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A CLASS OF RELAXED $\gamma - r$–COCOERCIVE NONLINEAR VARIATIONAL INEQUALITIES AND CONVERGENCE OF PROJECTION METHODS

RAM U. VERMA

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Abstract. Let K be a nonempty closed convex subset of a real Hilbert space H. Approximation-solvability of a class of relaxed γ - r -cocoercive nonlinear variational inequality (NVI) problems, based on the convergence of projection methods, is discussed as follows: find an elements $x^* \in K$ such that

$$\langle T(x^*), x - x^* \rangle \geqslant 0 \quad \forall x \in K$$

where $T : K \to H$ ia a nonlinear mapping on K.

1. Introduction

Recently, Nie et al. [2], based on Verma [3], introduced a new system of nonlinear strongly monotone variational inequalities and studied, based on the convergence of a system of iterative algorithms, the approximation-solvability of this system in a Hilbert space setting. Iterative procedures have been applied widely to problems arising from complementarity, convex quadratic programming, and variational problems. On the top of that, numerous numerical computations/experiments have been carried out in the context of the approximatino-solvability of strongly monotone variational inequalities, especially in $\mathbb{R}^n$. The author [4] introduced the class of partially relaxed monotone mappings, which is more general than the class of strongly monotone mappings, as well as cocoercive mappings, Zhu and Marcotte [9] studied a class of cocoercive variational inequalities based on the convergence of an iterative scheme. For more details on the approximation-solvability of general nonlinear variational inequalities, we refer to [1-9].

Here in this paper, first we intend to present the notion of the relaxed $\gamma - r$-cocoercive nonlinear mappings, and then we consider, based on the projection method, the approximation-solvability of a class of nonlinear relaxed $\gamma - r$-cocoercive variational inequalities in a Hilbert space setting. The obtained results complement the results of Verma [3, 4], He and He [1], Nie et al. [2] and others.

Let H be a real Hilbert space with the inner product $<,>$ and norm $||.||$. Let $T : K \to H$ be any mapping on K and K be a closed convex subset of H. We consider

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a class of nonlinear variational inequality (abbreviated as NVI) problems as follows: determine elements $x^* \in K$ such that

$$\langle T(x^*), x - x^* \rangle \geqslant 0 \quad \forall x \in K \tag{1.1}$$

The NVI (1.1) problem is equivalent to the following projection formula

$$x^* = P_K[x^* - \rho T(x^*)] \quad \text{for} \quad \rho > 0,$$

where P_K is the projection of H onto K.

Let K be a closed convex cone of H. The NVI (1.1) problem is equivalent to a class of nonlinear complementarities (abbreviated as NC): find the elements $x^* \in K$ such that $T(x^*) \in K^*$ and

$$\langle \rho T(x^*), x^* \rangle = 0 \quad \text{for} \quad \rho > 0, \tag{1.2}$$

where K^* is a polar cone to K defined by

$$K^* = \{f \in H : \langle f, x \rangle \geqslant 0 \quad \forall x \in K\}.$$

Now we need to recall the following auxiliary result, most commonly used in the context of approximation-solvability of nonlinear variational inequality problems based on iterative procedures.

LEMMA 1.1. *For an element $z \in H$, we have*

$$x \in K \quad \text{and} \quad \langle x - z, y - x \rangle \geqslant 0 \quad \forall y \in K \quad \text{if and only if} \quad x = P_K(z).$$

A mapping $T : H \to H$ is called monotone if for each $x, y \in H$, we have

$$\langle T(x) - T(y), x - y \rangle \geqslant 0.$$

A mapping $T : H \to H$ is called r-strongly monotone if for each $x, y \in H$, we have

$$\langle T(x) - T(y), x - y \rangle \geqslant r||x - y||^2 \quad \text{for a constant } r > 0.$$

This implies that

$$||T(x) - T(y)|| \geqslant r||x - y||,$$

that is, T is r-expansive, and when r = 1, it is expansive. The mapping T is called s-Lipschitz continuous (or Lipschitzian) if there exists a constant $s \geqslant 0$ such that

$$||T(x) - T(y)|| \leqslant s||x - y|| \quad \forall x, y \in H.$$

T is called μ-cocoercive if for each $x, y \in H$, we have

$$\langle T(x) - T(y), x - y \rangle \geqslant \quad \mu||T(x) - T(y)||^2 \quad \text{for a constant } \mu > 0.$$

Clearly, every μ-cocoercive mapping T is $(1/\mu)$-Lipschitz continuous.

We can easily see that the following implications on monotonicity, strong monotonicity and expansiveness hold:

$$\begin{array}{c}\text{strong monotonicity} \Rightarrow \text{monotonicity}\\ \Downarrow \\ \text{expansiveness}\end{array}$$

T is called relaxed γ -cocoercive if there exists a constant $\gamma > 0$ such that

$$\langle T(x) - T(y), x - y\rangle \geqslant (-\gamma)||T(x) - T(y)||^2 \quad \forall x, y \in H$$

T is said to be relaxed $\gamma - r$-cocoercive if there exist constants $\gamma, r > 0$ such that

$$\langle T(x) - T(y), x - y\rangle \geqslant (-\gamma)||T(x) - T(y)||^2 + r||x - y||^2 \quad \forall x, y \in H.$$

For $\gamma = 0$, T is r-strongly monotone. This class of mappings are more general than the class of strongly monotone mappings since the r-strong monotonicity implies the relaxed $\gamma - r$-cocoercivity. Based on this, we have the following implication:

$$\begin{array}{c}\text{strong } r\text{-monotonicity}\\ \Downarrow \\ \text{relaxed } \gamma - r\text{-cocerciveness}\end{array}$$

2. Convergence of Projection Methods

In this section we present the convergence of projection methods in the context of the approximation-solvability of the NVI (1.1) problem.

ALGORITHM 2.1. For an arbitrarily chosen initial point $\mathrm{x}^0 \in \mathrm{K}$ with compute the sequence $\{\mathrm{x}^k\}$ such that

$$x^{k+1} = (1 - a^k)x^k + a^k P_K[x^k - \rho T(x^k))],$$

where P_K is the projection of H onto K, $\rho > 0$ is a constant, and

$$0 \leqslant a^k \leqslant 1 \quad \text{and} \quad \sum_{k=0} a^k = \infty.$$

We now present, based on Algorithm 2.1, the approximation-solvability of the NVI (1.1) problem involving relaxed γ -r-cocoercive and μ -Lipschitz continuous mappings in a Hilbert space setting.

THEOREM 2.1. *Let H be a real Hilbert space and K a nonempty closed convex subset of H. Let $T : K \to H$ be relaxed γ -r-cocoercive and g- μ -Lipschitz continuous. Suppose that $x^* \in K$ is a solution to the NVI (1.1), the sequence $\{x^k\}$ is generated by Algorithm 2.1, and*

$$0 \leqslant a^k \leqslant 1 \quad \text{and} \quad \sum_{k=0}^{\infty} a^k = \infty.$$

Then the sequences $\{x^k\}$ converges to x^* for

$$0 < \rho < 2(r - \gamma\mu^2)/\mu^2.$$

Proof. Since x^* is a solution of the NVI (1.1) problem, it follows that

$$x^* = P_K[x^* - \rho T(x^*)].$$

Applying Algorithm 2.1, we have

$$\begin{aligned} ||x^{k+1} - x^*|| &= ||(1-a^k)x^k + a^k P_K[x^k - \rho T(x^k)] \\ &\quad - (1-a^k)x^* - a^k P_K[x^* - \rho T(x^*)]|| \\ &\leqslant (1-a^k)||x^k - x^*|| \\ &\quad + a^k ||P_K[x^k - \rho T(x^k)] - P_K[x^* - \rho T(x^*)]|| \\ &\leqslant (1-a^k)||x^k - x^*|| \\ &\quad + a^k ||x^k - x^* - \rho[T(x^k) - T(x^*)]||. \end{aligned} \tag{2.1}$$

Since T is relaxed γ -r-cocoercive and μ -Lipschitz continuous, we have

$$\begin{aligned} &||x^k - x^* - \rho[T(x^k) - T(x^*)]||^2 \\ &\quad = ||x^k - x^*||^2 - 2\rho\langle T(x^k) - T(x^*), x^k - x^*\rangle \\ &\qquad + \rho^2 ||T(x^k) - T(x^*)||^2 \\ &\quad \leqslant ||x^k - x^*||^2 + 2\rho\gamma||T(x^k) - T(x^*)||^2 - 2\rho r||x^k - x^*||^2 \\ &\qquad + (\rho^2\mu^2)||x^k - x^*||^2 \\ &\quad \leqslant ||x^k - x^*||^2 + 2\rho\gamma\mu^2||x^k - x^*||^2 + (\rho\mu)^2||x^k - x^*||^2 \\ &\qquad - 2\rho r||x^k - x^*||^2 \\ &\quad = [1 - 2\rho r + 2\rho\gamma\mu^2 + (\rho\mu)^2]||x^k - x^*||^2. \end{aligned}$$

As a result, we have

$$||x^{k+1} - x^*|| \leqslant (1-a^k)||x^k - x^*|| + a^k\theta||x^k - x^*||, \tag{2.2}$$

where $\theta = [1 - 2\rho r + 2\rho\gamma\mu^2 + (\rho\mu)^2]^{1/2}$.

It follows from (2.2) that

$$\begin{aligned} ||x^{k+1} - x^*|| &\leqslant [1 - (1-\theta)a^k]||x^k - x^*|| \\ &\leqslant \prod_{j=0}^{k} [1 - (1-\theta)a^j]||x^0 - x^*||, \end{aligned} \tag{2.3}$$

where $\theta = [1 - 2\rho r + 2\rho\gamma\mu^2 + (\rho\mu)^2]^{1/2} < 1$.

Since $\theta < 1$ and $\sum_{k=0}^{\infty} a^k$ is divergent, it implies in light of [7] that

$$\lim_{k\to\infty} \prod_{j=0}^{k} [1 - (1-\theta)a^j] = 0.$$

Hence, the sequence $\{x^k\}$ converges to x^* by (2.3) for

$$0 < \rho < 2(r - \gamma\mu^2)/\mu^2.$$

This completes the proof. □

COROLLARY 2.1. [3]. *Let H be a real Hilbert space and K a nonempty closed convex subset of H. Let T: K $\to$ H be r-strongly monotone and μ -Lipschitz continuous. Suppose that $x^* \in K$ is a solution to the NVI (1.1), the sequence $\{x^k\}$ is generated by Algorithm 2.1, and*

$$0 \leqslant a^k \leqslant 1 \quad \text{and} \quad \sum_{k=0}^{\infty} a^k = \infty.$$

Then sequence $\{x^k\}$ converges to x^* for

$$0 < \rho < 2r/\mu^2.$$

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Ram U. Verma
Department of Mathematics
University of Toledo
Toledo, Ohio 43606
USA
e-mail: rverma@pop3.utoledo.edu
e-mail: verma99@msn.com