Sons, New York, 1986.

[34] YANG BICHENG AND GAO MINGZHE, *On a best value of Hardy-Hilbert's inequality*, Advances in Math., 26:2 (1997), 159–164.

[35] GAO MINGZHE AND YANG BICHENG, *On the extended Hilbert's inequality*, Proc. Amer. Math. Soc., 126:3 (1998), 751–759.
[36] YANG BICHENG AND L. DEBNATH, *On new strengthened Hardy-Hilbert's inequality*, Internat. J. Math. Math. Sci., 21:2 (1998), 403–408.
[37] YANG BICHENG, *On a strengthened version of the more accurate Hardy-Hilbert's inequality*, Acta Mathematica Sinica, 42:6 (1999), 1103–1110.
[38] YANG BICHENG, *On a strengthened Hardy-Hilbert's inequality*, J. Ineq. in Pure and Applied Math., 1:2 (2000), Article 22.
[39] YANG BICHENG, *On a refinement of Hardy-Hilbert's inequality and its applications*, North. Math. J., 16:3 (2000), 279–286.
[40] YANG BICHENG, *On generalizations of Hardy-Hilbert's integral inequalities*, Acta Mathematica Sinica, 41:4 (1998), 839–844.
[41] YANG BICHENG, *On Hardy-Hilbert's integral inequality and its equivalent form*, North. Math. J., 19:2 (2003), 139–148.
[42] YANG BICHENG, *On a general Hardy-Hilbert's integral inequality with a best value*, Chinese Annals of Math., 21A:4 (2000), 401–408.
[43] YANG BICHENG, *On Hardy-Hilbert's integral inequality*, J. Math. Anal. Appl., 261 (2001), 295–306.
[44] WANG ZHUXI AND GUO DUNRIN, *An introduction to special functions*, Science Press, Beijing, 1979.
[45] YANG BICHENG, *On generalizations of Hardy-Hilbert's inequality and its equivalent form*, Journal of Mathematics, 23:4 (2003) (to appear).
[46] KUANG JICHANG, *On new extensions of Hilbert's integral inequality*, J. Math. Anal. Appl., 235 (1999), 608–614.
[47] HU KE, *On Hardy-Littlewood-Polya inequality*, Acta Mathematica Tcientia, 20 (2000), 684–687.
[48] YANG BICHENG, *On an extension of Hardy-Hilbert's inequality*, Chinese Annals of Math., 23A:2 (2002), 247–254.
[49] YANG BICHENG AND L. DEBNATH, *On a new generalization of Hardy-Hilbert's inequality and its applications*, J. Math. Anal. Appl., 233 (1999), 484–497.
[50] YANG BICHENG, *On a new extension of Hardy-Hilbert's inequality and its applications*, International Journal of Pure and Applied Mathematics, 5:1 (2003), 57–66.
[51] YANG BICHENG, *On a Multiple Hardy-Hilbert's inequality*, Chin. Annals. of Math., 24A:6 (2003) (to appear).
[52] HE LEPING, YU JIANGMING AND GAO MINGZHE, *An extension of Hilbert's integral inequality*, Journal of Shaoguan Univ. (Natural Science), 23:3 (2002), 25–30.

(Received June 3, 2002)

Bicheng Yang
Department of Mathematics
Guangdong Education College
Guangzhou, Guangdong 510303
People's Republic of China
e-mail: bcyang@pub.guangzhou.gd.cn

Themistocles M. Rassias
Department of Mathematics
National Technical University of Athens
Zografou Campus,15780
Athens, Greece
e-mail: trassias@math.ntua.gr