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A REMARK ON THE CONVERGENCE OF THE TIKHONOV REGULARIZATION WITHOUT MONOTONICITY

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Abstract. Relying on an idea of T. Pennanen, we use a localized version of maximal hypomonotonicity, and prove that it is enough to guarantee the local convergence of the Tikhonov regularization. In the process, we revisit this method which together with the help of the analysis developed by T. Pennanen allows us to obtain the desired result.

1. Introduction and preliminaries

In this note we deal with a method for finding zeroes of set-valued operators A in Hilbert spaces. Relying on the analysis developed by Pennanen for the proximal point algorithm, we establish the local convergence of the Tikhonov regularization provided that for $\gamma > 0$, the mapping $A^{-1} + \gamma I$ is maximal monotone when restricted to a neighborhood of $A^{-1}(0) \times \{0\}$.

To begin with, let us recall the following concepts which are of common use in the context of convex and nonlinearr analysis, see for example Rockafellar-Wets [4]. Throughout, $\mathcal{H}$ is a real Hilbert space, $\langle \cdot, \cdot \rangle$ denotes the associated scalar poduct and $|\cdot|$ stands for corresponding norm. An operator is said to be monotone if

$$\langle u - v, x - y \rangle \geqslant 0 \quad \text{whenever} \quad u \in A(x), v \in A(y).$$

It is said to be maximal monotone if, in addition, the graph, $gphA := \{(x,y) \in \mathcal{H} \times \mathcal{H} : y \in A(x)\}$ is not properly contained in the graph of any other monotone operator. It is well-known that for each $x \in \mathcal{H}$ and $\lambda > 0$ there is a unique $z \in \mathcal{H}$ such that $x \in (I + \lambda A)z$. The single-valued operator $J_\lambda^A := (I + \lambda A)^{-1}$ is called the resolvent of A of parameter λ. It is a nonexpansive mapping which is everywhere defined. Let us also recall that the inverse A^{-1} of A is the operator defined by $x \in A^{-1}(y) \Leftrightarrow y \in A(x)$.

In this paper we will focus our attention on the classical problem of finding a zero of a maximal monotone operator A on a real Hilbert $\mathcal{H}$, namely

$$\text{find} \quad x \in \mathcal{H} \quad \text{such that} \quad A(x) \ni 0 \tag{1.1}$$

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One of the fundamental approaches to solving (1.1) is the Tikhonov regularization. Specifically, having an arbitrary point $x_0 \in \mathcal{H}$, this method generates a sequence $\{x_n\}$ by solving the well-posed subproblem

$$0 \in Ax + \mu_n(x - x_0) \tag{1.2}$$

where μ_n is a regularization parameter having to go to zero. In other words $x_n = J^A_{\mu_n^{-1}}(x_0)$.

The fundamental advantage of Tikhonov's method is the strong convergence of the generated sequence to the solution of (1.1) which is of minimal norm in the solution set S. This is a common feature of all viscosity approximation methods. It is worth nothing that the sequence generates, for example, by the proximal point algorithm converges only weakly to a solution which is not characterized. Even when we have to solve infinite dimensional problems, numerical implementations of algorithms are certainly applied to finite-dimensional approximations. Neverthless, it is important to have convergence theory in infinite dimensional case, because it guarantees robustness and stability with respect to discretisation schemes used for obtaining finite dimensional approximations of infinite dimensional problems. Note that many real-world problems in economics and engineering are modeled in infinite-dimensional spaces. In the sequel we assume that the set of solution $S := A^{-1}(0)$ is nonempty. Indeed in a large number of variational and optimization problems the solution fails to be unique, for example when considering problems arising in phase transitions and linear mathematical programming. In such situations it important both for theoretical and numerical reasons, to describe methods which allow us to reach some particular solutions.

The rest of paper is organized as follows: In section 2, the concept of local maximal hypomonotonicity and some of its properties are recalled. In section 3, the Tikhonov regularization is revisited, its convergence is derived and used to prove the local convergence of the Tikhonov regularization.
Our analysis is based on the observation that the solution set of (1.1) coincides with that of the inclusion

$$\text{find} \quad x \in \mathcal{H} \text{ such that} \quad A_\gamma(x) \ni 0, \tag{1.3}$$

where $A_\gamma := (A^{-1} + \gamma I)^{-1}$ is the Yosida regularization of A with parameter $\gamma > 0$. As in [3], we apply an over-relaxed Tikhonov regularization to A_γ, and show that, with a judicious choice of relaxation parameters, this yields the classical Tikhonov regularization for A itself. The main interest is that for γ large enough the mapping A_γ can be locally monotone even when A is not, and the local maximal monotonicity is enough to guarantee the local convergence of the Tikhonov regularization.

2. Hypomonotone operators

In this section we state the definition of the hypomonotonicity as well as some of its key properties.

DEFINITION 2.1. Given a positive real γ and a subset $\mathcal{W} \in \mathcal{H} \times \mathcal{H}$, an operator $A : \mathcal{H} \to 2^{\mathcal{H}}$ is said to be

1. γ *-hypomonotone* if and only if

$$\langle x-y,u-v\rangle \geqslant -\gamma|x-y|^2 \quad \forall (x,u)\in gphA, (y,v)\in gphA,$$

2. *maximal* γ *-hypomonotone* if and only if A is γ -hypomonotone and in addition $gphA = gphA'$ whenever A' is γ -hypomonotone and $gphA \subset gphA'$.
3. γ *-hypomonotone* in $\mathcal{W}$ if and only if $gphA\cap\mathcal{W}$ is γ -hypomonotone
4. *maximal* γ *-hypomonotone* in $\mathcal{W}$ if and only if A is γ -hypomonotone in $\mathcal{W}$ and in addition $gphA\cap\mathcal{W}\subset gphA'\cap\mathcal{W}$.

The subdifferential of a lower C^2 function (i.e. a function which can be written as $g-h$ in a neighborhood of a point with g is finite and h is C^2), is hypomonotone (see [4]). We would like to emphasize that this property could be achieved by any extended function, $f:\mathcal{H}\to\mathbb{R}\cup\{+\infty\}$ satisfying $f(x)\geqslant -\frac{c}{2}(||x||^2+1)$ for some $c>0$, by means of its Lasry-Lions regularization, namely

$$f_{\lambda,\mu}(x)=\sup_{y\in\mathcal{H}}\inf_{u\in\mathcal{H}}(f(u)+\frac{1}{2\lambda}||u-y||^2-\frac{1}{2\mu}||y-x||^2).$$

More precisely, for all $0<\mu<\lambda<c^{-1}$, $f_{\lambda,\mu}$ is a $C^{1,1}$ function and is $\frac{1}{\mu}$ -paraconvex which amounts to saying that its gradient, $\bigtriangledown f_{\lambda,\mu}$, is hypomonotone with parameter $\frac{1}{\mu}$ (theorem 4.1, Attouch-Azé [1]). It is also easy to check that a locally Lipschitz continuous mapping is hypomonotone for every γ greater than the Lipschitz constant. Next we state a characterization of the hypomonotonicity and the maximal hypomonotonicity of an operator with the help of the monotonicity and maximal monotonicity of its Yosida approximate (see Iusem, Pannanen and Svaiter [2]).

PROPOSITION 2.1. *Let* $\gamma\geqslant 0$ *and* A *be a set-valued mapping. Then*
1. A^{-1} *is* γ *-hypomonotone if and only if* A_γ *is monotone,*
2. A^{-1} *is maximal* γ *-hypomonotone if and only if* A_γ *is maximal monotone.*

In view of the latter result and according to the fact that an operator has the same zeroes as its Yosida approximate, the set of the zeroes of a hypomonotone operator is thus closed and convex.

3. Local convergence results

Now, let us prove the convergence of the sequence generates by (1.2).

THEOREM 3.1. *Let* A *be a maximal monotone operator defined on a Hilbert space* $\mathcal{H}$. *Then for any* $x_0\in\mathcal{H}$ *one has*

$$\lim_{n\to+\infty} J^A_{\mu_n^{-1}}(x_0)=proj_S(x_0).$$

Proof. In view of

$$x_n=J^A_{\mu_n^{-1}}(x_0)\Leftrightarrow \mu_n(x_0-x_n)\in A(x_n),$$

in other words

$$|x_n|^2 \leqslant \langle\, x_n - x_0, z\rangle \,+ \langle\, x_0, x_n\rangle\,, \tag{3.4}$$

from which we infer that $\{x_n\}$ is bounded. Let $\overline{x}$ be a weak cluster point of the sequence $\{x_n\}$. By passing to the limit on a subsequence in the last inequality and by using the weak-lower semicontinuity of the norm, we obtain

$$|\overline{x}|^2 \leqslant \langle\, \overline{x} - x_0, z\rangle \,+ \langle\, x_0, \overline{x}\rangle\,,$$

that is

$$\langle\, x_0 - \overline{x}, z - \overline{x}\rangle \,\leqslant 0, \quad \text{in other words} \quad \overline{x} = proj_S(x_0).$$

As the limit does not depend on the subsequence, we obtain the weak convergence of the whole sequence to $\overline{x}$. On the other hand, by passing to the limit in (3.4), we obtain

$$\limsup_{n\to+\infty} |x_n|^2 \leqslant \langle\, \overline{x} - x_0, z\rangle \,+ \langle\, x_0, \overline{x}\rangle \quad \forall z \in S,$$

In particular for $z = \overline{x}$, we have that $\limsup_{n\to+\infty} |x_n|^2 \leqslant |\overline{x}|^2$ which ensures the norm convergence of $\{x_n\}$ to $\overline{x}$. We conclude by nothing that both the weak and the norm convergence yield the strong convergence of the sequence $\{x_n\}$. □

From this we easily infer the following corollary

COROLLARY 3.1. *In the situation of theorem 3.1. and assume that $\{\sigma_n\}$ is a scalar sequence converging to 1. Then the sequence $\{x_n\}$ generated by the rule*

$$x_n = \sigma_n J^A_{\mu_n^{-1}}(x_0) + (1 - \sigma_n)x_0 \tag{3.5}$$

converges strongly to $\overline{x} = proj_S(x_0)$.

Now we use a localized version of maximal monotonicity to show that it is enough to ensure the local convergence of the rule (3.5).
The following key result due to T. Pennanen [3] will be needed in the proof of the next proposition.

THEOREM 3.2. *Let $A : \mathcal{H} \to 2^{\mathcal{H}}$ be a set-valued mapping which is maximal monotone in $\mathcal{U} \times \mathcal{V}$, where $\mathcal{U}, \mathcal{V} \subset \mathcal{H}$ are open sets such that $0 \in \mathcal{V}$ and $S \cap \mathcal{U} \neq \emptyset$ and closed with $S \cap \mathcal{U} + B(0, \delta) \subset \mathcal{U}$ for some $\delta > 0$. Suppose that the sequences $\{\lambda_n\}$ and $\{\sigma_n\}$ satisfy*

$$\inf_n \lambda_n > 0, \quad \inf_n \sigma_n \geqslant 1 \quad \textit{and} \quad \sup_n \sigma_n < 2.$$

If $dist(x_0, S \cap \mathcal{U})$ is small enough, then there exist $\epsilon > 0$ and $\overline{x} \in S \cap \mathcal{U}$ such that $x_0 \in B(\overline{x}, \epsilon)$ and for every n

$$B(\overline{x}, \epsilon) \cap \left(\sigma_n J^A_{\lambda_n}(x_0) + (1 - \sigma_n)x_0\right) = \sigma_n J^{\overline{A}}_{\lambda_n}(x_0) + (1 - \sigma_n)x_0 \quad \forall x \in B(\overline{x}, \epsilon)$$

where $B(\overline{x}, \epsilon)$, stands for the open ball with center $\overline{x}$ and radius ϵ and $\overline{A}$ is a maximal monotone operator verifying $gph\overline{A} \cap \mathcal{U} \times \mathcal{V} = gphA \cap \mathcal{U} \times \mathcal{V}$.

In the light of the above results, we derive the following proposition.

PROPOSITION 3.1. *Set $\overline{x} := proj_{S\cap\mathcal{U}}x_0$ and assume, in addition to the hypothesis of the theorem above, that $\lim_{n\to+\infty} \sigma_n = 1$. If $d(x_0, S)$ is small enough, then there exists $\epsilon > 0$ such that $x_0 \in B(\overline{x}, \epsilon)$ and the rule*

$$x_n = B(\overline{x}, \epsilon) \cap (\sigma_n J^A_{\mu_n^{-1}}(x_0) + (1 - \sigma_n)x_0)$$

generates a unique sequence $\{x_n\}$ converging strongly to $\overline{x}$.

Proof. Followed by the fact that $\overline{A}^{-1}(0) = S \cap \mathcal{U}$ and by invoking corollary 3.1 and theorem 3.2. □

We are now able to give the local convergence result without monotonicity. First, we stress that the operator A and its Yosida regularization, A_γ, have the same zeroes. So, according to the fact that A_γ may be locally monotone even when A is not, we will use A_γ instead of A. Note that A_γ is locally maximal monotone if, for example, A^{-1} is locally maximal γ-hypomonotone. Applying in this case, the relaxed Tikhonov regularization to it, leads to a sequence converging to $\overline{x}$ since $A_\gamma^{-1}(0) = S$. Relying on the following formula

$$J^{A_\gamma}_\lambda(x) = \frac{\lambda}{\gamma + \lambda} J^A_{\gamma+\lambda}(x) + \frac{\gamma}{\gamma + \lambda} x \quad \lambda > 0 \quad \gamma \geqslant 0, \tag{3.6}$$

which easily follows from the equation $(A_\gamma)_\lambda = A_{\gamma+\lambda}$, and applying an appropriate relaxed Tikhonov regularization, we will obtain ordinary Tikhonov regularization.

THEOREM 3.3. *Let A be a set-valued operator such that A^{-1} is maximal hypomonotone with parameter $\gamma \geqslant 0$ in $\mathcal{U} \times \mathcal{V}$ where $\mathcal{U}, \mathcal{V} \subset \mathcal{H}$ are open sets such that $0 \in \mathcal{V}$ and $S \cap \mathcal{U}$ is nonempty and closed with $S \cap \mathcal{U} + B(0, \delta) \subset \mathcal{U}$ for some $\delta > 0$ and let $\overline{x} := proj_{S\cap\mathcal{U}}x_0$. If $dist(x_0, S \cap \mathcal{U})$ is small enough and $\inf_n \mu_n^{-1} > 2\gamma$, then there exists $\epsilon > 0$ such that $x_0 \in B(\overline{x}, \epsilon)$ and the rule*

$$x_n = B(\overline{x}, \epsilon) \cap J^A_{\mu_n^{-1}}(x_0) \tag{3.7}$$

defines a unique sequence $\{x_n\}$ converging strongly to $\overline{x}$.

Proof. The condition $\inf_n \mu_n^{-1} > 2\gamma$, which is automatically satisfied for indices n sufficiently large since $\lim_{n\to+\infty} \mu_n^{-1} = +\infty$ ensures that $\inf_n \lambda_n > 0, \inf_n \sigma_n \geqslant 1$ and $\sup_n \sigma_n < 2$, where $\lambda_n := \mu_n^{-1} - \gamma$ and $\sigma_n := \frac{\mu_n^{-1}}{\mu_n^{-1} - \gamma}$. Thus we may apply proposition 3.1 which gives the existence of ϵ such that $x_0 \in B(\overline{x}, \epsilon)$, and the rule

$$x_n = B(\overline{x}, \epsilon) \cap \sigma_n J^{A_\gamma}_{\lambda_n}(x_0) + (1 - \sigma)x_0$$

define a sequence $\{x_n\}$ with required properties. Furthermore, a simple calculation involving the formula (3.6) leads to the equality

$$\sigma_n J^{A_\gamma}_{\lambda_n}(x_0) + (1 - \sigma_n)x_0 = J^A_{\mu_n^{-1}}(x_0).$$

□

REMARK 3.1.

1. The relation (3.7) means that x_n is the unique solution in $B(\overline{x}, \epsilon)$ to the inclusion (1.2). Since the latter inclussion is not in general monotone, it is necessary to resrict the region when the problem is solved in order to make it well-posed.
2. In the case when A^{-1} is maximal γ-hypomonotone in the whole spacee, it is easy to derive the global convergence of the method under consideration.

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