Mathematical
Inequalities
& Applications
Volume 7, Number 2 (2004), 235–243

CONFORMAL BOUNDARY AND ALMOST OPEN BALLS

TIMO TOSSAVAINEN

(*communicated by C. Bandle*)

Abstract. We establish a generalization to an inequality that can be used to measure how badly the intersection of an open ball of the euclidean space with the conformal boundary, i.e. the metric boundary of a conformal deformation of the unit ball $\mathbb{B}^n$, fails to be open in the euclidean sense. As an application to this, we show among other things that every point of a conformal boundary can be approached along a euclidean geodesical arc in a bounded set with respect to the intrinsic metric of conformal boundary.

1. Introduction

The theories of conformal and quasiconformal mappings of subdomains of $\mathbb{R}^n$ constitute a remarkable part of the modern geometric function theory. In recent years, these theories have been generalized in many ways, and one of the most important new aspects to this matter is to study a conformal mapping f of the unit disk via thinking $|f'(z)|$ as a given density ρ on $\mathbb{B}^2$, and then consider the properties of f as properties on ρ. It is shown in [2] that a large part of the properties of f follow in this setting from two simple geometric conditions on ρ, see $HI(A)$ and $VG(B)$ below.

Also the quasiconformal mapping f of $\mathbb{B}^n$, $n \geqslant 2$ can be studied similarly using only $HI(A)$ and $VG(B)$, when the density is taken as the avarage of the Jacobian of f, see below. Moreover, the class of function satisfying the mentioned geometric conditions cover another types of functions, so in [2] we are given a remarkable generalization and unification of the theories of conformal and quasiconformal mappings.

If the given density ρ on $\mathbb{B}^n$, $n \geqslant 2$ satiesfies the mentioned conditions, it induces a metric d_ρ in $\mathbb{B}^n$ that extends also to the boundary $\partial\mathbb{B}^n$ of the unit ball, see below. The set of the boundary points that are at the finite distance from 0 with respect to d_ρ, builds up the conformal boundary $\partial_\rho\mathbb{B}^n$, or equivalently the ρ-boundary of unit ball, that is a metric space. This space correspondences to, for example, when ρ arises from a quasiconformal mapping f of $\mathbb{B}^n$, the boundary of the image $f(\mathbb{B}^n)$ of the unit ball, and is thus worth investigation, indeed.

The study of $\partial_\rho\mathbb{B}^n$ was initiated in [2], and then continued in [1], [7] and [5]. In these papers the emphasis was on the dimensional and connectivity properties of $\partial_\rho\mathbb{B}^n$ with respect to the intrinsic metric d_ρ. In this note, we concentrate primarily on what

Mathematics subject classification (2000): 30C65.

Key words and phrases: Conformal boundary, density estimate, Hausdorff content, quasiconformal mapping.

$\partial_\rho \mathbb{B}^n$ looks like locally in the euclidean sense. However, before stating our results we give the formal definition of $\partial_\rho \mathbb{B}^n$.

From now on let $n \geqslant 2$ if not stated otherwise. We consider a continuous function $\rho : \mathbb{B}^n \to (0, \infty)$ satisfying for some constants $A \geqslant 1$ and $B > 0$,

$$1/A \leqslant \frac{\rho(x)}{\rho(y)} \leqslant A \quad \text{whenever } x, y \in B(z, \frac{1}{2}(1-|z|)) \text{ for any } z \in \mathbb{B}^n, \qquad HI(A)$$

where $B(z, \frac{1}{2}(1-|z|))$ is a (Whitney) ball centred at z with radius $\frac{1}{2}(1-|z|)$, and

$$\mu_\rho(B_\rho(x,r)) \leqslant Br^n \quad \text{for all } x \in \mathbb{B}^n \text{ and } r > 0. \qquad VG(B)$$

Here $\mu_\rho(E) = \int_E \rho^n dm_n$, in which m_n is n-dimensional Lebesgue measure, and $B_\rho(x,r)$ is a ball centred at x with radius r with respect to the metric d_ρ, defined for $x, y \in \mathbb{B}^n$ by the formula

$$d_\rho(x,y) = \inf_\gamma \int_\gamma \rho ds,$$

where the infimum is taken over all curves in $\mathbb{B}^n$ with endpoints x and y. We will call a function ρ satisfying both $HI(A)$ and $VG(B)$ for some constants A and B *a conformal density*. The canonical example of a conformal density is given by $\rho(z) = |f'(z)|$ where f is a conformal mapping of the unit disk in the plane. For $n \geqslant 2$, also the avarage $a_f(x) = \left(\frac{1}{m_n(B_x)} \int_{B_x} J_f \, dm_n\right)^{1/n}$ of the Jacobian of a quasiconformal mapping $f : \mathbb{B}^n \to \mathbb{R}^n$ is a conformal density. Notice that $HI(A)$ requires that the density ρ satisfies a Harnack inequality and $VG(B)$ that the volume growth is at most euclidean.

The *ρ-boundary of the unit ball*, $\partial_\rho \mathbb{B}^n$, is defined as $\overline{(\mathbb{B}^n, d_\rho)} \setminus (\mathbb{B}^n, d_\rho)$. Here we take the abstract completion of the metric space $(\mathbb{B}^n, d_\rho)$. The metric d_ρ then extends in a natural way to this boundary; cf. Section 6 in [2]. Seen from another point of view, by the results in [2], $\partial_\rho \mathbb{B}^n$ can be identified with the set of those points $\varsigma \in \partial \mathbb{B}^n$ for which the integral of ρ along the ray $[0, \varsigma) = \{t\varsigma : 0 \leqslant t < 1\}$ is finite.

Next let us introduce more of our notation. Given $\varsigma \in \partial_\rho \mathbb{B}^n$, $\lambda \in (0,1)$, and $r \in (0,1]$ we write

$$\text{Cone}(\varsigma, \lambda, r) = \bigcup \{B\big(t\varsigma, \lambda(1-t)\big) : 1-r \leqslant t < 1\}. \tag{1.1}$$

By *the Hausdorff content* H_α^∞ (of a set D) we mean the number

$$H_\alpha^\infty(D) = \inf\{\sum_{k=1}^\infty r_k^\alpha : D \subset \bigcup_{k=1}^\infty B(x_k, r_k)\}.$$

Notice that H_α^∞ is an outer measure in $\mathbb{R}^n$; cf. [8].

The main result in this paper is the following theorem. It is a generalization of the density estimate that originally appears as Theorem 3.1 in [7]. That estimate could cover only radii $r \leqslant 1$.

THEOREM 1.2. *Let $0 < r \leqslant \frac{3}{2}$ and $\varsigma \in \partial_\rho \mathbb{B}^n$. Choose $0 < \alpha \leqslant n-1$ and set $\delta = diam_\rho(Cone(\varsigma, \frac{4}{5}, \min\{1, r\}))$ and $A_k(\varsigma) = B(\varsigma, r/2^k) \backslash B(\varsigma, r/2^{k+1})$. Then there* *exists a constant* $C = C(A,B,\alpha,n) > 0$ *such that*

$$\sum_{k=0}^{\infty} \frac{H_\alpha^\infty\big(\partial\mathbb{B}^n \cap A_k(\varsigma)\backslash B_\rho(\varsigma,\hat{r})\big)}{2^{-k\alpha}} \leqslant \frac{Cr^\alpha}{\Big(\log(1+\frac{\hat{r}}{2\delta})\Big)^{n-1}}$$

for $\hat{r} \geqslant 2\delta$.

It is relatively easy to construct ρ-boundaries that are not open subsets of $\partial\mathbb{B}^n$; cf. Section 2 in [7]. Thus the relative topology of (the euclidean topology of) $\partial\mathbb{B}^n$ in $\partial_\rho\mathbb{B}^n$ is not euclidean in general. In that sense, Theorem 1.2 reveals us many interesting things. Firstly, choosing $r \geqslant \sqrt{2}$ and a pair of antipodal points $\pm\varsigma \in \partial_\rho\mathbb{B}^n$, and then letting $\hat{r} \to \infty$, we get another proof for the fact that $\partial\mathbb{B}^n\backslash\partial_\rho\mathbb{B}^n = D$ where D has α-dimensional Hausdorff measure zero for every $\alpha > 0$. Moreover, Theorem 1.2 gives us a concrete estimate on the speed how fast $B_\rho(\varsigma,\hat{r})$ covers 'the most of' $\partial\mathbb{B}^n \cap B(\varsigma,r)$ as $\hat{r} \to \infty$. The conclusion is that $B(\varsigma,r) \cap \partial_\rho\mathbb{B}^n$ are almost open subsets of $\partial\mathbb{B}^n$ in the euclidean sense.

Using, among other things, the facts how the H_α^∞-measure of a set varies in Lipschitz continuous mappings we also prove the following consequence of Theorem 1.2 that trivially fails in the case $n = 2$.

THEOREM 1.3. *Let* $n \geqslant 3$, *and assume that* ρ *is a conformal density on* $\mathbb{B}^n$. *Then, for any* $\varsigma, \vartheta \in \partial_\rho\mathbb{B}^n$, *there is a continuous mapping* $\gamma : ([0,1],|\cdot|) \to (\partial_\rho\mathbb{B}^n,|\cdot|)$ *such that* $\gamma(0) = \varsigma$, $\gamma(1) = \vartheta$, *and length*$(\gamma) \leqslant 2|\varsigma - \vartheta|$. *Moreover, there is a finite constant* $L = L(A,B,n,\varsigma,\vartheta) > 0$ *such that* $d_\rho(0,\gamma(t)) \leqslant L$ *for every* $t \in [0,1]$.

The first part of this theorem may be proved using the above mentioned fact that $\partial\mathbb{B}^n\backslash\partial_\rho\mathbb{B}^n = D$, where D is a Borel set of Hausdorff dimension zero, but because of the latter assertion of the theorem, we prefer to give another proof that is also based on Theorem 1.2.

Finally, we show that Theorems 1.2 and 1.3 allow us to conclude the following geometrical results.

COROLLARY 1.4. *Let* $n \geqslant 3$. *Every* $\varsigma \in \partial_\rho\mathbb{B}^n$ *can be approached in* $\partial_\rho\mathbb{B}^n$ *along a geodesic arc of* $\partial\mathbb{B}^n$ *that is bounded in the sense of* d_ρ. *Moreover, there is a sequence* (r_j) *with* $r_j \to 0$ *as* $j \to \infty$ *such that* $S_j \subset \partial_\rho\mathbb{B}^n$ *for every* j, *where* $S_j = \{x \in \partial\mathbb{B}^n : |x - \varsigma] = r_j\}$, *and*

$$d_\rho(\varsigma,S_j) \to 0 \text{ as } j \to \infty. \tag{1.5}$$

2. Proof of Theorem 1.2

We start by noticing that in $HI(A)$ the constant $1/2$ can be replaced with any fixed $\lambda \in (\frac{1}{2},1)$ so that the analogous bounds for $\rho(x)/\rho(y)$ hold with some constant A' that depends only on A and n. Namely, every point of a thicker cone can be reached with a finite chain of intersecting Whitney balls from a cone such as defined in (1.1). Thus there exists a constant $C = C(A,n) > 0$ such that

$$\mathrm{diam}_\rho(\mathrm{Cone}(\varsigma, \frac{4}{5}, \epsilon)) \leqslant C\mathrm{diam}_\rho([(1-\epsilon)\varsigma, \varsigma)) \quad \text{whenever } \epsilon \in (0,1]. \tag{2.1}$$

Since $\int_{[0,\varsigma)} \rho ds < \infty$ for every $\varsigma \in \partial_\rho \mathbb{B}^n$, our δ is finite for every r. For more details, see Lemma 1.3 in [7].

Next recall the definition of modulus, which is an outer measure.

The modulus mod Γ of family Γ of curves in $\mathbb{B}^n$ is the number

$$\inf_{\tilde{\rho}} \int_{\mathbb{B}^n} \tilde{\rho}^n dm_n,$$

where the infimum is taken over all Borel measurable densities $\tilde{\rho} : \mathbb{B}^n \to [0,\infty]$ for which $\int_\gamma \tilde{\rho} ds \geqslant 1$ whenever $\gamma \in \Gamma$. For other properties of modulus, see for example [8].

In order to shorten our proof, we cover the easier part of it by refering to a lemma that is proved in [2].

LEMMA 2.2. *(Lemma 3.2 in [2]) There exists a constant $C(B,n) > 0$ with the following property. Let F be a nonempty subset of $\mathbb{B}^n$ and suppose $L \geqslant \delta > 0$. Assume that diam$_\rho(F) \leqslant \delta$ and that Γ is a family of curves in $\mathbb{B}^n$ so that γ has one endpoint in F and length$_\rho(\gamma) \geqslant L$ for every $\gamma \in \Gamma$. Then*

$$\mathit{mod}\,\Gamma \leqslant \frac{C}{(\log(1+L/\delta))^{n-1}}.$$

To prove Theorem 1.2, let Γ be the family of all rectifiable paths in $\mathbb{B}^n$ that connect $E = \partial\mathbb{B}^n \cap B(\varsigma, r) \setminus B_\rho(\varsigma, \hat{r})$ and $F = \mathrm{Cone}(\varsigma, \frac{4}{5}, \min\{1, r\})$ to one another. By inequality in (2.1) and Lemma 2.2, it follows that there is a constant $C = C(A,B,n) > 0$ such that

$$\mathrm{mod}\,\Gamma \leqslant \frac{C}{\Big(\log(1+\frac{\hat{r}}{2\delta})\Big)^{n-1}}. \tag{2.3}$$

To show that mod Γ has a lower bound, define Γ_k, $k = 0,1,2,\ldots$, to be the subfamily of all paths that connect the cone F to the intersection $E \cap A_k(\varsigma)$ in $\mathbb{B}^n \cap A_k(\varsigma)$. Then the families Γ_k are distinct and $\cup_k \Gamma_k \subset \Gamma$. It suffices to show that there exists a constant $C' = C'(n)$ such that

$$\frac{H^\infty_\alpha(E \cap A_k(\varsigma))}{2^{-k\alpha}} \leqslant C' \frac{r^\alpha}{\alpha^{n-1}} \,\mathrm{mod}\,\Gamma_k, \tag{2.4}$$

since summing over k and using the fact that the modulus is an outer measure, implies that

$$\sum_{k=0}^{\infty} \frac{H^\infty_\alpha(E \cap A_k(\varsigma))}{2^{-k\alpha}} \leqslant C' \frac{r^\alpha}{\alpha^{n-1}} \,\mathrm{mod}\,\Gamma \tag{2.5}$$

Finally, combining (2.3) and (2.5) leads us to the desired inequality.

To prove formula (2.4), let $\tilde{\rho}$ be a non-negative Borel function in $\mathbb{R}^n$ (for simplicity, assume that $\tilde{\rho} = 0$ in $\mathbb{R}^n \setminus \mathbb{B}^n$) such that

$$\int_\gamma \tilde{\rho} ds \geqslant 1 \quad \text{for all } \gamma \in \Gamma.$$

Let $\varphi \in E \cap A_k(\varsigma)$ and $0 < l < L < \infty$. For all $\omega \in \partial\mathbb{B}^n$, we have by Hölder's inequality that

$$\begin{aligned}(\int_l^L \tilde{\rho}(\varphi + t\omega)dt)^n &= (\int_l^L \tilde{\rho}(\varphi + t\omega)t^{\frac{n-1}{n}}t^{\frac{1-n}{n}}dt)^n \\ &\leqslant (\log\frac{L}{l})^{n-1}\int_l^L \tilde{\rho}(\varphi + t\omega)^n t^{n-1}dt.\end{aligned}$$

We shall show that if mod Γ_k had no lower bound, then there would be ω such that a ray from φ to direction ω hits F and has $\tilde{\rho}$-length less than one. This would be a contradiction with our assumption on Borel function $\tilde{\rho}$.

Therefore, for each $j = 1, 2, \dots$, write $l_j = \frac{r/2^k}{j^{(n+3)/\alpha}}$, and let $A_j(\varphi) = B(\varphi, l_j) \setminus B(\varphi, l_{j+1})$, and set $I_j(\omega) = (\int_{l_{j+1}}^{l_j} \tilde{\rho}(\varphi + t\omega)dt)^n$. Then by the inequality above and the fact that $\log(1+x) \leqslant x$ for $x > 0$, we have

$$\begin{aligned}\int_{\partial\mathbb{B}^n} I_j(\omega)dm_{n-1} &= \int_{\partial\mathbb{B}^n} \Big(\int_{l_{j+1}}^{l_j} \tilde{\rho}(\varphi + t\omega)\, dt\Big)^n dm_{n-1} \\ &\leqslant \int_{\partial\mathbb{B}^n} \Big(\log(1+1/j)^{\frac{n+3}{\alpha}}\Big)^{n-1} \int_{l_{j+1}}^{l_j} \tilde{\rho}(\varphi + t\omega)^n t^{n-1} dt\, dm_{n-1} \\ &\leqslant (\frac{n+3}{j\alpha})^{n-1} \int_{\partial\mathbb{B}^n}\int_{l_{j+1}}^{l_j} \tilde{\rho}(\varphi + t\omega)^n t^{n-1} dt\, dm_{n-1} \\ &= (\frac{n+3}{j\alpha})^{n-1}\int_{A_j(\varphi)} \tilde{\rho}(x)^n dx.\end{aligned}$$

Suppose now that for each j,

$$\int_{A_j(\varphi)} \tilde{\rho}(x)^n dx \leqslant C\frac{\alpha^{n-1}}{r^\alpha/2^{k\alpha}} l_j^\alpha, \tag{2.6}$$

where the constant $C = C(n)$ will be chosen later. Write $\mathrm{Bad}_j(s) = \{\omega \in \partial\mathbb{B}^n : I_j(\omega) \geqslant s\}$, and fix a constant $C' > 0$ such that $\sqrt[n]{C'} < 1/\sum_{j=1}^\infty \frac{1}{j^2}$.

Then by the calculation above and remembering that $l_j = \frac{r/2^k}{j^{(n+3)/\alpha}}$,

$$m_{n-1}(\mathrm{Bad}_j(s)) \leqslant \int_{\partial\mathbb{B}^n} \frac{I_j(\omega)}{s} dm_{n-1} \leqslant \frac{C}{s}\frac{(n+3)^{n-1}}{j^{n+3+n-1}},$$

where C is the constant in (2.6), and especially

$$\begin{aligned} m_{n-1}(\cup_j \mathrm{Bad}_j(C' j^{-2n})) &\leqslant \sum_j \frac{C}{C'} \frac{j^{2n}(n+3)^{n-1}}{j^{n+3+n-1}} \\ &\leqslant \frac{C}{C'}(n+3)^{n-1} \sum_j \frac{1}{j^2}. \end{aligned}$$

This can be made arbitrarily small by choosing C appropriately. Observe that the m_{n-1}-measure of the directions in which a ray from φ of the euclidean length $r/2^k$ hits F in $A_k(\varsigma)$, has always a strictly positive lower bound, since the constant $4/5$ is large enough in the definition of F. More precisely, for every k, the cone F is thick enough so that it can be seen in a strictly positive angle inside $A_k(\varsigma) \cap \mathbb{B}^n$ from every φ.

For each $\omega \in \partial \mathbb{B}^n$ not in $\bigcup_j \mathrm{Bad}_j(C' j^{-2n}))$ we have

$$\begin{aligned} \int_0^{\hat{r}/2^k} \tilde{\rho}(\varphi + t\omega) dt &= \sum_j I_j(\omega)^{1/n} \leqslant \sum_j (C' \frac{1}{j^{2n}})^{1/n} \\ &= (C')^{1/n} \sum_j \frac{1}{j^2} < 1. \end{aligned}$$

Recall what we have been doing. We assumed that the density $\tilde{\rho}$ satisfies $\int_\gamma \tilde{\rho} ds \geqslant 1$ over any path $\gamma \in \Gamma_k$. But the calculation above shows that, under the additional assumption on $\int_{A_j(\varphi)} \tilde{\rho}(x)^n dx$, we find paths in Γ_k that have $\tilde{\rho}$-length less than one. So, we conclude that there must be indices $j \in \mathbb{N}$ such that

$$\int_{A_j(\varphi)} \tilde{\rho}(x)^n dx > C \frac{\alpha^{n-1}}{r^\alpha / 2^{k\alpha}} l_j^{\,\alpha}.$$

By the Besicovitch covering theorem we may cover $E \cap A_k(\varsigma)$ with countably many balls $B(\varphi_m, l_{j_m})$ of the above type (we simply take the whole ball instead of the annulus $A_j(\varphi_m)$) and so that only a bounded number depending on n of them overlap at any point of $\mathbb{B}^n$. Then, by taking the infimum of Borel functions $\tilde{\rho}$ of appropriate type we get

$$H_\alpha^\infty(E \cap A_k(\varsigma)) \leqslant \sum_m l_{j_m}{}^\alpha \leqslant C' \frac{r^\alpha / 2^{k\alpha}}{\alpha^{n-1}} \mathrm{mod}\Gamma_k$$

as desired. Inequality (2.4) follows and the theorem is proven.

3. Applications of Theorem 1.2

Now let $n \geqslant 3$ and $\alpha = 1$. In order to verify other results given in Introduction, we establish first a few lemmas. To that end, recall that function $f : X \to Y$ is *Lipschitz continuous with constant* C if $|f(x) - f(y)| \leqslant C|x - y|$ for every $x, y \in X$.

LEMMA 3.1. *Let* $f:\mathbb{R}^n\to\mathbb{R}^n$ *be Lipschitz continuous function with constant* C. *Then for all* $D\subset\mathbb{R}^n$,

$$H_1^\infty(f(D))\leqslant CH_1^\infty(D).$$

Proof. Let $\{B(x_j,r_j)\}$ be a cover of D. Then $\{f(B(x_j,r_j))\}$ is a cover of $f(D)$ and

$$f(B(x_j,r_j))\subset B(f(x_j),\tilde r_j)\subset B(f(x_j),Cr_j),$$

where $\tilde r_j=\sup_{|x_j-x|\leqslant r_j}|f(x_j)-f(x)|$. Thus

$$\sum_j\tilde r_j\leqslant\sum_j(Cr_j)=C\sum_j r_j.$$

Taking the infimum over all covers of D leads us to the desired result. $\square$

Let $A_k(\varsigma)$, δ, r and ς be as in Theorem 1.2. Write

$$S(\varsigma,r)=\{x\in\partial\mathbb{B}^n:\ |\varsigma-x|=r\}.$$

Notice that the radial projection P_k along $\partial\mathbb{B}^n$ of $\partial\mathbb{B}^n\cap A_k(\varsigma)$ onto $S(\varsigma,r)$ is Lipschitz continuous with constant 2^{k+1}. Thus, by Lemma 3.1, we have the following corollary of Theorem 1.2.

Lemma 3.2. There exists a constant $C=C(A,B,n)>0$ such that

$$\sum_{k=0}^\infty H_1^\infty\Big(P_k(\partial\mathbb{B}^n\cap A_k(\varsigma)\backslash B_\rho(\varsigma,\hat r))\Big)\leqslant\frac{Cr}{\Big(\log(1+\frac{\hat r}{2\delta})\Big)^{n-1}}$$

for $\hat r\geqslant 2\delta$.

Now we are ready to prove Theorem 1.3. For given $\varsigma,\vartheta\in\partial_\rho\mathbb{B}^n$, choose $r=\frac{3}{4}|\varsigma-\vartheta|$. Then, by the symmetry,

$$H_1^\infty\big(S(\varsigma,r)\cap B(\vartheta,r)\big)=H_1^\infty\big(B(\varsigma,r)\cap S(\vartheta,r)\big)\geqslant Cr \tag{3.3}$$

for some constant $C>0$.

Next, choose $\hat r>0$ in Lemma 3.2 large enough to have both

$$\sum_{k=0}^\infty H_1^\infty\Big(P_k\big(\partial\mathbb{B}^n\cap A_k(\varsigma)\backslash B_\rho(\varsigma,\hat r)\big)\Big)<\frac{C}{3}r$$

and

$$\sum_{k=0}^\infty H_1^\infty\Big(P_k\big(\partial\mathbb{B}^n\cap A_k(\vartheta)\backslash B_\rho(\vartheta,\hat r)\big)\Big)<\frac{C}{3}r,$$

where C is the constant in (3.3). Then we conclude that there are $\sigma\in S(\varsigma,r)$ and $\theta\in S(\vartheta,r)$ such that circular arcs $[\varsigma,\sigma)$ and $[\vartheta,\theta)$ of $\partial\mathbb{B}^n$ intersect, and we have

$$[\varsigma,\sigma)\cup[\vartheta,\theta)\subset B_\rho(\varsigma,\hat r)\cup B_\rho(\vartheta,\hat r). \tag{3.4}$$

Now γ is obtained by gluing together suitable pieces of $[\varsigma,\sigma)$ and $[\vartheta,\theta)$. Since $\operatorname{diam}\big([\varsigma,\sigma)\cup[\vartheta,\theta)\big)\leqslant 2r$, the upper bound for the length (γ) follows from the definition of r and elementary geometry. The last assertion of the theorem follows from (3.4), the triangle inequality and the fact that both $d_\rho(0,\varsigma)$ and $d_\rho(0,\vartheta)$ are finite whenever $\varsigma,\vartheta\in\partial_\rho\mathbb{B}^n$.

The first claim of Corollary 1.4 follows trivially from the proof of Theorem 1.3. In order to prove the second claim and (1.5), we record yet another lemma.

LEMMA 3.5. *For every fixed $0<\beta<1$ and $\hat r\geqslant 2\delta$, there exists a constant $k_0=k_0(\beta,\hat r)$ such that, for all $k\geqslant k_0$,*

$$\frac{H_1^\infty\big(\partial\mathbb{B}^n\cap A_k(\varsigma)\backslash B_\rho(\varsigma,\hat r)\big)}{H_1^\infty(\partial\mathbb{B}^n\cap A_k(\varsigma))}<\beta.$$

Proof. If there are infinitely many $A_k(\varsigma)$'s such that

$$H_1^\infty\big(\partial\mathbb{B}^n\cap A_k(\varsigma)\backslash B_\rho(\varsigma,\hat r)\big)\geqslant\beta H_1^\infty(\partial\mathbb{B}^n\cap A_k(\varsigma))\geqslant\frac{Cr}{2^k},$$

where $C=C(n)>0$ is a constant, then for any $C'>0$,

$$\sum_{k=0}^{\infty}\frac{H_1^\infty\big(\partial\mathbb{B}^n\cap A_k(\varsigma)\backslash B_\rho(\varsigma,\hat r)\big)}{2^{-k}}>C'r,$$

which contradicts Theorem 1.2. □

To conclude the second claim of Corollary 1.4, notice that the spherical projection of $\partial\mathbb{B}^n\cap A_k(\varsigma)$ onto any fixed geodesical arc $[\varsigma,\sigma)$ of $\partial\mathbb{B}^n$, where $\sigma\in S(\varsigma,r)$, is Lipschitz continuous with constant 1. On the other hand, $H_1^\infty([\varsigma,\sigma)\cap A_k(\varsigma))\geqslant\frac{C''r}{2^k}$, where $C''=C''(n)>0$ is constant, for every k.

For fixed $\delta>0$ and $\hat r\geqslant 2\delta$, we find a series of S_j's in $B_\rho(\varsigma,\hat r)$ such that $d(\varsigma,S_j)\to 0$ as $j\to\infty$, by using Lemma 3.1 together with Lemma 3.5 and $\beta>0$ small enough. To find a series of S_j's such that also the claim in (1.5) is satisfied, we proceed as follows. Let $\hat r_j=2^{-j}$. Since $\int_{[0,\varsigma)}\rho ds<\infty$, and by (2.1), we can choose for every j, a strictly positive r_j such that

$$\delta_j=\operatorname{diam}_\rho(\operatorname{Cone}(\varsigma,\frac{4}{5},r_j))\leqslant\frac{1}{2}\hat r_j.$$

Thus for every j, we find a suitable S_j by repeating the above argument for δ_j and $\hat r_j$.

REFERENCES

[1] M. BONK AND P. KOSKELA, *Conformal metrics and the size of boundary*, American J. Math., **6** (2002) 1247–1287.

[2] M. BONK, P. KOSKELA AND S. ROHDE, *Conformal metrics on the unit ball in euclidean space*, Proc. London Math. Soc. (3) **77** (1998) 635–664.

[3] L. C. EVANS AND R. F. GARIEPY, *Measure Theory and Fine Properties of Functions* - CRC Press, Boca Raton New York London Tokyo, 1992.

[4] P. KOSKELA AND S. ROHDE, *Hausdorff dimension and mean porosity*, Math. Ann. (4) **309** (1998) 593–609.
[5] P. KOSKELA AND T. TOSSAVAINEN, *Pathwise connectivity of a conformal boundary*, Bull. London Math. Soc., **35** (2003), 645–650.
[6] C. A. ROGERS, *Hausdorff Measures - Cambridge University Press*, London, 1970.
[7] T. TOSSAVAINEN, *On the connectivity properties of the* ρ*-boundary of the unit ball*, Ann. Acad. Sci. Fenn. Math. Diss. **123** (2000).
[8] J. VÄISÄLÄ, *Lectures on* n*-dimensional Quasiconformal Mappings* - Lecture Notes in Math. 229, Springer-Verlag, 1971.

(Received June 4, 2003)

Timo Tossavainen
Department of Teacher Education
University of Joensuu
P.O. Box 55, 57101 Savonlinna
Finland
e-mail: timo.tossavainen@joensuu.fi