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A–STATISTICAL CONVERGENCE OF APPROXIMATING OPERATORS

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Abstract. In this paper we provide various approximation results concerning the classical Korovkin theorem via A-statistical convergence. We also study the rates of A-statistical convergence of approximating positive linear operators and give some examples.

1. Introduction

Approximation theory has important applications in the theory of polynomial approximation, in various areas of functional analysis, numerical solutions of differential and integral equations [1], [4], [12], [17]. Most of the classical approximation operators tend to converge to the value of the function being approximated. However, at points of discontinuity, they often converge to the average of the left and right limits of the function. There are, however, some sharp exceptions such as the interpolation operator of Hermite-Fejer (see [2]). These operators do not converge at points of simple discontinuity. For such a misbehavior, the matrix summability methods of Cesáro type are strong enough to correct the lack of convergence (see [3]). The Cesáro summability method also corrects Gibbs phenomenon of some non-positive approximation operators such as the partial sums of Fourier series (see [18], [22], [23]). In recent years another form of regular (non-matrix) summability transformation has shown to be quite effective in "summing" non-convergent sequences which may have unbounded subsequences [7], [8]. The aim of this paper is to investigate their use in approximation theory settings.

Let K be a subset of $\mathbb{N}$, the set of all natural numbers. The density of K is defined by $\delta(K) := \lim_n \frac{1}{n} \sum_{k=1}^{n} \chi_K(k)$ provided the limit exists, where χ_K is the characteristic function of K. A sequence $x := (x_k)$ is called statistically convergent to a number L if, for every $\varepsilon > 0$, $\delta\{k \in \mathbb{N} : |x_k - L| \geqslant \varepsilon\} = 0$ [6] (also see [8]). Let $A := (a_{nk})$, $n, k = 1, 2, \ldots$, be an infinite summability matrix. For a given sequence $x := (x_k)$, the A-transform of x, denoted by $Ax := ((Ax)_n)$, is given by

$$(Ax)_n = \sum_{k=1}^{\infty} a_{nk} x_k,$$

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provided the series converges for each n. We say that A is regular if $\lim_n (Ax)_n = L$ whenever $\lim x = L$ [14]. Assume now that A is a non-negative regular summability matrix. A sequence $x = (x_k)$ is called A-statistically convergent to L provided that for every $\varepsilon > 0$,

$$\lim_n \sum_{k:|x_k - L| \geqslant \varepsilon} a_{nk} = 0.$$

We denote this limit by $st_A - \lim x = L$ [7] (also see [5], [11], [16], [21]).

Recently some Korovkin type approximation theorems have been studied via statistical convergence in [13]. In the following we will consider the analogs of the classical Korovkin theorem via A-statistical convergence by using an arbitrary interval of $\mathbb{R}$. Furthermore, we examine some A-statistical rates of convergence of positive linear operators.

2. A-Statistical convergence of positive linear operators

In this section, using A-statistical convergence, we prove a Korovkin type theorem on an arbitrary interval of $\mathbb{R}$. Throughout I will be an interval of $\mathbb{R}$, and $C(I)$ will be the linear space of all real-valued continuous functions on I. If we take $I = [a,b]$, then $C(I)$ is a Banach space with norm $\|f\|_{C[a,b]} := \sup_{x\in[a,b]} |f(x)|$, for $f \in C[a,b]$. Let g be a non-negative increasing function on $[0,\infty)$ with $g(0) = 1$, then the set $C_g(I)$ is given by

$$C_g(I) := \left\{ f \in C(I) : \lim_{\substack{|x|\to\infty \\ x\in I}} \frac{|f(x)|}{[g(|x|)]^c} = 0 \text{ for any } c > 0 \right\}.$$

When $I = [a,b]$, our notation $C_g(I)$ will stand for $C[a,b]$ with $g(x) \equiv 1$. It is clear that $C_g(I)$ is a linear space. We will be concerned with Korovkin type results dealing with the problem of approximating a function f by a sequence $\{L_n(f,x)\}$ of positive linear operators over $C_g(I)$ (see, for instance, [15]). With this terminology the classical Korovkin theorem shows that:

"Let $\{L_n\}$ be a sequence of positive linear operators from $C[a,b]$ into $C[a,b]$. Then the following two statements are equivalent:

(i) $\lim_n \|L_n(f,x) - f(x)\|_{C[a,b]} = 0$ for all $f \in C[a,b]$.

(ii) $\lim_n \|L_n(f_i,x) - f_i(x)\|_{C[a,b]} = 0$ for $i = 0,1,2$ where $f_0(y) = 1$, $f_1(y) = y$ and $f_2(y) = y^2$."

If I is an arbitrary interval of $\mathbb{R}$ and if we apply A-statistical limit operator instead of limit operator in the Korovkin theorem, then we may conclude the following analogs.

THEOREM 1. *Let I be an arbitrary interval of $\mathbb{R}$. For an $x \in I$, let $\{\mu_{n,x} : n \geqslant 1\}$ be a collection of measures defined on $(I,\mathcal{B})$, where $\mathcal{B}$ is the sigma field of Borel measurable subsets of I. Let g be a function such that $f_2(y) = y^2$ is in $C_g(I)$ and for any $\delta > 0$, $\sup_{n\in\mathbb{N}} \int_{I\setminus I_\delta} g(|y|)d\mu_{n,x}(y) < \infty$, where $I_\delta := [x-\delta, x+\delta] \cap I$. Let $\{L_n\}$ be* *defined by*

$$L_n(f,x) = \int_I f(y)d\mu_{n,x}(y),\ n \in \mathbb{N} \ \text{and}\ f \in C_g(I).$$

Let $A=(a_{nk})$ be a non-negative regular summability matrix. Then the following two statements are equivalent:

(i) $st_A - \lim_n |L_n(f,x) - f(x)| = 0$ *for all* $f \in C_g(I)$.

(ii) $st_A - \lim_n |L_n(f_i,x) - f_i(x)| = 0$ *for* $i = 0,1,2$ *where* $f_0(y) = 1$, $f_1(y) = y$ *and* $f_2(y) = y^2$.

Proof. The implication $(i) \Rightarrow (ii)$ is clear. So we will prove the implication $(ii) \Rightarrow (i)$. Let $f \in C_g(I)$ and fix $x \in I$. Since f is continuous on I, for every $\varepsilon > 0$ there exists a real number $\delta > 0$ such that $|f(y) - f(x)| < \varepsilon$ for y satisfying $|x-y| \leqslant \delta$. Now we can write

$$|f(y)-f(x)| = |f(y)-f(x)|\,\chi_{I_\delta}(y) + |f(y)-f(x)|\,\chi_{I\setminus I_\delta}(y).$$

Using this, positivity and linearity of the operator, we get

$$\begin{aligned}
|L_n(f,x)-f(x)| &= |L_n(f-f(x)f_0,x) - f(x)(L_n(f_0,x)-f_0(x))| \\
&\leqslant L_n(|f-f(x)f_0|,x) + |f(x)|\,|L_n(f_0,x)-f_0(x)| \\
&= \int_{I_\delta} |f(y)-f(x)|\,d\mu_{n,x}(y) + \int_{I\setminus I_\delta} |f(y)-f(x)|\,d\mu_{n,x}(y) \\
&\quad + |f(x)|\,|L_n(f_0,x)-f_0(x)| \\
&\leqslant \varepsilon \int_I d\mu_{n,x}(y) + |f(x)|\,|L_n(f_0,x)-f_0(x)| \\
&\quad + \int_{I\setminus I_\delta} |f(y)-f(x)|\,d\mu_{n,x}(y) \\
&= \varepsilon L_n(f_0,x) - \varepsilon f_0(x) + \varepsilon f_0(x) + |f(x)|\,|L_n(f_0,x)-f_0(x)| \\
&\quad + \int_{I\setminus I_\delta} |f(y)-f(x)|\,d\mu_{n,x}(y).
\end{aligned}$$

So we have the inequality

$$\begin{aligned}
|L_n(f,x)-f(x)| &\leqslant \varepsilon + (\varepsilon + |f(x)|)\,|L_n(f_0,x)-f_0(x)| \\
&\quad + \int_{I\setminus I_\delta} |f(y)-f(x)|\,d\mu_{n,x}(y). \qquad (1)
\end{aligned}$$

It follows from the Cauchy-Bunyakowsky-Schwarz inequality that

$$\begin{aligned}\int\limits_{I\setminus I_\delta} |f(y)-f(x)|\,d\mu_{n,x}(y) &= \int\limits_{I\setminus I_\delta} \chi_{I\setminus I_\delta}(y)\,|f(y)-f(x)|\,d\mu_{n,x}(y)\\ &\leqslant \left[\int\limits_{I} \chi_{I\setminus I_\delta}(y)d\mu_{n,x}(y)\right]^{\frac{1}{p}} \left[\int\limits_{I\setminus I_\delta} |f(y)-f(x)|^q\,d\mu_{n,x}(y)\right]^{\frac{1}{q}}\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $p>1$. By hypothesis and the definition of the function g we conclude that $f\in C_g(I)$ implies $f^q\in C_g(I)$ and also that there exists a number K such that

$$\left[\int\limits_{I\setminus I_\delta} |f(y)-f(x)|^q\,d\mu_{n,x}(y)\right]^{\frac{1}{q}} < K. \tag{2}$$

Now define the function $\varphi: I\to\mathbb{R}$ by $\varphi(y)=\frac{(y-x)^2}{\delta^2}$, so we have

$$\begin{aligned}\mu_{n,x}(I\backslash I_\delta) &= \int\limits_I \chi_{I\setminus I_\delta}(y)d\mu_{n,x}(y)\\ &\leqslant \int\limits_I \varphi(y)d\mu_{n,x}(y)\\ &= \frac{1}{\delta^2}\left[(L_n(f_2,x)-f_2(x))-2x(L_n(f_1,x)-f_1(x))+x^2(L_n(f_0,x)-f_0(x))\right].\end{aligned}$$

Using the inequality $|x+y|^\alpha\leqslant|x|^\alpha+|y|^\alpha$ for each $\alpha\in(0,1]$, we have

$$\begin{aligned}[\mu_{n,x}(I\setminus I_\delta)]^{\frac{1}{p}} &\leqslant \frac{1}{\delta^{\frac{2}{p}}}\left\{x^{\frac{2}{p}}\,|L_n(f_0,x)-f_0(x)|^{\frac{1}{p}}+2^{\frac{1}{p}}\,|x|^{\frac{1}{p}}\,|L_n(f_1,x)-f_1(x)|^{\frac{1}{p}}\right.\\ &\quad\left.+\,|L_n(f_2,x)-f_2(x)|^{\frac{1}{p}}\right\}.\end{aligned} \tag{3}$$

Combining (1) with (2) and (3) we conclude that

$$\begin{aligned}|L_n(f,x)-f(x)| &\leqslant \varepsilon+(\varepsilon+|f(x)|)\,|L_n(f_0,x)-f_0(x)|+\frac{K}{\delta^{\frac{2}{p}}}\left\{x^{\frac{2}{p}}\,|L_n(f_0,x)-f_0(x)|^{\frac{1}{p}}\right.\\ &\quad\left.+2^{\frac{1}{p}}\,|x|^{\frac{1}{p}}\,|L_n(f_1,x)-f_1(x)|^{\frac{1}{p}}+|L_n(f_2,x)-f_2(x)|^{\frac{1}{p}}\right\}.\end{aligned}$$

Taking $B(x):=\max\left\{\varepsilon+|f(x)|,\frac{K}{\delta^{\frac{2}{p}}},\,K\left(\frac{|x|}{\delta}\right)^{\frac{2}{p}},\,K\left(\frac{2|x|}{\delta^2}\right)^{\frac{1}{p}}\right\}$, we have

$$\begin{aligned}|L_n(f,x)-f(x)|\leqslant\varepsilon+B(x)&\left\{|L_n(f_0,x)-f_0(x)|+|L_n(f_0,x)-f_0(x)|^{\frac{1}{p}}\right.\\ &\left.+\,|L_n(f_1,x)-f_1(x)|^{\frac{1}{p}}+|L_n(f_2,x)-f_2(x)|^{\frac{1}{p}}\right\}\end{aligned} \tag{4}$$

for each $n\in\mathbb{N}$. Given $r>0$, choose $\varepsilon>0$ such that $\varepsilon<r$. Define

$$\begin{aligned} D &:= \Big\{n : |L_n(f_0,x)-f_0(x)| + |L_n(f_0,x)-f_0(x)|^{\frac{1}{p}} + |L_n(f_1,x)-f_1(x)|^{\frac{1}{p}} \\ &\quad + |L_n(f_2,x)-f_2(x)|^{\frac{1}{p}} \geqslant \frac{r-\varepsilon}{B(x)}\Big\}, \\ D_1 &:= \Big\{n : |L_n(f_0,x)-f_0(x)| \geqslant \frac{r-\varepsilon}{4B(x)}\Big\}, \\ D_2 &:= \Big\{n : |L_n(f_0,x)-f_0(x)|^{\frac{1}{p}} \geqslant \frac{r-\varepsilon}{4B(x)}\Big\}, \\ D_3 &:= \Big\{n : |L_n(f_1,x)-f_1(x)|^{\frac{1}{p}} \geqslant \frac{r-\varepsilon}{4B(x)}\Big\}, \\ D_4 &:= \Big\{n : |L_n(f_2,x)-f_2(x)|^{\frac{1}{p}} \geqslant \frac{r-\varepsilon}{4B(x)}\Big\}. \end{aligned}$$

Then it is easy to see that $D \subset D_1 \cup D_2 \cup D_3 \cup D_4$. Thus (4) yields that

$$\sum_{k:|L_k(f,x)-f(x)|\geqslant r} a_{nk} \leqslant \sum_{k\in D} a_{nk} \leqslant \sum_{k\in D_1} a_{nk} + \sum_{k\in D_2} a_{nk} + \sum_{k\in D_3} a_{nk} + \sum_{k\in D_4} a_{nk}$$

and taking limit as $n \to \infty$ part (i) follows. □

An argument similar to the above proof leads to the following

COROLLARY 2. *If I is a closed and bounded interval, then the following statements are equivalent:*

(i) $st_A - \lim_n \|L_n(f,x) - f(x)\|_{C[a,b]} = 0$ *for all* $f \in C[a,b]$.

(ii) $st_A - \lim_n \|L_n(f_i,x) - f_i(x)\|_{C[a,b]} = 0$ *for* $f_i(x) = x^i$, $i = 0,1,2$.

Furthermore, when A is replaced by the identity matrix one obtains the classical Korovkin theorem.

3. Rates of A-statistical convergence

In the classical summability settings rates of summation have been introduced in several ways (see for instance, [9], [19], [20]). The concept of statistical rates of convergence, for nonvanishing two null sequences, is studied in [10]. In this section we introduce various ways of defining rates of convergence in the A-statistical sense. Unfortunately no single definition seems to have become the “standard” for the comparison of rates of summability transforms. The situation becomes even more uncharted when one considers rates of A-statistical convergence. In this section our aim is to propose several different ways one may define rates of A-statistical transforms.

DEFINITION 1. Let $A = (a_{nk})$ be a non-negative regular summability matrix and let (a_n) be a positive non-increasing sequence. We say that the sequence $x = (x_k)$ is A-statistically convergent to the number L with the rate of $o(a_n)$ if for every $\varepsilon > 0$,

$$\lim_n \frac{1}{a_n} \sum_{k:|x_k-L|\geqslant\varepsilon} a_{nk} = 0.$$

In this case we write

$$x_k - L = st_A - o(a_k), \quad (\text{as } k \to \infty).$$

DEFINITION 2. Let the matrix $A = (a_{nk})$ and the sequence (a_n) be the same as in Definition 1. We say that the sequence (x_k) is A-statistically bounded with the rate of $O(a_n)$ if for every $\varepsilon > 0$,

$$\sup_n \frac{1}{a_n} \sum_{k:|x_k| \geqslant \varepsilon} a_{nk} < \infty.$$

In this case we write

$$x_k = st_A - O(a_k), \quad (\text{as } k \to \infty).$$

In the above two definitions the "rate" is more controlled by the entries of the summability method rather than the terms of the sequence (x_k). For instance, when one takes the identity matrix I, if $a_{nn} = o(a_n)$ then $x_k - L = st_A - o(a_k)$ for any convergent sequence $(x_k - L)$ regardless of how slowly it goes to zero. To avoid such an unfortunate situation one may borrow the concept of convergence in measure from measure theory to define the rate of convergence as follows.

DEFINITION 3. Let the matrix $A = (a_{nk})$ and the sequence (a_n) be the same as in Definition 1. We say that the sequence (x_k) is A-statistically convergent to L with the rate of $o_m(a_n)$ if for every $\varepsilon > 0$,

$$\lim_n \sum_{k:|x_k - L| \geqslant \varepsilon a_k} a_{nk} = 0.$$

In this case we write

$$x_k - L = st_A - o_m(a_k), \quad (\text{as } k \to \infty).$$

DEFINITION 4. Let the matrix $A = (a_{nk})$ and the sequence (a_n) be the same as in Definition 1. We say that the sequence (x_k) is A-statistically bounded with the rate of $O_m(a_n)$ if for there is a positive number M such that

$$\lim_n \sum_{k:|x_k| \geqslant M a_k} a_{nk} = 0.$$

In this case we write

$$x_k = st_A - O_m(a_k), \quad (\text{as } k \to \infty).$$

It is perhaps possible to define more variants of the above definitions. Instead of presenting an exhaustive list, we will now show that, as far as approximation theory is concerned, all four definitions lead to analogous results. For this purpose, first let us mention a few standard concepts from approximation theory. Let $f \in C(I)$. The modulus of continuity of f, denoted by $w(f, \delta)$, is defined to be

$$w(f, \delta) = \sup_{\substack{x,y \in I \\ |x-y| < \delta}} |f(x) - f(y)|.$$

The modulus of continuity of the function f in $C(I)$ gives the maximum oscillation of

f in any interval of length not exceeding $\delta > 0$. It is well known that a necessary and sufficient condition for a function $f \in C[a,b]$ is

$$\lim_{\delta\to 0} w(f,\delta) = w(f,0) = 0.$$

It is also well known that for any constants $c > 0$, $\delta > 0$,

$$w(f,c\delta) \leqslant (1+[c])w(f,\delta) \tag{5}$$

where $[c]$ is defined to be the greatest integer less than or equal to c.

We will need the following lemma.

LEMMA 3. *Let $x = (x_k)$ and $y = (y_k)$ be two sequences. Assume that $A = (a_{nk})$ is a non-negative regular summability matrix. Let (a_n) and (b_n) be positive non-increasing sequences. If for some real numbers L_1, L_2, we have $x_k - L_1 = st_A - o(a_k)$ and $y_k - L_2 = st_A - o(b_k)$ as $k \to \infty$, then the following hold:*

(i) $(x_k - L_1) \pm (y_k - L_2) = st_A - o(c_k)$,

(ii) $(x_k - L_1)(y_k - L_2) = st_A - o(c_k)$

where $c_n = \max\{a_n, b_n\}$. Similar conclusions hold with little "o" replaced by big "O".

Proof. (i) Using hypothesis we see that

$$\frac{1}{c_n}\sum_{k:|(x_k-L_1)\pm(y_k-L_2)|\geqslant\varepsilon} a_{nk} \leqslant \frac{1}{a_n}\sum_{k:|x_k-L_1|\geqslant\frac{\varepsilon}{2}} a_{nk} + \frac{1}{b_n}\sum_{k:|y_k-L_2|\geqslant\frac{\varepsilon}{2}} a_{nk}$$

from which the result follows immediately.

(ii) Apply the fact that if $\alpha\beta \geqslant \varepsilon$, then $\alpha \geqslant \frac{\sqrt{\varepsilon}}{2}$ or $\beta \geqslant \frac{\sqrt{\varepsilon}}{2}$ for any α, $\beta \geqslant 0$. $\square$

The above proof can easily be modified to prove the following analog.

LEMMA 4. *Let $x = (x_k)$ and $y = (y_k)$ be two sequences. Assume that $A = (a_{nk})$ is a non-negative regular summability matrix. Let (a_n) and (b_n) be positive non-increasing sequences. If for some real numbers L_1, L_2, we have $x_k - L_1 = st_A - o_m(a_k)$ and $y_k - L_2 = st_A - o_m(b_k)$ as $k \to \infty$, then the following hold:*

(i) $(x_k - L_1) \pm (y_k - L_2) = st_A - o_m(c_k)$, *where* $c_k = \max\{a_k, b_k\}$.

(ii) $(x_k - L_1)(y_k - L_2) = st_A - o_m(a_k b_k)$

Similar conclusions hold with little "o_m" replaced by big "O_m".

Now we will find the rates of A-statistical convergence of the sequence of positive linear operators in Theorem 1.

THEOREM 5. *Let I be an arbitrary interval of $\mathbb{R}$. For an $x \in I$, let $\{\mu_{n,x} : n \geqslant 1\}$ be a collection of measures defined on $(I,\mathcal{B})$. Let g be a function such that $f_2(y) = y^2$ is in $C_g(I)$ and for any $\delta > 0$, $\sup_{n\in\mathbb{N}} \int_{I\setminus I_\delta} g(|y|)d\mu_{n,x}(y) < \infty$, where $I_\delta := [x-\delta, x+\delta]\cap I$. Let $\{L_n\}$ be defined by*

$$L_n(f,x) = \int_I f(y)d\mu_{n,x}(y),\ n\in\mathbb{N} \quad \text{and} \quad f \in C_g(I).$$

Suppose that $A=(a_{nk})$ *is a non-negative regular summability matrix and assume the operators* L_n *satisfy the conditions*

(i) $L_n(f_0,x)-f_0(x)=st_A-o(a_n(x))$ *with* $f_0(y)=1$,

(ii) $w(f,\alpha_n(x))=st_A-o(b_n(x))$ *with* $\alpha_n(x)=\sqrt{L_n(\varphi_x,x)}$, *and* $\varphi_x(y)=(y-x)^2$,

where $(a_n(x))$ *and* $(b_n(x))$ *are positive non-increasing sequences. Then*

$$L_n(f,x)-f(x)=st_A-o(c_n(x)),\ \text{as}\quad n\to\infty,$$

where $c_n(x)=\max\{a_n(x),b_n(x)\}$. *Similar results hold when little "o" is replaced by big "O".*

Proof. Using the definition of $L_n(f,x)$ and applying (5), for any $\delta>0$ we have

$$\begin{aligned}
|L_n(f,x)-f(x)| &= \left|\int_I f(y)d\mu_{n,x}(y)-f(x)\right| \\
&\leqslant \int_I |f(y)-f(x)|\,d\mu_{n,x}(y)+|f(x)|\,|L_n(f_0,x)-f_0(x)| \\
&\leqslant \int_I w\left(f,\delta\frac{|y-x|}{\delta}\right)d\mu_{n,x}(y)+|f(x)|\,|L_n(f_0,x)-f_0(x)| \\
&\leqslant \int_I \left(1+\left[\frac{|y-x|}{\delta}\right]\right)w(f,\delta)d\mu_{n,x}(y)+|f(x)|\,|L_n(f_0,x)-f_0(x)|.
\end{aligned}$$

Therefore

$$\begin{aligned}
|L_n(f,x)-f(x)| &\leqslant w(f,\delta)\int_I\left(1+\frac{\varphi_x(y)}{\delta^2}\right)d\mu_{n,x}(y)+|f(x)|\,|L_n(f_0,x)-f_0(x)| \\
&\leqslant w(f,\delta)\left\{L_n(f_0,x)+\frac{L_n(\varphi_x,x)}{\delta^2}\right\}+|f(x)|\,|L_n(f_0,x)-f_0(x)|.
\end{aligned}$$

Put $\delta=\alpha_n(x):=\sqrt{L_n(\varphi_x,x)}$, so we get

$$\begin{aligned}
|L_n(f,x)-f(x)| &\leqslant w(f,\alpha_n(x))\{1+L_n(f_0,x)\}+|f(x)|\,|L_n(f_0,x)-f_0(x)| \\
&\leqslant 2w(f,\alpha_n(x))+w(f,\alpha_n(x))\,|L_n(f_0,x)-f_0(x)| \\
&\quad +|f(x)|\,|L_n(f_0,x)-f_0(x)|. \qquad (6)
\end{aligned}$$

Given $\varepsilon>0$, by (6) we have

$$\begin{aligned}
&\frac{1}{c_n(x)}\sum_{k:|L_k(f,x)-f(x)|\geqslant\varepsilon}a_{nk} \\
&\qquad\leqslant \frac{1}{b_n(x)}\sum_{k:2w(f,\alpha_k(x))\geqslant\frac{\varepsilon}{3}}a_{nk}+\frac{1}{c_n(x)}\sum_{k:w(f,\alpha_k(x))|L_k(f_0,x)-f_0(x)|\geqslant\frac{\varepsilon}{3}}a_{nk} \\
&\qquad\quad +\frac{1}{a_n(x)}\sum_{k:|f(x)||L_k(f_0,x)-f_0(x)|\geqslant\frac{\varepsilon}{3}}a_{nk}.
\end{aligned}$$

Now conditions (i), (ii) and Lemma 3 yield the proof. □

The following analog also holds.

THEOREM 6. *Let I be an arbitrary interval of $\mathbb{R}$. For an $x \in I$, let $\{\mu_{n,x} : n \geqslant 1\}$ be a collection of measures defined on $(I, \mathcal{B})$. Let g be a function such that $f_2(y) = y^2$ is in $C_g(I)$ and for any $\delta > 0$, $\sup_{n\in\mathbb{N}} \int_{I\setminus I_\delta} g(|y|)d\mu_{n,x}(y) < \infty$, where $I_\delta := [x-\delta, x+\delta]\cap I$. Let $\{L_n\}$ be defined by*

$$L_n(f,x) = \int_I f(y)d\mu_{n,x}(y), \ n \in \mathbb{N} \quad \text{and} \quad f \in C_g(I).$$

Suppose that $A = (a_{nk})$ is a non-negative regular summability matrix and assume the operators L_n satisfy the conditions

(i) $L_n(f_0,x) - f_0(x) = st_A - o_m(a_n(x))$ with $f_0(y) = 1$,

(ii) $w(f, \alpha_n(x)) = st_A - o_m(b_n(x))$ with $\alpha_n(x) = \sqrt{L_n(\varphi_x, x)}$, and $\varphi_x(y) = (y-x)^2$,

where $(a_n(x))$ and $(b_n(x))$ are positive non-increasing sequences. Then

$$L_n(f,x) - f(x) = st_A - o_m(c_n(x)), \quad \text{as} \quad n \to \infty,$$

where $c_n(x) = \max\{a_n(x), b_n(x), a_n(x)b_n(x)\}$. Similar results hold when little "o_m" is replaced by big "O_m".

4. Concluding remarks

In this section, we first explain what type of positive linear approximation operators may have the property that they are not convergent to the function, and yet they are A-statistically convergent. Then we discuss A-statistical rates for some classical approximation operators.

Let $P_n(f,x)$ be a positive linear operator which can be decomposed as follows

$$P_n(f,x) = A_n(f,x) + u_n(x)\,B_n(f,x),$$

where $A_n(f,x)$ is convergent to $f(x)$ pointwise (or norm-wise), and $B_n(f,x)$ is bounded in n pointwise (or norm-wise) and $u_n(x)$ is pointwise (or norm-wise) A-statistically null sequence. Then $P_n(f,x)$ will be A-statistically convergent pointwise (or norm-wise). To show how one can easily construct such operators, first note that if $A = (a_{nk})$ is a non-negative regular matrix such that $\lim_n \max_k |a_{nk}| = 0$, then A-statistical convergence is stronger than convergence [16]. Assume now that (u_n) is an A-statistically null sequence but not convergent. Without loss of generality we may assume that (u_n) is a non-negative; otherwise we would replace (u_n) by $(|u_n|)$. Now define (P_n) on $C[a,b]$ by $P_n(f,x) = (1+u_n)L_n(f,x)$ where (L_n) is the sequence of positive linear operators in the Korovkin theorem. Now observe that (L_n) being convergent and (u_n) being A-statistically null, their product will also be A-statistically null. Hence (P_n) will not be convergent to f but will be A-statistically convergent to f.

Now we provide an example showing the A-statistical rate of convergence of some classical approximation operators. Consider a sequence of positive linear operators $L_n(f,x)$ over $I=[a,b]$ obeying the conditions of Korovkin's theorem such as the classical positive linear approximation operators among which the Bernstein polynomials operator is a typical prototype. In this case we get the following A-statistical rate.

COROLLARY 7. *If a sequence of operators (L_n) obeys the conditions of Korovkin's theorem then we have*

$$\|L_n(f,x)-f(x)\|_{C[a,b]} = st_A - O\left(\frac{1}{n}\right) \quad \text{for all} \quad f \in C[a,b]$$

where A is the Cesáro summability method.

Proof. The result follows from the fact that $L_n(f)$ converges to f implying that for any $\varepsilon > 0$, the inequality $\|L_n(f)-f\|_{C[a,b]} > \varepsilon$ takes place for at most finitely many values of n. □

Again we may obtain analogs of the above result if we use the other definitions of rates of convergence. Similar results could be stated in terms of pointwise limits of some positive linear operators that are defined over $C_g(I)$ when I is an unbounded interval. We omit the straight forward details.

In the end we should remark that our primary focus dealt with positive linear operators. However A-statistical convergence concept could be used for nonpositive approximation operators such as the Dirichlet operator for 2π-periodic functions. It is a well known fact that the Fejer kernel, being the Cesaro summability transform of the Dirichlet kernel, is able to correct the lack of convergence behavior of the Dirichlet operator. It is still an open question as to how useful is the A-statistical summability concept in correcting the lack of convergence of such nonpositive operators.

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