Mathematical
Inequalities
& Applications
Volume 7, Number 2 (2004), 255–265

HARDY–LITTLEWOOD MAXIMAL OPERATOR ON $L^{p(x)}(\mathbb{R})$

ALEŠ NEKVINDA

(*communicated by L. Pick*)

Abstract. We consider Hardy-Littlewood maximal operator on the general Lebesgue space $L^{p(x)}(\mathbb{R}^n)$ with variable exponent. A sufficient condition on the function p is known for the boundedness of the maximal operator on $L^{p(x)}(\Omega)$ with an open bounded Ω. Our main aim is to find an additional condition to p to guarantee the boundedness of the maximal operator on $L^{p(x)}(\mathbb{R}^n)$. From this point of view we put an emphasis on the behavior of functions p near the infinity. We find a sufficient condition on p such that the maximal operator is bounded on $L^{p(x)}(\mathbb{R}^n)$. We also construct a function p for which the maximal operator is unbounded.

1. Introduction

The generalized Lebesgue space $L^{p(x)}$ and the corresponding Sobolev space $W^{1,p(x)}$ have attracted more and more interest in recent years. We refer to [8] for the establishment of the fundamental properties of these spaces, to [3] for some properties of the norm on $L^{p(x)}$, and to [6] for inequalities of Sobolev type. Further motivation for the study of these spaces is provided in [11, 12] by means of mathematical models of electrorheological fluids which involve nonlinear systems of partial differential equations with coefficients of variable rate of growth.

A crucial difference between $L^{p(x)}$ and the classical Lebesgue spaces is that $L^{p(x)}$ is not, in general, invariant under translation(see [8], Ex. 2.9). Because of this, serious problems arise with regard to convolutions, the density of smooth functions in $W^{1,p(x)}$ (see [5] and [13]) and the boundedness of the Hardy-Littlewood maximal operator.

In [4] and [9] a discrete version ℓ^{p_n} of $L^{p(x)}$ is introduced and some interesting properties of these spaces are proved. Especially, a necessary and sufficient condition to bounded sequences $\{p_n\}$, $\{q_n\}$ is found to guarantee the equivalence of norms in spaces ℓ^{p_n} and ℓ^{q_n}.

In [2] L.Diening proved that the maximal operator is bounded on $L^{p(x)}(\Omega)$ for a bounded Ω provided $-|p(x)-p(y)|\ln|x-y| \leqslant C$ for $|x-y| \leqslant \frac{1}{2}$, $x,y \in \Omega$. In [10] the authors showed that the maximal operator can be unbounded if there exists $x \in \Omega$ such that $-\lim_{y\to x}|p(x)-p(y)|\ln|x-y| = \infty$.

We will investigate in this paper the Hardy-Littlewood maximal operator on $L^{p(x)}(\mathbb{R}^n)$. It is not difficult to prove the boundedness of maximal operator provided

Mathematics subject classification (2000): 46E30, 26D15.

Key words and phrases: maximal operator, Banach function space, Lebesgue spaces, variable exponent.

The author was supported by MSM 210000010.

Paper MIA-07-28

$p(x)$ is constant on $\{x \in \mathbb{R}^n; |x| \geqslant R\}$ for some positive R and p satisfies the condition from [2] on $\{x \in \mathbb{R}^n; |x| \leqslant R\}$. We find a more general condition to the behavior of $p(x)$ then constancy near the infinity to guarantee the boundedness of maximal operator on $L^{p(x)}(\mathbb{R}^n)$. We show that the class of functions $p(x)$ satisfying this sufficient condition is contained in a wider class of functions which have a finite limit at the infinity in the sense of Lebesgue measure. Moreover, we construct an example of a function $p(x)$ from this wider class such that the maximal operator is unbounded on $L^{p(x)}(\mathbb{R})$.

DEFINITION 1.1. Let $f \in L^1_{loc}(\mathbb{R}^n)$. Define the Hardy-Littlewood maximal function Mf by

$$(Mf)(x) = \sup_{x\in Q} \frac{1}{|Q|} \int_Q |f(t)| \mathrm{d}t$$

where Q are cubes which sides are parallel to the coordinates.

Remind the well-known theorem on the maximal operator. The proof can be found for instance in [7], Theorem 21.76.

PROPOSITION 1.2. *Let $r \in \mathbb{R}$, $1 < r \leqslant \infty$, then there exists $M_r > 0$ such that*

$$\int_{\mathbb{R}^n} \big(Mf(x)\big)^r dx \leqslant M_r \int_{\mathbb{R}^n} |f(x)|^r dx.$$

Let $B \subset \mathbb{R}^n$ be a measurable set. Denote by $\mathfrak{M}(B)$ the set of all measurable functions on B and adopt the notation $|B|$ for the Lebesgue measure. Let $\mathfrak{B}(B)$ denote the set of all functions $p \in \mathfrak{M}(B)$ such that $1 \leqslant p(x)$, ess sup$\{x \in B; p(x) < \infty\}$.

DEFINITION 1.3. We say that a normed linear subspace X of $\mathfrak{M}(B)$ is a Banach function space if the following five axioms are satisfied:

(i) the norm $\|f\|_X$ is defined for all $f \in \mathfrak{M}(B)$, and $f \in X$ if and only if $\|f\|_X < \infty$;
(ii) $\|f\|_X = \|\,|f|\,\|_X$ for all $f \in \mathfrak{M}(B)$;
(iii) $0 \leqslant f_n \nearrow f$ a.e. in B, then $\|f_n\|_X \nearrow \|f\|_X$;
(iv) if $|E| < \infty$, then $\chi_E \in X$ where χ_E denotes the characteristic function of E;
(v) for every $E \subset B$ with $|E| < \infty$ there exists a constant C_E such that $\int_E f(t)v(t)dt \leqslant C_E\|f\|_X$ for all $f \in X(B)$.

DEFINITION 1.4. Let $p(x) \in \mathfrak{M}(B)$, $1 \leqslant p(x) < \infty$. Denote for $f \in \mathfrak{M}(B)$ the Luxemburg norm by

$$\|f\|_{p(x)} = \inf\{\lambda > 0; \int_B \left|\frac{f(x)}{\lambda}\right|^{p(x)} \mathrm{d}x \leqslant 1\}$$

Define a space $L^{p(x)}(B)$ by

$$L^{p(x)}(B) = \{f \in \mathfrak{M}(B); \|f\|_{p(x)} < \infty\}.$$

In [4], the following lemma is proved.

LEMMA 1.5. *The space $L^{p(x)}(B)$ is a Banach function space.*

The following lemma is proved in [8] (see Theorem 2.4).

LEMMA 1.6. *Let* $p \in \mathfrak{B}(\mathbb{R}^n)$. *Then*

$$L^{p(x)}(\mathbb{R}^n) = \{f \in \mathfrak{M}(\mathbb{R}^n); \int_{\mathbb{R}^n} |f(x)|^{p(x)} \mathfrak{d}x < \infty\}.$$

To prove the next lemma we used the idea of the proof of Theorem 1.8 in [1].

LEMMA 1.7. *Let* $p \in \mathfrak{B}(\mathbb{R}^n)$ *and let* $X = L^{p(x)}(\mathbb{R}^n)$. *Then the following statements are equivalent.*

(i) Then there exists a constant $C > 0$ *such that* $\|Mf\|_X \leqslant C\|f\|_X$ *for all* $f \in X$.

(ii) $\int_{\mathbb{R}^n} |Mf(x)|^{p(x)} dx < \infty$ *provided* $\int_{\mathbb{R}^n} |f(x)|^{p(x)} dx \leqslant 1$.

Proof. The implication (i) $\Rightarrow$ (ii) is an easy consequence of Lemma 1.6. Prove the opposite implication. Assume the contrary. Then there exists a sequence of functions $f_n \geqslant 0$ with $\|f_n\|_X \leqslant 1$ and $\|Mf_n\|_X \geqslant n^3$. Set $f = \frac{6}{\pi^2}\sum_{k=1}^{\infty} \frac{f_n}{n^2}$. Clearly, $\|f\|_X \leqslant 1$ and as an easy consequence of the boundedness of p we obtain $\int_{\mathbb{R}^n} |f(x)|^{p(x)} \mathrm{d}x \leqslant 1$. On the other hand, we obtain $\|Mf\|_X \geqslant \frac{6}{\pi^2}\frac{\|f_n\|_X}{n^2} \geqslant \frac{6n}{\pi^2}$ for each n. Thus, $\|Mf\|_X = \infty$ and $\int_{\mathbb{R}^n} |Mf(x)|^{p(x)} \mathrm{d}x = \infty$ by Lemma 1.6. □

2. Boundedness of maximal operator

Adopt in the next the notation $B_R(x) = \{y \in \mathbb{R}^n; |x-y| < R\}$, $B_R := B_R(0)$, $B_R^c = \mathbb{R}^n \setminus B_R$ and set $p_* = \inf\{p(x); x \in \mathbb{R}^n\}$, $p^* = \sup\{p(x); x \in \mathbb{R}^n\}$.

DEFINITION 2.1. Let $p \in \mathfrak{B}(\mathbb{R}^n)$. Say that $p \in \mathcal{L}$ if there is a constant $K > 0$ such that

$$|p(x) - p(y)| \leqslant \frac{K}{-\ln|x-y|}$$

for $x, y \in \mathbb{R}^n$, $0 < |x-y| \leqslant \frac{1}{2}$.

Given $\Omega \subset \mathbb{R}^n$ and $p : \mathbb{R}^n \to \mathbb{R}$ we adopt the notation $p_{m,\Omega} = \operatorname{ess\,inf}\{p(x); x \in \Omega\}$ and $p_{M,\Omega} = \operatorname{ess\,sup}\{p(x); x \in \Omega\}$.

In [2] (see Lemma 3.2) the following lemma is proved.

LEMMA 2.2. *Let* $p \in \mathcal{L}$. *Then there exists a constant* $C > 0$ *such that the estimate*

$$|B|^{p_{m,B} - p_{M,B}} \leqslant C$$

holds for any ball.

By an easy modification we obtain the following lemma.

LEMMA 2.3. *Let* $p \in \mathcal{L}$. *Then there exists a constant* $C > 0$ *such that the estimate*

$$|B_r(x)|^{p_{m,B_r(x)} - p(x)} \leqslant C$$

holds for any ball.

Proof. If $|B_r(x)| \geqslant 1$ then the fact $p_{m,B_r(x)} - p(x) \leqslant 0$ clearly gives

$$|B_r(x)|^{p_{m,B_r(x)}-p(x)} \leqslant 1.$$

If $|B_r(x)| \leqslant 1$ then by Lemma 2.2 we obtain

$$\begin{aligned} |B_r(x)|^{p_{m,B_r(x)}-p(x)} &= |B_r(x)|^{p_{m,B_r(x)}-p_{M,B_r(x)}} |B_r(x)|^{p_{M,B_r(x)}-p(x)} \\ &\leqslant |B_r(x)|^{p_{m,B_r(x)}-p_{M,B_r(x)}} \leqslant C \end{aligned}$$

which finishes the proof. □

Define a centered maximal operator by

$$(M_{(c)}f)(x) = \sup_{R>0} \frac{1}{B_R(x)} \int_{B_R(x)} |f(t)| \mathrm{d}t.$$

Lars Diening proved in [2] (see Lemma 3.3) the following lemma.

LEMMA 2.4. *Let $p \in \mathcal{L}$. Then there is a constant $C(p) > 0$ such that for each $x \in \mathbb{R}^n$ and for each function f with $\int_{\mathbb{R}^n} |f(t)|^{p(t)} dt \leqslant 1$ the inequality*

$$\big(M_{(c)}f(x)\big)^{p(x)} \leqslant C(p)\big(M_{(c)}\big(|f(.)|^{p(.)}\big)(x) + 1\big)$$

holds.

It is not difficult to observe that there is a constant $D > 0$ such that the inequalities

$$D^{-1}Mf(x) \leqslant M_{(c)}f(x) \leqslant DMf(x) \tag{2.1}$$

hold for each $f \in L^1_{loc}$ and $x \in \mathbb{R}^n$.

DEFINITION 2.5. Let $p \in \mathcal{L}$. Say that a function f belongs to a class $\mathcal{G}_p$ (write $p \in \mathcal{G}_p$) if $f(x) = 0$ or $|f(x)| \geqslant 1$ for each $x \in \mathbb{R}^n$ and

$$\int_{\mathbb{R}^n} |f(x)|^{p(x)} dx \leqslant 1.$$

Using the analogous idea as in the proof of Lemma 3.2 in [2] we obtain the following lemma which is similar to Lemma 2.4.

LEMMA 2.6. *Let $p \in \mathcal{L}$. Then there exists a constant $C_p > 0$ such that the inequality*

$$|Mf(x)|^{p(x)} \leqslant C_p M(|f(.)|^{p(.)})(x)$$

holds for all $f \in \mathcal{G}_p$ and $x \in \mathbb{R}^n$.

Proof. Set for $r > 0$

$$M_r f(x) = \frac{1}{|B_r(x)|} \int_{|B_r(x)|} |f(y)| dy.$$

Suppose $f \in \mathcal{G}_p$ and $x \in \mathbb{R}^n$. By Jensen's inequality we obtain

$$\big(M_r f(x)\big)^{p(x)} = \Big(\frac{1}{|B_r(x)|}\int_{|B_r(x)|}|f(y)|dy\Big)^{p(x)} \tag{2.2}$$

$$\leqslant \Big(\frac{1}{|B_r(x)|}\int_{|B_r(x)|}|f(y)|^{p_{m,B_r(x)}}dy\Big)^{\frac{p(x)}{p_{m,B_r(x)}}} := I.$$

Since $f \in \mathcal{G}_{p(.)}$ and $p_{m,B_r(x)} \leqslant p(y)$ for $y \in B_r(x)$ we have $|f(y)|^{p_{m,B_r(x)}} \leqslant |f(y)|^{p(y)}$ which gives

$$I \leqslant |B_r(x)|^{-\frac{p(x)}{p_{m,B_r(x)}}}\Big(\int_{B_r(x)}|f(y)|^{p(y)}dy\Big)^{\frac{p(x)}{p_{m,B_r(x)}}}.$$

Using again the assumption $f \in \mathcal{G}_{p(.)}$ we have

$$\int_{B_r(x)}|f(y)|^{p(y)}dy \leqslant \int_{\mathbb{R}^n}|f(y)|^{p(y)}dy \leqslant 1$$

and since $\frac{p(x)}{p_{m,B_r(x)}} \geqslant 1$ we obtain

$$I \leqslant |B_r(x)|^{-\frac{p(x)}{p_{m,B_r(x)}}}\int_{B_r(x)}|f(y)|^{p(y)}dy$$

$$= |B_r(x)|^{\frac{p_{m,B_r(x)}-p(x)}{p_{m,B_r(x)}}}\Big(\frac{1}{|B_r(x)|}\int_{B_r(x)}|f(y)|^{p(y)}dy\Big).$$

By Lemma 2.3 we have

$$|B_r(x)|^{\frac{p_{m,B_r(x)}-p(x)}{p_{m,B_r(x)}}} \leqslant C^{\frac{1}{p_{m,B_r(x)}}} \leqslant \max\{1,C\}^{\frac{1}{p_{m,B_r(x)}}} \leqslant \max\{1,C\}.$$

Thus, $I \leqslant \max\{1,C\}M_r\big(|f(.)|^{p(.)}\big)(x)$ which gives with 2.2

$$\big|M_r f(x)\big|^{p(x)} \leqslant \max\{1,C\}M_r\big(|f(.)|^{p(.)}\big)(x).$$

Taking supremum on both sides we obtain

$$\big|M_{(c)} f(x)\big|^{p(x)} \leqslant \max\{1,C\}M_{(c)}\big(|f(.)|^{p(.)}\big)(x).$$

which gives with 2.1 the assertion of our lemma. □

LEMMA 2.7. *Let $p \in \mathcal{L}$, $1 < p_*$ and $f \in \mathcal{G}_p$. Then*

$$\int_{\mathbb{R}^n}|Mf(x)|^{p(x)}dx < \infty.$$

Proof. Set $q(x) = \frac{p(x)}{p_*}$ and $h(x) = |f(x)|^{q(x)}$. Then $h \in L^{p_*}$ and according to Theorem 1.2 we have

$$\int_{\mathbb{R}^n}|Mh(x)|^{p_*}\mathrm{d}x \leqslant M_{p_*}\int_{\mathbb{R}^n}|h(x)|^{p_*}\mathrm{d}x < \infty.$$

It follows with the easy fact $q \in \mathcal{L}$ and with Lemma 2.6

$$\begin{aligned}\int_{\mathbb{R}^n} |Mf(x)|^{p(x)}\mathrm{d}x &= \int_{\mathbb{R}^n} (|Mf(x)|^{q(x)})^{p_*}\mathrm{d}x \leqslant C^{p_*}(q)\int_{\mathbb{R}^n}\Big(M\big(f(.)^{q(.)}\big)(x)\Big)^{p_*}\mathrm{d}x \\ &\leqslant C^{p_*}(q)\int_{\mathbb{R}^n}\big(M(f(.)^{q(.)})(x)\big)^{p_*}\mathrm{d}x \leqslant C^{p_*}(q)\int_{\mathbb{R}^n}\big(Mh(x)\big)^{p_*}\mathrm{d}x \\ &\leqslant C^{p_*}(q)M_{p_*}\int_{\mathbb{R}^n}\big(h(x)\big)^{p_*}\mathrm{d}x < \infty\end{aligned}$$

which finishes the proof. □

Now, we will find a condition to the behavior of $p(x)$ near to infinity to guarantee the boundedness of M.

DEFINITION 2.8. Let $\varepsilon \in \mathfrak{M}(\mathbb{R}^n)$. Denote $\mathbb{P}(\varepsilon) = \{x \in \mathbb{R}^n; \varepsilon(x) > 0\}$. Say that $\varepsilon \in \mathscr{P}$ if there exists a positive real number c such that

$$\int_{\mathbb{P}(\varepsilon)} \varepsilon(x)c^{1/\varepsilon(x)}\mathrm{d}x < \infty. \tag{2.3}$$

REMARK 2.9. It is easy to see that $|\varepsilon| \in \mathcal{P}$ if and only if $\varepsilon \in \mathcal{P}$ and $-\varepsilon \in \mathcal{P}$.

LEMMA 2.10. *Let $K > 0$ and $\alpha \in \mathfrak{M}(\mathbb{R}^n)$ satisfy $0 \leqslant \alpha(x) \leqslant K$ for $x \in \mathbb{R}^n$. Let $\varepsilon \in \mathcal{P}$. Then $\alpha\varepsilon \in \mathcal{P}$.*

Proof. Let c satisfy (2.3). Without loss of generality we can assume $c \leqslant 1$. Set $d = c^K$. Let us estimate

$$\int_{\mathbb{P}(\alpha\varepsilon)} \alpha(x)\varepsilon(x)d^{1/(\alpha(x)\varepsilon(x))}\mathrm{d}x.$$

Since $0 \leqslant \alpha(x) \leqslant K$, we have $d = c^K \leqslant c^{\alpha(x)}$ and using the simple fact that $\mathbb{P}(\alpha\varepsilon) \subset \mathbb{P}(\varepsilon)$ we obtain

$$\begin{aligned}\int_{\mathbb{P}(\alpha\varepsilon)} \alpha(x)\varepsilon(x)d^{1/(\alpha(x)\varepsilon(x))} &= \int_{\mathbb{P}(\alpha\varepsilon)} \alpha(x)\varepsilon(x)(c^K)^{1/(\alpha(x)\varepsilon(x))}\mathrm{d}x \\ &\leqslant K\int_{\mathbb{P}(\alpha\varepsilon)} \varepsilon(x)(c^{\alpha(x)})^{1/(\alpha(x)\varepsilon(x))}\mathrm{d}x \leqslant K\int_{\mathbb{P}(\varepsilon)} \varepsilon(x)c^{1/\varepsilon(x)}\mathrm{d}x < \infty,\end{aligned}$$

which finishes the proof. □

LEMMA 2.11. *Let $M \subset \mathbb{R}^n$, $\varepsilon(x) \leqslant 1$ and $\varepsilon \in \mathscr{P}$. Assume $0 \leqslant g(x) \leqslant 1$, $\int_M g(x)dx < \infty$. Then $\int_M g(x)^{1-\varepsilon(x)}dx < \infty$.*

Proof. Denote $F_1 = \{x \in M; \varepsilon(x) < 0\}$, $F_2 = \{x \in M \setminus F_1; g(x) > \varepsilon(x)c^{1/\varepsilon(x)}\}$ and $F_3 = \{x \in M \setminus F_1; g(x) \leqslant \varepsilon(x)c^{1/\varepsilon(x)}\}$. Since F_1, F_2, F_3 are pair-wise disjoint and $M = F_1 \cup F_2 \cup F_3$ we have

$$\int_M g(x)^{1-\varepsilon(x)}\mathrm{d}x = \int_{F_1} g(x)^{1-\varepsilon(x)}\mathrm{d}x + \int_{F_2} g(x)^{1-\varepsilon(x)}\mathrm{d}x + \int_{F_3} g(x)^{1-\varepsilon(x)}\mathrm{d}x. \tag{2.4}$$

Let $x \in F_1$. Since $\varepsilon(x) < 0$ we have $g(x)^{-\varepsilon(x)} \leqslant 1$ and so,

$$\int_{F_1} g(x)^{1-\varepsilon(x)}\mathrm{d}x \leqslant \int_{F_1} g(x)\mathrm{d}x < \infty. \tag{2.5}$$

Let $x \in F_2$. Since $\varepsilon(x) > 0$ we have $\varepsilon(x)^{-\varepsilon(x)} \leqslant e^{1/e}$ which gives with $g(x) > \varepsilon(x)c^{1/\varepsilon(x)}$ that $g(x)^{1-\varepsilon(x)} \leqslant g(x)\big(\varepsilon(x)c^{1/\varepsilon(x)}\big)^{-\varepsilon(x)} < c^{-1}e^{1/e}g(x)$ and

$$\int_{F_2} g(x)^{1-\varepsilon(x)}\mathrm{d}x \leqslant \frac{e^{1/e}}{c}\int_{F_2} g(x)\mathrm{d}x < \infty. \tag{2.6}$$

Let $x \in F_3$. Then $0 \leqslant g(x) \leqslant \varepsilon(x)c^{1/\varepsilon(x)}$ and since $\varepsilon \in \mathcal{P}$ we have by (2.3)

$$\int_{F_3} g(x)^{1-\varepsilon(x)}\mathrm{d}x \leqslant \int_{F_3} \varepsilon(x)c^{1/\varepsilon(x)}\big(\varepsilon(x)c^{1/\varepsilon(x)}\big)^{-\varepsilon(x)}\mathrm{d}x \leqslant \frac{e^{1/e}}{c}\int_{F_3} \varepsilon(x)c^{1/\varepsilon(x)}dx < \infty$$

which proves with (2.4), (2.5) and (2.6) the lemma. □

LEMMA 2.12. *Let $p, q \in \mathfrak{B}(\mathbb{R}^n)$ and $|p-q| \in \mathcal{P}$. Assume that $0 \leqslant f(x) \leqslant 1$ a.e. in $\mathbb{R}^n$. Then $\int_{\mathbb{R}^n} f(x)^{p(x)}dx < \infty$ if and only if $\int_{\mathbb{R}^n} f(x)^{q(x)}dx < \infty$.*

Proof. Assume $\int_{\mathbb{R}^n} f(x)^{p(x)}\mathrm{d}x < \infty$ and set $g(x) = f(x)^{p(x)}$, $\varepsilon(x) = \frac{p(x)-q(x)}{p(x)}$. By Lemma 2.10 we obtain $|\varepsilon| \in \mathcal{P}$ and by Remark 2.9 we have $\varepsilon \in \mathcal{P}$. Since $\varepsilon(x) < 1$ and $\int_{\mathbb{R}^n} g(x)\mathrm{d}x < \infty$ we obtain by Lemma 2.11

$$\int_{\mathbb{R}^n} f(x)^{q(x)}\mathrm{d}x = \int_{\mathbb{R}^n} g(x)^{1-\varepsilon(x)}\mathrm{d}x < \infty$$

and the proof follows. □

Let us introduce a class of functions p which describes a behavior of p at the infinity.

DEFINITION 2.13. Let $p \in \mathfrak{B}(\mathbb{R}^n)$. Say that $p \in \mathcal{N}$ if there is a real number $p_\infty > 1$ such that $|p - p_\infty| \in \mathcal{P}$.

The following theorem is the main result of this paper.

THEOREM 2.14. *Let $p \in \mathcal{L} \cap \mathcal{N}$, $1 < p_*$. Then M is bounded on $L^{p(x)}(\mathbb{R}^n)$.*

Proof. Assume $\int_{\mathbb{R}^n} |f(x)|^{p(x)}\mathrm{d}x \leqslant 1$. By Lemma 1.7 it suffices to prove

$$\int_{\mathbb{R}^n} |Mf(x)|^{p(x)}dx < \infty.$$

Set $f_1 = f\chi_{\{x;|f(x)|\geqslant 1\}}$, $f_2 = f\chi_{\{x;|f(x)|<1\}}$. Then $f = f_1 + f_2$ and so, $Mf(x) \leqslant Mf_1(x) + Mf_2(x)$. It gives

$$\int_{\mathbb{R}^n} |Mf(x)|^{p(x)}\mathrm{d}x \leqslant 2^{p^*-1}\Big(\int_{\mathbb{R}^n} |Mf_1(x)|^{p(x)}\mathrm{d}x + \int_{\mathbb{R}^n} |Mf_2(x)|^{p(x)}\mathrm{d}x\Big). \tag{2.7}$$

Since $f_1 \in \mathcal{G}_{p(.)}$ we obtain by Lemma 2.6

$$\int_{\mathbb{R}^n} |Mf_1(x)|^{p(x)} \mathrm{d}x < \infty. \tag{2.8}$$

Now, let us estimate $\int_{\mathbb{R}^n} |Mf_2(x)|^{p(x)} \mathrm{d}x$. Remark that

$$\int_{\mathbb{R}^n} |f_2(x)|^{p(x)} dx \leqslant 1.$$

Since $|f_2(x)| < 1$ for all $x \in \mathbb{R}^n$ and $|p(x) - p_\infty| \in \mathcal{P}$ we obtain by Lemma 2.12

$$\int_{\mathbb{R}^n} |f_2(x)|^{p_\infty} dx < \infty.$$

By Proposition 1.2 we have

$$\int_{\mathbb{R}^n} |Mf_2(x)|^{p_\infty} dx < \infty.$$

Now, the fact $|f_2(x)| < 1$ for all $x \in \mathbb{R}^n$ immediately yields $|Mf_2(x)| \leqslant 1$ for all $x \in \mathbb{R}^n$ and thus, we can use again Lemma 2.12 to obtain

$$\int_{\mathbb{R}^n} |Mf_2(x)|^{p(x)} dx < \infty.$$

which finishes with (2.7) and (2.8) the proof. □

3. Counter-example

Let us define a wider class of functions p than $\mathcal{N}$.

DEFINITION 3.1. Let $p \in \mathfrak{B}(\mathbb{R}^n)$. Say that $p \in \mathcal{A}$ if there exists a real number $p_\infty > 1$ such that

$$|\{x \in \mathbb{R}^n; |p(x) - p_\infty| \geqslant \delta\} \cap B_r^c| \to 0 \text{ provided } r \to \infty$$

for each $\delta > 0$.

LEMMA 3.2. *The inclusion $\mathcal{N} \subset \mathcal{A}$ holds.*

Proof. Assume $p \in \mathcal{N}$ and set $\varepsilon(x) = |p(x) - p_\infty|$. Then we can find $0 < c \leqslant 1$ such that

$$\int_{\mathbb{R}^n} \varepsilon(x) c^{1/\varepsilon(x)} \mathrm{d}x < \infty. \tag{3.1}$$

Assume $\varepsilon \notin \mathcal{A}$. Then there exist $\delta > 0$, $\eta > 0$ such that $|\{x \in \mathbb{R}^n; \varepsilon(x) \geqslant \delta\} \cap B_r^c| \geqslant \eta$ for each $r > 0$ and consequently, setting $M = \{x \in \mathbb{R}^n; \varepsilon(x) \geqslant \delta\}$ we obtain $|M| = \infty$. Since $c \leqslant 1$ we have $c^{1/\delta} \leqslant c^{1/\varepsilon(x)}$ for all $x \in M$ and so,

$$\int_{\mathbb{R}^n} \varepsilon(x) c^{1/\varepsilon(x)} \mathrm{d}x \geqslant \int_M \varepsilon(x) c^{1/\varepsilon(x)} \mathrm{d}x \geqslant \delta c^{1/\delta} \int_M \mathrm{d}x = \infty$$

which is a contradiction with (3.1). $\square$

It is not so difficult to find a Lipschitz function $p \in \mathfrak{B}(\mathbb{R})$ with $1 < p_*$ such that M is unbounded on $L^{p(x)}(\mathbb{R})$. In [4] the example of a bounded Lipschitz function p, $1 < p_*$, is found such that average operators T_a given by $Tf(x) = \frac{1}{a}\int_x^{x+a} f(t)\mathrm{d}t$ are unbounded on $L^{p(x)}([0,\infty))$ for all $a > 0$. Extending p to $(-\infty, 0]$ by $p(-x) = p(x)$ we immediately obtain that M is unbounded on $L^{p(x)}(\mathbb{R})$. But this function p from [4] satisfies $p \notin \mathcal{A}$. From this point of view it would be interesting to find a function $p \in \mathcal{A} \cap \mathcal{L}$ such that M is unbounded on $L^{p(x)}(\mathbb{R})$. So, the rest is devoted to a construction of such p which is even Lipschitz.

Let $q > 1$ and $\{n_k\}_{k=0}^{\infty}$ be a fixed increasing sequence of integers, $n_0 = 0$. Fix a sequence $\{\varepsilon_k\}_{k=0}^{\infty}$ of real numbers, $\varepsilon_k \searrow 0$, $\varepsilon_0 = 1$ and define a function $p : \mathbb{R} \to [1,\infty)$ by

$$p(x) = \begin{cases} q + \varepsilon_0 \text{ if } x = 0; \\ q + \varepsilon_k \text{ if } x \in [1 + n_{k-1}, n_k],\ k \in \mathbb{N}; \\ p \text{ is linear if } x \in [n_{k-1}, n_{k-1} + 1]; \\ p(-x) \text{ for each } x < 0. \end{cases} \tag{3.2}$$

Clearly, $p_* = p_\infty = q > 1$, $p^* = q + \varepsilon_0 < \infty$. Since $\lim_{|x|\to\infty} p(x) = q$ we have $p \in \mathcal{A}$. Moreover, since p is continuous and

$$\max\{|p'_-(x)|, |p'_+(x)|\} \leqslant \max\{\varepsilon_k; k \geqslant 0\} \leqslant 1$$

for all $x \in \mathbb{R}$ we have that p is Lipschitz function and so, $p \in \mathcal{L}$. It proves $p \in \mathcal{A} \cap \mathcal{L}$.

LEMMA 3.3. *Let the sequence $\{n_k\}_{k=0}^{\infty}$ satisfy $n_1 - n_0 \geqslant 2$ and let $\{n_k - n_{k-1}\}_{k=1}^{\infty}$ be non-decreasing. Assume that $(n_k - n_{k-1})^{\varepsilon_k - \varepsilon_{k-1}}$ is unbounded. Then M is unbounded on $L^{p(x)}(\mathbb{R})$.*

Proof. Set $\delta_k := n_k - n_{k-1}$. Since $\delta_k^{\varepsilon_k - \varepsilon_{k-1}}$ is unbounded then $(\delta_k - 1)^{(\varepsilon_k - \varepsilon_{k-1})/(q+\varepsilon_k)}$ is unbounded, too. Then there exists a sequence of positive real numbers $\{b_k\}_{k=1}^{\infty}$ such that

$$\sum_{k=1}^{\infty} b_k \leqslant 1 \text{ and } \sum_{k=1}^{\infty} (\delta_k - 1)^{(\varepsilon_k - \varepsilon_{k-1})/(q+\varepsilon_k)} b_k = \infty. \tag{3.3}$$

Set

$$f(x) = \sum_{k=0}^{\infty} \left(\frac{b_k}{\delta_k - 1}\right)^{1/q+\varepsilon_k} \chi_{[1+n_{k-1}, n_k]}(x)$$

Estimate $I = \int_{\mathbb{R}} |f(x)|^{p(x)}\mathrm{d}x$. Clearly,

$$I = \sum_{k=1}^{\infty} \int_{1+n_{k-1}}^{n_k} \left(\left(\frac{b_k}{\delta_k - 1}\right)^{1/(q+\varepsilon_k)}\right)^{(q+\varepsilon_k)} \mathrm{d}x = \sum_{k=1}^{\infty} (\delta_k - 1)\frac{b_k}{\delta_k - 1} \leqslant 1.$$

Now, fix $k \geqslant 1$ and $x \in [1+n_k, n_{k+1}]$. Then

$$\begin{aligned} Mf(x) &\geqslant \tfrac{1}{(x-n_{k-1}-1)} \int_{n_{k-1}+1}^{x} f(t)\mathrm{d}t \\ &\geqslant \tfrac{1}{(x-n_{k-1}-1)} \int_{n_{k-1}+1}^{n_k} \left(\tfrac{b_k}{\delta_k-1}\right)^{1/(q+\varepsilon_k)} \mathrm{d}t \\ &= \tfrac{\delta_k-1}{x-n_{k-1}-1}\left(\tfrac{b_k}{\delta_k-1}\right)^{1/(q+\varepsilon_k)}. \end{aligned}$$

Thus, denoting $J = \int_{\mathbb{R}} (Mf(x))^{p(x)}\mathrm{d}x$ we have

$$\begin{aligned} J &\geqslant \sum_{k=1}^{\infty} \int_{1+n_k}^{n_{k+1}} \left(\frac{\delta_k-1}{x-n_{k-1}-1}\left(\frac{b_k}{\delta_k-1}\right)^{1/(q+\varepsilon_k)}\right)^{(q+\varepsilon_{k+1})} \mathrm{d}x \qquad (3.4) \\ &= \sum_{k=1}^{\infty} (\delta_k-1)^{q+\varepsilon_{k+1}} \left(\frac{b_k}{\delta_k-1}\right)^{(q+\varepsilon_{k+1})/(q+\varepsilon_k)} \int_{1+n_k}^{n_{k+1}} \frac{\mathrm{d}x}{(x-n_{k-1}-1)^{(q+\varepsilon_{k+1})}}. \end{aligned}$$

Using the assumption $2 \leqslant \delta_k \leqslant \delta_{k+1}$ we obtain

$$\begin{aligned} &\int_{1+n_k}^{n_{k+1}} \frac{\mathrm{d}x}{(x-n_{k-1}-1)^{(q+\varepsilon_{k+1})}} \\ &= \frac{1}{q+\varepsilon_{k+1}-1}\left(\frac{1}{\delta_k^{q+\varepsilon_{k+1}-1}} - \frac{1}{(\delta_k+\delta_{k+1}-1)^{q+\varepsilon_{k+1}-1}}\right) \\ &\geqslant \frac{1}{q+\varepsilon_{k+1}-1}\left(\frac{1}{\delta_k^{q+\varepsilon_{k+1}-1}} - \frac{1}{(2\delta_k-1)^{q+\varepsilon_{k+1}-1}}\right) \\ &\geqslant \frac{1}{q+\varepsilon_{k+1}-1}\left(\frac{1}{\delta_k^{q+\varepsilon_{k+1}-1}} - \frac{1}{(2\delta_k-\frac{\delta_k}{2})^{q+\varepsilon_{k+1}-1}}\right) \\ &\geqslant \frac{1}{q+\varepsilon_0-1}\left(1-\left(\frac{2}{3}\right)^{q-1}\right)\frac{1}{\delta_k^{q+\varepsilon_{k+1}-1}} \\ &= \frac{1}{q+\varepsilon_0-1}\left(1-\left(\frac{2}{3}\right)^{q-1}\right)\left(\frac{\delta_k-1}{\delta_k}\right)^{q+\varepsilon_{k+1}-1}\frac{1}{(\delta_k-1)^{q+\varepsilon_{k+1}-1}} \\ &\geqslant \frac{2^{1-q-\varepsilon_0}}{q+\varepsilon_0-1}\left(1-\left(\frac{2}{3}\right)^{q-1}\right)\frac{1}{(\delta_k-1)^{q+\varepsilon_{k+1}-1}} = \frac{C(q,\varepsilon_0)}{(\delta_k-1)^{q+\varepsilon_{k+1}-1}}. \end{aligned}$$

By (3.3) it is $b_k \leqslant 1$ which gives with (3.4)

$$\begin{aligned} J &\geqslant C(q,\varepsilon_0)\textstyle\sum_{k=1}^{\infty}(\delta_k-1)\left(\tfrac{b_k}{\delta_k-1}\right)^{(q+\varepsilon_{k+1})/(q+\varepsilon_k)} \\ &= C(q,\varepsilon_0)\textstyle\sum_{k=1}^{\infty}(\delta_k-1)^{(\varepsilon_k-\varepsilon_{k+1})/(q+\varepsilon_k)} b_k^{(q+\varepsilon_{k+1})/(q+\varepsilon_k)} \\ &\geqslant C(q,\varepsilon_0)\textstyle\sum_{k=1}^{\infty}(\delta_k-1)^{(\varepsilon_k-\varepsilon_{k+1})/(q+\varepsilon_k)} b_k = \infty \end{aligned}$$

and the lemma is proved. □

THEOREM 3.4. *There exists $p \in \mathcal{A} \cup \mathcal{L}$ such that M is unbounded on $L^{p(x)}(\mathbb{R})$.*

Proof. Let $q > 1$ be an arbitrary. Take a sequence of integers $\{\delta_k\}_{k=1}^{\infty}$ such that $\delta_1 \geqslant 1$, $\delta_k \nearrow \infty$ and set $\varepsilon_k = \frac{1}{k}$. Define a sequence $\{n_r\}_{r=1}^{\infty}$ by

$$n_r = \begin{cases} n_0 = 0; \\ n_r = \sum_{k=1}^{r} 2\delta_k^{k(k-1)}, \ r \geqslant 1. \end{cases}$$

Since $n_r - n_{r-1} = 2\delta_r^{r(r-1)}$ we have $n_1 - n_0 \geqslant 2$ and $n_r - n_{r-1}$ is non-decreasing. Moreover,

$$(n_r - n_{r-1})^{\varepsilon_r - \varepsilon_{r-1}} = \left(2\delta_r^{r(r-1)}\right)^{1/r(r-1)} \geqslant \delta_r.$$

Thus, $(n_r - n_{r-1})^{\varepsilon_r - \varepsilon_{r-1}}$ is unbounded. Let p be given by (3.2). Since p is Lipschitz we have $p \in \mathcal{A} \cap \mathcal{L}$. By Lemma 3.3 the operator M is unbounded on $L^{p(x)}(\mathbb{R})$ which finishes the proof. □

REFERENCES

[1] C. BENNET, R. SHARPLEY, *Interpolations of operators*, Pure and Apl. Math., **129**, Academic Press, New York, 1988

[2] L. DIENING, *Maximal function on Generalised Lebesgue Spaces* $L^{p(\cdot)}$ University of Freiburg, preprint, 2002, 2000

[3] D. E. EDMUNDS, J. LANG, A. NEKVINDA, *On* $L^{p(x)}$ *norms*, Proc. R. Soc. Lond. A **455**, 1999, 219–225

[4] D. E. EDMUNDS, A. NEKVINDA, *Averaging operators on* $\ell^{\{p_n\}}$ *and* $L^{p(x)}$, Math. Inequal. Appl., to appear

[5] D. E. EDMUNDS J. RÁKOSNÍK, *Density of smooth functions in* $W^{k,p(x)}(\Omega)$, Proc. Royal Soc. London A, **437**, 1993, 153–167

[6] D. E. EDMUNDS, J. RÁKOSNÍK, *Sobolev embeddings with variable exponent*, Studia Math. **143**, 2000, 267–293

[7] E. HEWITT, K. STROMBERG, *Real and Abstract Analysis*, Springer-Verlag Berlin Heidelberg New York, 1965

[8] O. KOVÁČIK, J. RÁKOSNÍK, *On spaces* $L^{p(x)}$ *and* $W^{k,p(x)}$, Czech. Math. J., **41**, 1996, 167–177

[9] A. NEKVINDA, *Equivalence of* $\ell^{\{p_n\}}$ *norms and shift operators*, Math. Inequal. Appl. **5**, 4, 2002, 711–724

[10] L. PICK, M. RUŽIČKA, *An example of a space* $L^{p(x)}$ *on which the Hardy–Littlewood maximal operator is not bounded*, Expositiones Mathematicae **19**, 2001, 369–371

[11] M. RUŽIČKA, it Electrorheological fluids: mathematical modelling and existence theory, Habilitationsschrift, Universität Bonn (1998)

[12] M. RUŽIČKA, *Flow of shear dependent electrorheological fluids*, C. R. Acad. Sci. Paris Série, **I 329**, 1999, 393–398

[13] S. G. SAMKO, *The density of* $C_0^{\infty}(\mathbb{R}^n)$ *in generalized Sobolev spaces* $W^{m,p(x)}(\mathbb{R}^n)$, Soviet Math. Doklady, **60**, 1999, 382–385

(Received December 20, 2003)

Aleš Nekvinda
Department of Mathematics, Faculty of Civil Engineering
Czech Tehnical University, Thákurova 7
16629 Prague 6, Czech Republic
e-mail: nekvinda@fsv.cvut.cz