Mathematical
Inequalities
& Applications
Volume 7, Number 2 (2004), 245–253

MAXIMAL FUNCTION ON GENERALIZED LEBESGUE SPACES $L^{p(\cdot)}$

L. DIENING

(*communicated by L. Pick*)

Abstract. We prove the boundedness of the Hardy–Littlewood maximal function on the generalized Lebesgue space $L^{p(\cdot)}(\mathbb{R}^d)$ under a continuity assumption on p that is weaker than uniform Hölder continuity. We deduce continuity of mollifying sequences and density of $C^\infty(\overline{\Omega})$ in $W^{1,p(\cdot)}(\Omega)$.

1. Introduction

In the last years there has been an increasing interest in the study of complex materials, where the underlying energy cannot be expressed in terms of classical Lebesgue spaces. One example of such materials are electrorheological fluids. These are special fluids that undergo a significant change when disposed to an electric field. For fluids of this type the viscosity may vary by a factor of 1000. (See Růžička [13] for a model of electrorheological fluids.) In such a case the underlying energy is given by $\int |\mathbf{Du}^{p(x)}|\,dx$, where $\mathbf{Du}$ denotes the symmetric part of the gradient of the velocity field $\mathbf{u}$ and p is a material function which depends on the electric field. The same type of energy also appears in a model proposed by Zhikov [18] for another type of fluids, where the stress tensor depends on a distribution of temperature T. The right spaces for the underlying energy of such materials are the so called generalized Lebesgue spaces $L^{p(\cdot)}$ (also known as $L^{p(x)}$ spaces) which are a special case of generalized Orlicz spaces. These spaces and the generalized Sobolev spaces $W^{k,p(\cdot)}$ have been studied among others by Hudzik [7], Musiełak [11], Kováčik, Rákosník [8] and Růžička [13]. The above energies also appear in the investigations of variational integrals with non–standard growth (see e.g. Zhikov [17] and Marcellini [9]).

While many results for the classical Lebesgue Spaces (like separability, uniform convexity, and embeddings of type $L^{p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$) also hold in the generalized case, there are some results which fail to hold (translation is discontinuous on $L^{p(\cdot)}$, see also below). Unfortunately there are still a good deal of conjectured propositions, of which it is not known whether they are true or not (like continuity of singular integrals). Most of the results of this category do only exist in a slightly weaker form, that is, with an ϵ-defect. For example, for a long time it has only been known that $W^{1,p(\cdot)}(\Omega) \hookrightarrow$

Mathematics subject classification (2000): 42B25, 46E30.

Key words and phrases: maximal function, generalized Lebesgue spaces, generalized Orlicz spaces, mollifier, electrorheological fluids.

Paper MIA-07-27

$L^{p^*(\cdot)-\epsilon}(\Omega)$ continuously for all $\epsilon > 0$ with $\frac{1}{p^*(x)} = \frac{1}{p(x)} - \frac{1}{d}$. Only recently it has been proved in [5] that $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p^*(\cdot)}(\Omega)$ continuously if p is Lipschitz continuous. In order to employ more sophisticated methods regarding questions of existence and regularity of solutions to the fluid models above, it is important to provide optimal results also in the case of generalized Lebesgue and Sobolev spaces. In this context the Hardy–Littlewood maximal operator is quite an essential tool. In this article we will prove that if p satisfies the uniform local continuity condition

$$|p(x) - p(y)| \leqslant \frac{C}{-\ln|x-y|} \tag{1}$$

and is constant outside some large ball, then the Hardy–Littlewood maximal operator is continuous on $L^{p(\cdot)}(R^d)$. To demonstrate the strength of this result we will deduce continuity of mollifying sequences in $W^{1,p(\cdot)}(R^d)$ although the convolution with an L^1 function is in general not continuous on $L^{p(\cdot)}$. Furthermore, we deduce density of $C^\infty(\overline{\Omega})$ in $W^{1,p(\cdot)}(\Omega)$ for Lipschitz domains. Note that the modulus of continuity in 1 also appears when examining functionals of $p(x)$–growth, i.e. $\int f(x, \nabla u)\,dx$ with $|z|^{p(x)} \leqslant f(x,z) \leqslant L(1+|z|^{p(x)})$ for some $L \geqslant 1$. See for example Zhikov [19] on Lavrentiev's phenomenon and Acerbi and Mingione [1] on regularity of minimizers. Moreover, L. Pick and M. Růžička [12] have recently presented a counterexample for the boundedness of M on $L^{p(\cdot)}$ for general p. They show: If p has a point x_0 of very rapid increase, i.e. $-|p(x)-p(x_0)| \cdot \ln|x-x_0| \to \infty$ for $x \to x_0$ (so the continuity condition above is just violated), then M cannot be bounded on $L^{p(\cdot)}$. In this sense the modulus of continuity for p as chosen above is the limiting one for the maximal operator. Nevertheless it remained an open question if the uniform local continuity condition 1 is sufficient for the boundedness of the Hardy–Littlewood maximal operator. The positive answer to this question is given in this paper assuming that p is constant outside some large ball. The mentioned results regarding functionals with $p(x)$–growth strongly indicate that the considered modulus of continuity is the inherent one for systems with $p(x)$–growth.

2. Notation and Basic Facts

We will now introduce the spaces $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$ and state their fundamental properties (see e.g. [3, 7, 10, 13]). Hereby Ω denotes a measurable subset of R^d. For a measurable function $p : R^d \to [1,\infty)$ (called the exponent or exponent on R^d) we define $L^{p(\cdot)}(\Omega)$ to consist of measurable functions $f : \Omega \to R$ such that the modular $\rho_p(f) := \int_\Omega |f(x)|^{p(x)}\,dx$ is finite. If $p^+ := \sup p < \infty$ then we call p a bounded exponent. In this case the expression $||f||_{p(\cdot)} := \inf \lambda > 0 : \rho_p(f/\lambda) < 1$ defines a norm on $L^{p(\cdot)}(\Omega)$. This makes $L^{p(\cdot)}(\Omega)$ a Banach space. Moreover, $||f||_{p(\cdot)} \leqslant 1$ if and only if $\rho_p(f) \leqslant 1$. If p is constant, then $L^{p(\cdot)}(\Omega)$ coincides with the classical Lebesgue space L^p. If $p^- := \inf p > 1$, then $L^{p(\cdot)}(\Omega)$ is uniformly convex and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$, where $\frac{1}{p} + \frac{1}{p'} = 1$. Further let $W^{1,p(\cdot)}(\Omega)$ denote the space of measurable functions $f : \Omega \to R$ such that f and the distributional derivative ∇f are in $L^{p(\cdot)}(\Omega)$. The norm $||f||_{1,p(\cdot)} := ||f||_{p(\cdot)} + ||\nabla f||_{p(\cdot)}$ makes $W^{1,p(\cdot)}$ a Banach space.

By $W_0^{1,p(\cdot)}(\Omega)$ we denote the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. By B we denote an arbitrary ball in R^d. We write $B(x)$ for a ball centered at x and B_r for a ball with radius r. For $f \in L^1_{loc}(R^d)$ we denote

$$M_B f := \fint_B |f(y)|\, dy,$$

where $\fint_B$ is the mean value integral over B. By Mf we denote the Hardy–Littlewood maximal function of f, i.e.

$$Mf(x) := \sup_{B(x)} M_{B(x)} f,$$

where the supremum is taken over all balls centered at x. Moreover, we will use C as a generic constant, i.e. its value may change from line to line.

The spaces $L^{p(\cdot)}$ share even more properties with the classical Lebesgue spaces: separability, modified versions of Hölder's inequality, embeddings ($L^{p(\cdot)} \hookrightarrow L^{q(\cdot)}$ for $p \geqslant q$ on bounded domains) and complex interpolation hold. But $L^{p(\cdot)}$ lacks other fundamental properties. So for every non–constant exponent p a function $f \in L^{p(\cdot)}$ and a null sequence of translations τ_{h_n} exist, such that $\tau_{h_n} f \notin L^{p(\cdot)}$. For p continuous this result has been proved in [8]. Nevertheless we remark that it remains true for all bounded exponents. This will be proven in Lemma 2.3 below.

DEFINITION 2.1. Let p and q be exponents on Ω we say that p is non-weaker than q if and only if $\phi_p(x,z) := z^{p(x)}$ is non-weaker than $\phi_q(x,z) := z^{q(x)}$ in the sense of Musiełak [11], i.e. there exist constants $K_1, K_2 > 0$ and $h \in L^1(\Omega)$, $h \geqslant 0$, such that for a.a. $x \in \Omega$ and all $z \geqslant 0$

$$\phi_q(x,z) \leqslant K_1 \phi_p(x, K_2 z) + h(x). \tag{2}$$

We write $q \preccurlyeq p$ and $\phi_q \preccurlyeq \phi_p$.

LEMMA 2.2. *Let p, q be bounded exponents on Ω, then the following conditions are equivalent*

(a) $L^{p(\cdot)} \hookrightarrow L^{q(\cdot)}$,

(b) $q \preccurlyeq p$,

(c) $q \leqslant p$ *a.e.* *and* $\limsup_{\lambda \to 0^+} \int_\Omega \lambda^{\frac{p(x)q(x)}{p(x)-q(x)}}\, dx = 0$,

where $\lambda^{\frac{p(x)q(x)}{p(x)-q(x)}} := 0$ *for* $p(x) = q(x)$ *and* $0 \leqslant \lambda < 1$.

Proof. 2.2 $\Leftrightarrow$ 2.2: This follows directly from Theorem 8.5 of [11].
2.2 $\Rightarrow$ 2.2 : Let $K_1, K_2 > 0$ and $h \in L^1(\Omega)$, $h \geqslant 0$ be such that

$$z^{q(x)} \leqslant K_1 (K_2 z)^{p(x)} + h(x). \tag{3}$$

Let $x \in \Omega$ with $h(x) < \infty$, then the limit $z \to \infty$ implies $q(x) \leqslant p(x)$. Since h is finite a.e. it follows that $q \leqslant p$ a.e.

Define $r : \Omega \to R$ by $\frac{1}{r} := \frac{1}{q} - \frac{1}{p}$, then $r := \frac{pq}{p-q}$, where $r(x) = \infty$ if $p(x) = q(x)$. Moreover, r is measurable and $r : \Omega \to [1, \infty]$. Since $p \geqslant 1$ there exists $R \geqslant 1$ such that

$$z^{q(x)} \leqslant \tfrac{1}{2}(Rz)^{p(x)} + h(x). \tag{4}$$

Let $\lambda := 1/R$, then $0 < \lambda < 1$. For all $x \in \Omega$ with $p(x) = q(x)$ by convention

$$\lambda^{\frac{p(x)q(x)}{p(x)-q(x)}} := 0.$$

Now assume $x \in \Omega$ with $p(x) > q(x)$. From 4 there follows with $z = R^{\frac{p(x)}{q(x)-p(x)}}$

$$R^{\frac{p(x)q(x)}{q(x)-p(x)}} \leqslant \tfrac{1}{2}\left(RR^{\frac{p(x)}{q(x)-p(x)}}\right)^{p(x)} + h(x) = \tfrac{1}{2}R^{\frac{p(x)q(x)}{q(x)-p(x)}} + h(x).$$

Hence $R^{\frac{p(x)q(x)}{q(x)-p(x)}} \leqslant 2h(x)$. Thus for all $x \in \Omega$ with $p(x) > q(x)$ there follows

$$\lambda^{\frac{p(x)q(x)}{p(x)-q(x)}} = R^{\frac{p(x)q(x)}{q(x)-p(x)}} \leqslant 2h(x) < \infty.$$

Overall we have shown for all $x \in \Omega$ that

$$\int_\Omega \lambda^{\frac{p(x)q(x)}{p(x)-q(x)}}\, dx \leqslant 2\int_\Omega |h(x)|\, dx.$$

This proves (c).

2.2 $\Rightarrow$ 2.2 : From $q \leqslant p$ it follows that $z^{q(x)} \leqslant z^{p(x)}$ for all $z \geqslant 1$. So 2 is satisfied for all $z \geqslant 1$ as long as $K_1, K_2 \geqslant 1$ and $h \geqslant 0$. Now fix $0 < \lambda < 1$ such that $\int_\Omega \lambda^{\frac{p(x)q(x)}{p(x)-q(x)}}\, dx < \infty$. Let $0 \leqslant z \leqslant 1$, then

$$\begin{aligned} z^{q(x)} &= (z/\lambda)^{q(x)}\lambda^{q(x)} \\ &\leqslant (z/\lambda)^{p(x)} + \lambda^{\frac{p(x)q(x)}{p(x)-q(x)}} \\ &= \left(\tfrac{1}{\lambda}z\right)^{p(x)} + \lambda^{\frac{p(x)q(x)}{p(x)-q(x)}}, \end{aligned}$$

where we have used Young's inequality pointwise, i.e. $ab \leqslant a^\delta + b^{\delta'}$, with $\delta = p/q$ and $\delta = p/(p-q)$. Let $K_1 := 1$, $K_2 := 1/\lambda$, $h(x) := \lambda^{\frac{p(x)q(x)}{p(x)-q(x)}}$, then $h \in L^1(\Omega)$ and 3 is fulfilled. This proves 2.2. ☐

LEMMA 2.3. *Let p be a bounded exponent on $L^{p(\cdot)}(R^d)$, which is non–constant. Then there exists $h \in R^d \setminus 0$ such that the translation operator $(\tau_h f)(x) := f(x-h)$ is not continuous on $L^{p(\cdot)}(R^d)$. Moreover, there exists $f \in L^{p(\cdot)}(R^d)$ with $\tau_h f \notin L^{p(\cdot)}(R^d)$.*

Proof. We prove the first part by contradiction. Assume that τ_h is continuous on $L^{p(\cdot)}$ for every $h \in R^d$. Then $||\tau_h f||_{p(\cdot)} = ||f||_{\tau_{-h}p(\cdot)}$. This implies that we have the embeddings $L^{p(\cdot)}(R^d) \hookrightarrow L^{\tau_{-h}p(\cdot)}(R^d)$. From Lemma 2.2 we deduce that $p \geqslant \tau_h p$ almost everywhere. Since h is arbitrary, this implies p is constant. This is a contradiction.

The construction of an $f \in L^{p(\cdot)}(R^d)$ with $\tau_h f \notin L^{p(\cdot)}(R^d)$ is now standard: Assume that τ_h is not continuous on $L^{p(\cdot)}$. Choose $f_j \in L^{p(\cdot)}$ with $f_j \geqslant 0$, $||f_j||_{p(\cdot)} \leqslant 2^{-j}$ and $||\tau_h f_j||_{p(\cdot)} = 1$. Define $f := \sum_{j=1}^\infty f_j$, then $||f||_{p(\cdot)} \leqslant 1$ and $||\tau_h f||_{p(\cdot)} = \infty$. ☐

3. Results

We begin with the (dis-)continuity of the convoluting operator.

THEOREM 3.1. *Let $\Omega \subset R^d$ be a bounded, measurable set. Let Let p,r be bounded exponents (on Ω) with $p^- > 1$ and $r^- > 1$. Then the convolution $* : (f,g) \mapsto f * g$ is continuous as a mapping $L^{p(\cdot)}(\Omega) \times L^1(R^d) \to L^{r(\cdot)}(\Omega)$ if and only if $p^- \geqslant r^+$.*

Proof. "$\Leftarrow$": Use $L^{p^+}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega) \hookrightarrow L^{p^-}(\Omega)$ and the theory for classical Lebesgue spaces.

"$\Rightarrow$": Assume $p^- < r^+$, then there exists $f \in L^{p(\cdot)}$ and a translation τ_h such that $\tau_h f \notin L^{r(\cdot)}$ (see Lemma 2.3). For $\phi \in C_0^\infty(R^d)$, $\phi \geqslant 0$, and $\int \phi \, dx = 1$ define ϕ_ϵ by $\phi_\epsilon(x) := \epsilon^{-d}\phi((x-h)/\epsilon)$, then $f * \phi_\epsilon \to \tau_h f$ in $L^1(R^d)$. By assumption on the convolution holds $\|f * \phi_\epsilon\|_{r(\cdot)} \leqslant C$. Since $L^{p(\cdot)}(\Omega)$ is reflexive, there exists a subsequence converging $L^{p(\cdot)}(\Omega)$–weakly to $\tau_h f$. This contradicts $\tau_h f \notin L^{r(\cdot)}$. $\square$

Let $p : R^d \to [1,\infty)$ be a bounded exponent. For all balls $B \subset R^d$ define $p_B^- := essinf_B p(x)$ and $p_B^+ := esssup_B p(x)$.

LEMMA 3.2. *Let $p : R^d \to R$ be continuous. The following conditions are equivalent:*

(i) p is uniformly continuous with $|p(x)-p(y)| \leqslant \frac{C_0}{-\ln|x-y|}$ for all $0 < |x-y| \leqslant \frac{1}{2}$.

(ii) For all open balls B we have $|B|^{p_B^- - p_B^+} \leqslant C_1$.

Proof. $(i) \Rightarrow (ii)$: Note that there exists r_0 with $0 < r_0 < \frac{1}{2}$, such that

$$\frac{d}{2} \leqslant (\ln|B_r|)/(\ln(2r)) = \ln(Cr^d)/(\ln(2r)) \leqslant 2\,d. \tag{5}$$

for all $0 < r < r_0$. Now let $r > 0$ and $x,y \in B_r$. If $0 < r \leqslant r_0$, then $r < r_0 < \frac{1}{2}$ implies $|B_r| \leqslant |B_{r_0}| \leqslant 1$ and $|p_{B_r}^- - p_{B_r}^+| \leqslant \frac{C_0}{-\ln(2r)}$. Therefore

$$|B_r|^{p_{B_r}^- - p_{B_r}^+} \leqslant |B_r|^{\frac{C_0}{\ln(2r)}} = \exp\left(\frac{C_0 \ln|B_r|}{\ln(2r)}\right) \overset{5}{\leqslant} \exp(2\,C_0\,d).$$

If $r \geqslant r_0$, then $|B_r|^{p_{B_r}^- - p_{B_r}^+} \leqslant |B_{r_0}(0)|^{p^- - p^+}$.

$(ii) \Rightarrow (i)$: Let $x,y \in R^d$ with $0 < |x-y| < \frac{1}{2}$. Then there exists B_r with $x,y \in B_r$ and $\frac{|x-y|}{2} < r < |x-y|$.

Since $|B_r| \leqslant (2r)^d$,

$$(4|x-y|)^{-|p(x)-p(y)|} \leqslant (2r)^{-|p(x)-p(y)|} \leqslant |B_r|^{\frac{-|p(x)-p(y)|}{d}} \leqslant |B_r|^{\frac{p_B^- - p_B^+}{d}} \leqslant C_1^{\frac{1}{d}}.$$

Since $p^+ < \infty$, this proves $|x-y|^{-|p(x)-p(y)|} \leqslant C$ for some $C > 1$. We take the logarithm of this inequality to deduce $|p(x)-p(y)| \leqslant \frac{\ln C}{-\ln|x-y|}$. $\square$

In the following lemma we make use of the Lorentz space $L^{1,\infty}(R^d)$ (see e.g. [2] for a definition).

LEMMA 3.3. *Let p be a bounded exponent on R^d which satisfies one of the conditions of Lemma 3.2. Then there exists a constant $C(p) > 0$ such that, for all $\|f\|_{p(\cdot)} \leqslant 1$,*

$$(Mf(x))^{\frac{p(x)}{p^-}} \leqslant C(p) \left(M\left(|f|^{\frac{p}{p^-}} \right)(x) + 1 \right) \qquad \textit{for all } x \in R^d.$$

Proof. Let $q := p/p^-$, then q is also a bounded exponent which satisfies one of the conditions of Lemma 3.2. Let $\|f\|_{p(\cdot)} \leqslant 1$, then $\rho_p(f) \leqslant 1$. If $r \geqslant \frac{1}{2}$, then

$$(M_B f)^{q(x)} \leqslant \left(\fint_B \left(|f(y)|^{p(y)} + 1 \right) dy \right)^{q(x)} \leqslant \left(|B|^{-1}\rho_p(f) + 1 \right)^{q(x)} \leqslant \left(|B_{\frac{1}{2}}(0)|^{-1} + 1 \right)^{q^+}.$$

If $0 < r < \frac{1}{2}$, then $|B| \leqslant (2r)^d < 1$ and

$$\begin{aligned}(M_B f)^{q(x)} &\leqslant \left(\fint_B |f(y)|^{q_B^-} dy \right)^{\frac{q(x)}{q_B^-}} \\ &\leqslant \left(\fint_B \left(|f(y)|^{q(y)} + 1 \right) dy \right)^{\frac{q(x)}{q_B^-}} \\ &\leqslant |B|^{-\frac{q(x)}{q_B^-}} 3^{q^+} \left(\frac{1}{3} \int_B \left(|f(y)|^{q(y)} + 1 \right) dy \right)^{\frac{q(x)}{q_B^-}}.\end{aligned}$$

Since $\frac{1}{3} \int_B (|f(y)|^{q(y)} + 1)\, dy \leqslant \frac{1}{3} \int_B (|f(y)|^{p(y)} + 2)\, dy \leqslant \frac{1}{3}\rho_p(f) + \frac{2}{3}|B| < 1$, we have the inequalities

$$\begin{aligned}(M_B f)^{q(x)} &\leqslant |B|^{-\frac{q(x)}{q^-B}} 3^{q^+} \left(\frac{1}{3} \int_B |f(y)|^{q(y)} dy + \frac{2}{3}|B| \right) \\ &\leqslant |B|^{\frac{q_B^- - q_B^+}{q_B^-}} 3^{q^+-1} \left(\fint_B |f(y)|^{q(y)} dy + 2 \right) \\ &\overset{\text{Lemma 3.2}}{\leqslant} C_0\, 3^{q^+-1} \left(M_B(|f|^q) + 2 \right).\end{aligned}$$

Taking the supremum over all balls $B(x)$ proves the lemma. □

LEMMA 3.4. *Let p be a bounded exponent on R^d which satisfies one of the conditions of Lemma 3.2 and is constant outside some ball $B_R(0)$. Then there exist a constant $C(p) > 0$ and $h \in L^{1,\infty}(R^d) \cap L^\infty(R^d)$ such that, for all $\|f\|_{p(\cdot)} \leqslant 1$,*

$$(Mf(x))^{\frac{p(x)}{p^-}} \leqslant C(p)\, M\left(|f|^{\frac{p}{p^-}} \right)(x) + h(x) \qquad \textit{for a.a. } x \in R^d.$$

Proof. Let $\|f\|_{p(\cdot)} \leqslant 1$, then $\rho_p(f) \leqslant 1$. Split f into $f_0 := \chi_{B_R} f$ and $f_1 := \chi_{R^d \setminus B_R}$. Let p_∞ denote the value of p on the complement of $B_R := B_R(0)$. Further, let $q := p/p^-$ and $q_\infty := p_\infty/p^-$, then q satisfies the equivalent conditions of Lemma 3.2 and hence also the assumptions of Lemma 3.3. Thus, for all $x \in B_{2R} := B_{2R}(0)$

$$(Mf(x))^{q(x)} \leqslant C(q) M(|f|^q) + C(q). \tag{6}$$

Now let $x \in R^d \setminus B_{2R}$, then $|x| - R \geqslant \frac{1}{2}|x|$ and $|B_{|x|-R}| \geqslant C|x|^d$. So

$$\begin{aligned}
(Mf_0(x))^{q(x)} &\leqslant \left(\sup_{r>|x|-R} \frac{1}{|B_r|} \int_{B_r(x)} |f_0(y)|\, dy \right)^{q(x)} \leqslant \left(\frac{1}{|B_{|x|-R}|} \int_{B_R} |f(y)|\, dy \right)^{q(x)} \\
&\leqslant \left(\frac{C}{|x|^d} \int_{B_R} |f(y)|\, dy \right)^{q(x)} \leqslant \left(\frac{C}{|x|^d} \int_{B_R} |f(y)|^{p(y)} + 1\, dy \right)^{q(x)} \\
&\leqslant \frac{C(q)}{|x|^d},
\end{aligned} \tag{7}$$

where we have used that supp $f_0 \subset B_R$. Furthermore for $x \in R^d \setminus B_{2R}$

$$(Mf_1(x))^{q(x)} = (Mf_1(x))^{q_\infty} \leqslant M(|f_1|^{q_\infty})(x) \leqslant M(|f|^q)(x). \tag{8}$$

Overall, there holds for all $x \in R^d$,

$$\begin{aligned}
(Mf(x))^{q(x)} &\leqslant \chi_{B_{2R}} (Mf(x))^{q(x)} + \chi_{R^d \setminus B_{2R}} (M(f_0)(x) + M(f_1)(x))^{q(x)} \\
&\leqslant \chi_{B_{2R}} (Mf(x))^{q(x)} + C(q)\, \chi_{R^d \setminus B_{2R}} \Big((Mf_0(x))^{q(x)} + (Mf_1(x))^{q(x)} \Big) \\
&\overset{6,7,8}{\leqslant} C(q)\, M(|f|^q)(x) + \underbrace{\chi_{B_{2R}} C(q) + \chi_{R^d \setminus B_{2R}} C(q)\, |x|^{-d}}_{=:h}.
\end{aligned}$$

The fact that $h \in L^{1,\infty}(R^d) \cap L^\infty(R^d)$ proves the lemma. $\square$

THEOREM 3.5. *Let p be as in Lemma 3.4 with $p^- > 1$. Then M is bounded on $L^{p(\cdot)}(R^d)$, i.e.*

$$\|Mf\|_{p(\cdot)} \leqslant C(p)\, \|f\|_{p(\cdot)}.$$

Proof. Since Mf is positive homogenous, i.e. $M(\lambda f) = \|\lambda\| Mf$, it suffices to show $\|Mf\|_{p(\cdot)} \leqslant C$ for all f with $\|f\|_{p(\cdot)} \leqslant 1$. Since $p^+ < \infty$, it is also sufficient to prove $\rho_p(Mf) \leqslant C$ for all $\|f\|_{p(\cdot)} \leqslant 1$. Let $f \in L^{p(\cdot)}$ with $\|f\|_{p(\cdot)} \leqslant 1$, then $\rho_p(f) \leqslant 1$. Let $q := p/p^-$. By Lemma 3.4 there exists $h \in L^{1,\infty}(R^d) \cap L^\infty(R^d)$ such that $(Mf)^q \leqslant C(p) M(\||f\||^q) + h$. Thus,

$$\rho_p(Mf) = \|(Mf)^q\|_{p^-}^{p^-} \leqslant \big(C(p)\, \|M(|f|^q)\|_{p^-} + \|h\|_{p^-} \big)^{p^-}.$$

Since $p^- > 1$ the mapping $f \mapsto Mf$ is continuous on $L^{p^-}(R^d)$. Hence

$$\rho_p(Mf) \leqslant \Big(C(p)\, \||f|^q\|_{p^-} + \|h\|_{p^-} \Big)^{p^-} = \Big(C(p) \rho_p(f)^{\frac{1}{p^-}} + \|h\|_{p^-} \Big)^{p^-} \leqslant C(p).$$

This proves the theorem. $\square$

COROLLARY 3.6. *Let p be a bounded exponent on R^d such that M is bounded on $L^{p(\cdot)}(R^d)$ (e.g. let p be as in Theorem 3.5). Let $\phi : R^d \to R$ be an integrable function and set $\phi_\epsilon(x) := \epsilon^{-d}\phi(x/\epsilon)$ for all $\epsilon > 0$. Assume that the least decreasing radial majorant ψ of ϕ is integrable, i.e. $A := \int_{R^d} \sup_{|y|\geqslant|x|} |\psi(y)|\, dx < \infty$. Then*

(i) $\sup_{\epsilon>0} |(f * \phi_\epsilon)(x)| \leqslant 2A\, Mf(x)$ *for all* $f \in L^{p(\cdot)}(R^d)$.

(ii) If $\int_{R^d} \phi(x)\, dx = 1$, then, for $f \in L^{p(\cdot)}(R^d)$, we have $f * \phi_\epsilon \xrightarrow{\epsilon} f$ almost everywhere and in $L^{p(\cdot)}(R^d)$. Furthermore,

$$||f * \phi_\epsilon||_{p(\cdot)} \leqslant C(A,p)||Mf||_{p(\cdot)} \leqslant C(A,p)||f||_{p(\cdot)}.$$

Proof. Split $f \in L^{p(\cdot)}(R^d)$ into $f = f_0 + f_1$ with $f_0 = f\chi_{|f|>1}$ and $f_1 = f\chi_{|f|\leqslant 1}$. Then $f_0 \in L^1(R^d)$, $f_1 \in L^\infty(R^d)$, and $|f_j| \leqslant |f|$ for $j = 0, 1$. Then from [16], Theorem 2, p. 62 we immediately deduce that

$$\sup_{\epsilon>0} |(f_j * \phi_\epsilon)(x)| \leqslant A\, Mf_j(x) \leqslant A\, Mf(x) \qquad \text{for } j = 0, 1.$$

This proves (3.6). Now assume $\int_{R^d} \phi(x)\, dx = 1$. Then from the same theorem in [16] we deduce $\phi_\epsilon * f_j \to f_j$ almost everywhere, $j = 0, 1$. This proves $\phi_\epsilon * f \to f$ almost everywhere. By assumption on p there holds $||Mf||_{p(\cdot)} \leqslant C||f||_{p(\cdot)}$ which implies $\rho_p(Mf) < \infty$. So by (3.6) and dominated convergence $\lim_{\epsilon\to 0} \rho_p(\varphi_\epsilon * f - f) = 0$. Since $p^+ < \infty$, this implies $\lim_{\epsilon\to 0} ||\varphi_\epsilon * f - f||_{p(\cdot)} = 0$. This proves the corollary. $\square$

See [14] for a similar result on mollifiers $\phi \in C_0^\infty(R^d)$ on bounded domains Ω, i.e. $f \in L^{p(\cdot)}(\Omega)$, without the use of the Hardy–Littlewood maximal function. As an application of Corollary 3.6, we deduce

THEOREM 3.7. *Let Ω be a bounded domain with Lipschitz boundary and let p be as in Corollary 3.6. Then $C^\infty(\overline{\Omega})$ is dense in $W^{1,p(\cdot)}(\Omega)$.*

Proof. Using the results of Corollary 3.6 one can follow exactly the proof in [6], where the case of classical Sobolev spaces is treated. $\square$

This generalizes the result of [4] which was stated for p uniformly Lipschitz.

Acknowledgments. The author gratefully acknowledges inspiring discussions with M. Růžička (University of Freiburg, Germany) and L. Pick (Charles University Prague, Czech Republic).

REFERENCES

[1] EMILIO ACERBI AND GIUSEPPE MINGIONE, *Regularity results for a class of functionals with non-standard growth*, Arch. Ration. Mech. Anal. **156** (2001), no. 2, 121–140.

[2] JÖRAN BERGH AND JÖRGEN LÖFSTRÖM, *Interpolation spaces. An introduction*, Springer-Verlag, Berlin, 1976, Grundlehren der Mathematischen Wissenschaften, No. 223.

[3] LARS DIENING, *Theoretical and numerical results for electrorheological fluids*, Ph.D. thesis, University of Freiburg, Germany, 2002.

[4] DAVID E. EDMUNDS AND JIŘÍ RÁKOSNÍK, *Density of smooth functions in* $W^{k,p(x)}(\Omega)$, Proc. Roy. Soc. London Ser. A **437** (1992), no. 1899, 229–236.

[5] ———, *Sobolev embeddings with variable exponent*, Studia Math. **143** (2000), no. 3, 267–293.

[6] LAWRENCE C. EVANS AND RONALD F. GARIEPY, *Measure theory and fine properties of functions*, CRC Press, Boca Raton, FL, 1992.

[7] HENRYK HUDZIK, *The problems of separability, duality, reflexivity and of comparison for generalized Orlicz-Sobolev spaces* $W^k_M(\Omega)$, Comment. Math. Prace Mat. **21** (1980), no. 2, 315–324.

[8] ONDREJ KOVÁČIK AND JIŘÍ RÁKOSNÍK, *On spaces* $L^{p(x)}$ *and* $W^{k,p(x)}$, Czechoslovak Math. J. **41(116)** (1991), no. 4, 592–618.

[9] PAOLO MARCELLINI, *Regularity and existence of solutions of elliptic equations with* p,q *-growth conditions*, J. Differential Equations **90** (1991), no. 1, 1–30.

[10] J. MUSIEŁAK AND W. ORLICZ, *On modular spaces*, Studia Math. **18** (1959), 49–65.

[11] JULIAN MUSIEŁAK, *Orlicz spaces and modular spaces*, Springer-Verlag, Berlin, 1983.

[12] LUBOŠ PICK AND MICHAEL RŮŽIČKA, *An example of a space* $L^{p(x)}$ *on which the Hardy-Littlewood maximal operator is not bounded*, Expo. Math. **19** (2001), no. 4, 369–371.

[13] MICHAEL RŮŽIČKA, *Electrorheological fluids: modeling and mathematical theory*, Springer-Verlag, Berlin, 2000.

[14] S. G. SAMKO, *Density* $C_0^\infty(\mathbf{R}^n)$ *in the generalized Sobolev spaces* $W^{m,p(x)}(\mathbf{R}^n)$, Dokl. Akad. Nauk **369** (1999), no. 4, 451–454.

[15] STEFAN G. SAMKO, *Convolution and potential type operators in* $L^{p(x)}(\mathbf{R}^n)$, Integral Transform. Spec. Funct. **7** (1998), no. 3-4, 261–284.

[16] ELIAS M. STEIN, *Singular integrals and differentiability properties of functions*, Princeton University Press, Princeton, N.J., 1970.

[17] V. V. ZHIKOV, *Averaging of functionals of the calculus of variations and elasticity theory*, Izv. Akad. Nauk SSSR Ser. Mat. **50** (1986), no. 4, 675–710, 877.

[18] ———, *Meyer-type estimates for solving the nonlinear Stokes system*, Differ. Uravn. **33** (1997), no. 1, 107–114, 143.

[19] VASILIĬ V. ZHIKOV, *On Lavrentiev's phenomenon*, Russian J. Math. Phys. **3** (1995), no. 2, 249–269.

(Received March 25, 2003)

L. Diening
Section of Applied Mathematics
University of Freiburg
Eckerstr. 1, 79104 Freiburg
Germany