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OPIAL'S INEQUALITY FOR ZERO–AREA CONSTRAINT

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(*communicated by D. Hinton*)

Abstract. We prove that $\int_0^1 |yy'| \leqslant \frac{1}{4}\int_0^1 y'^2$ provided $\int_0^1 y = 0$. The nontrivial part is the sharp constant $\frac{1}{4}$. Cases of equality can be abstracted from the proof. This is an (apparently much deeper) variant of a result by Opial, who proved the same estimate under Dirichlet boundary conditions on y.

1. Introduction: Outline of Result and Open Problems

The purpose of this note is to give a proof of the following theorem:

THEOREM 1. *Let y be a function with square integrable first derivative on $[0,1]$, satisfying the area constraint $\int_0^1 y(x)dx = 0$. Then the inequality*

$$\mathcal{W}[y] := \int_0^1 |yy'|\,dx \leqslant \frac{1}{4}\int_0^1 y'^2\,dx =: \frac{1}{4}\mathcal{E}[y] \tag{1}$$

holds. The constant $\frac{1}{4}$ is sharp, and equality holds if and only if $y(x) = const(x-\frac{1}{2})$.

The salient feature of this result is the sharp constant $\frac{1}{4}$; with the non-sharp constant $\frac{1}{\pi}$, it would be a trivial corollary of Cauchy-Schwarz and the one-dimensional Poincaré inequality. The inequality is a variant of an inequality by Opial [13], which has been reproved subsequently by a variety of different, elementary methods [12], [1], [9], [11], [14], and which is quoted below as Lemma 3.a. In contrast to Opial's version, where a boundary condition $y(0) = y(1) = 0$ kills the constant functions, the present theorem, under the constraint $\int y = 0$, appears to be much more complicated, and has been an open problem for quite a while. Finding sharp constants in integral inequalities of various kinds has found a lot of interest recently; see, e.g., [6] and the references therein. (No attempt to give a representative overwiew of this literature is made here, but pioneering work by Talenti [15] and Brascamp, Lieb [2] should at least be mentioned, as should be the recent survey article by Gardner [7].)

REMARK 1. [Cases of Equality] In (1), the *only* cases of equality are the functions $y(x) = \text{const}(x-\frac{1}{2})$. This has first been shown by Brown and Plum [5], under the

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assumption that an optimizer (say, maximizing $\mathcal{W}$ under area 0 and fixed $\mathcal{E}$) exists. Theorem 1. verifies their hypothesis now. They design an appropriate variation of a supposed optimizer, and exploit the Euler-Lagrange equation thus obtained. Their variation is $y \to y_\varepsilon = (1 + k_\varepsilon) \cdot (y \circ \psi_\varepsilon)$ with ψ_ε a near-identity diffeomorphism $[0, 1] \to [0, 1]$ and k_ε separately constant on $\{y > 0\}$ and $\{y < 0\}$ respectively.

The author learnt about the problem from Richard Brown, who has also posed it in an open problem section at the 8th meeting on general inequalities in Noszvaj, near Eger, Hungary, September 2002.

Following step by step through the proof of Theorem 1. gives an alternative, direct, argument for the classification of the cases of equality. We will put these arguments in a sequence of remarks, such as not to overload the existence proof, which is quite involved already.

Our existence proof is not functional analytic in nature. Desirable as a functional analytic proof may be (it would likely generalize to higher space dimensions), attempts towards such a proof have been elusive so far. The critical difficulty is the lack of continuity of the term $\int |yy'|$ with respect to the weak $W^{1,2}$ topology. The problem of the existence of an optimizer does not appear to be amenable to direct methods; or rather, as the main obstruction to such a proof are oscillations, the natural question arises whether Young measures with a corresponding relaxation of the underlying variational problem should do the trick. This does not seem hopeless, and a few comments pointing in this direction will be given in Remark 6 below. However, Young measures are not a tool in the present proof.

Let us stress this open question of an abstract existence proof in higher dimensions: For a bounded domain $\Omega \subset \mathbb{R}^n$ (say, with sufficient smoothness for the Poincaré inequality to hold), does there exist a function that saturates the inequality $\int_\Omega |u\nabla u| \leqslant c_\Omega \int_\Omega |\nabla u|^2$ (and what is the sharp constant c_Ω for particular domains like balls and rectangles)? This question can be asked both for the constraint $\int u = 0$ and for Dirichlet boundary conditions.

Our proof for $\Omega = [0, 1]$ is constructive in nature: Beginning from any function y, a series of transformations is carried out that brings y closer to saturating the inequality. More precisely, they decrease $\mathcal{E} - 4\mathcal{W}$. In the final of these transformations (Section 5.), the estimates rely on the fact that we decrease $\mathcal{E} - 4\mathcal{W}$, whereas trying to show a decrease of $\mathcal{E}/\mathcal{W}$ appears to be hopelessly complicated. Any attempt to prove the existence of an optimizer in a the sharp inequality $\int_\Omega |u\nabla u| \leqslant c_\Omega \int |\nabla u|^2$ cannot be a straightforward generalization of our proof, unless the sharp constant c_Ω is anticipated and given explicitly.

The transformations mentioned are mainly (but not exclusively) of a rearrangement nature. The construction is a blend of rather different operations, none of which seems to be dispensable, and only some of which are likely to generalize to higher dimensions. The author hopes that this work stimulates research on such higher dimensional existence proof, and also that the approach by Brown and Plum [5] will find a similar generalization, in particular as a more abstract existence proof is unlikely to produce the cases of equality.

The proof of Theorem 1. will be given in Sections 3.–5.. The discussion of the cases of equality will be found in remarks.

2. Thm 1 as a sharp-constant result

Another question, concerning methods, is worth raising, when Theorem 1. is viewed as a sharp constant result rather than an existence result for an optimizer: an L^1 function y satisfying $\int y = 0$, but not identically zero, can be interpreted (after normalization) as a pair of (absolutely continuous) probability distributions $y_+ \, dx$ and $y_- \, dx$. A rather refined method of rearrangement (that is not related to the symmetrization type rearrangements) has been developed by optimizing transportation costs from one probability distribution to the other. This method has recently been used to obtain sharp constants in Sobolev and Gagliardo-Nirenberg inequalities [6], eventually using the concept of displacement convexity [10] and the existence of an optimal transport map [3]. The idea seems most promising in cases where the exact constants are arithmetically simple numbers and the optimizers are geometrically simple, so that certain expressions in terms of the presumed optimizers saturate each one in a chain of inequalities at the same time. Theorem 1. has the ingredients of simple optimizers and nice sharp constant, plus a natural setting in terms of two probability densities. On the other hand, these methods are related to the isoperimetric inequality, as explained in Theorem 3 of [6] (this proof of the isoperimetric inequality is due to Gromov [8]). The fact that our Theorem 1. is not a higher dimensional result that would reduce to a triviality in one dimension seems to disfavor such a connection.

Can a proof be designed exploiting this structure, in the spirit of [6]? Possible, but the author's attempts in this direction have not borne fruit so far. — For a survey that illuminates the connections between sharp constants, geometric (like Brunn-Minkowski, isoperimetric) inequalities, and mass transport, see [7].

3. Normalizations and Basic Dichotomy; Transformations M and σ

It is sufficient to show inequality (1) for C^1 functions, by a density argument. By a further density argument using the Weierstrass approximation theorem, it suffices to show the inequality for polynomials, and even, for non-constant polynomials. We will actually work in a larger class that only retains one crucial feature of polynomials (convenient for exposition, albeit presumably dispensable, at the price of technicalities): We assume that y is piecewise C^1 with a *finite* number of local extrema, and a finite number of zeros (a type of approximation that has already been used in Opial's original paper [13]). This will permit us to carry out certain rearrangements combinatorially, rather than dealing with measure theoretic technicalities. Incidentally, the assumptions also imply that y is not constant on any open interval.

We can also make the following normalizations:

$$y(1) \geqslant |y(0)| \tag{2}$$

or even

$$y(1) > |y(0)| \tag{2'}$$

Indeed, for (2), if $|y(1)| < |y(0)|$ replace $x \mapsto y(x)$ with $x \mapsto y(1-x)$; then, if $y(1) < 0$, replace y with $-y$. It is justified to omit the case $|y(1)| = |y(0)|$ in (2′), because this case is immediately dealt with by the following lemma:

LEMMA 1. *(a) If $y(0) = y(1) = 0$, then (1) holds, with equality if and only if $y = \text{const}\min\{x, 1-x\}$.*
(b) If $y(0) = y(1) \neq 0$ and y has some zero in $]0,1[$, then (1) holds, with equality if and only if the 1–periodic continuation of y coincides with the 1–periodic continuation of $y(x) = \text{const}|x - \frac{1}{2}|$ up to a phase shift.
(c) If $y(0) = -y(1) \neq 0$, then (1) holds, with equality if and only if, for $x \in [0,1]$, $y(x) = \text{const}(1 - 2|x - x_0|)$ ($x_0 \in [0,1]$). Among the cases of equality, only $y(x) = \text{const}(1-2x)$ satisfies the area constraint $\int_0^1 y(x)\,dx = 0$.

Proof. Part (a) is the classical Opial inequality [13],[12],[1],[9]. Part (b) is the same inequality, applied to the 1-periodic continuation of y, on an interval of length 1 bounded by zeros of the continued y. Part (c) is part (b), applied to the absolute value $|y|$. Alternatively, it follows from the much more general theorem by Brown, Fink, Hinton [4, Thm. 1(v)]. They also show that there is a second extremal saturating the inequality; but only the one mentioned here happens to satisfy the area constraint. □

In order to prove (1) under the area constraint, we apply to y a sequence of normalizations and various kinds of rearrangements, each of which either decreases the energy $\mathcal{E}$ while leaving $\mathcal{W}$ invariant, or increases $\mathcal{W}$ while leaving $\mathcal{E}$ invariant. These normalizations and rearrangements come in two kinds: either they preserve the area constraint, in which case we may continue to use the features achieved by them as further assumptions without loss of generality, or else they may violate the area constraint. In this second case, we will always be able to apply Lemma 1. immediately, with inequality (1) arising as an immediate corollary of the lemma.

Our normalizations and rearrangements have a successive improvement of the monotonicity properties of y in mind.

First, the set $y^{-1}(\{0\})$ dissects the unit interval into finitely many open (relative $[0,1]$) subintervals in each of which it holds either $y > 0$, or $y < 0$, with y vanishing on their boundaries. (This is not to claim vanishing at $x = 0$ or $x = 1$, because in the relative topology on $[0,1]$, these points count as interior, not boundary.) By rearranging these pieces in different order, we can achieve that there are two numbers $x_\pm$, namely $0 \leqslant x_- < x_+ < 1$ such that

$$
\begin{array}{lll}
y(x) > 0 & \text{for} & 0 \leqslant x < x_- \quad (\text{empty if } x_- = 0) \\
y(x) \leqslant 0 & \text{for} & x_- \leqslant x \leqslant x_+ \\
y(x) \geqslant 0 & \text{for} & x_+ < x \leqslant 1
\end{array}
\tag{3}
$$

The originally first and last intervals $[0,?[$ and $]?,1]$ of the dissection remain at their original positions. Also note that $x_- > 0$ if and only if $y(0) > 0$. This dissecting and reassembling preserves continuity and changes neither the energy $\mathcal{E}[y] := \int_0^1 y'^2\,dx$ nor the total variation of y^2, which is twice $\mathcal{W}[y] = \int_0^1 |yy'|\,dx$. We will also need the total variation of y on certain subintervals: We define

$$\mathcal{V}_I[y] := \int_I |y'|\,dx\ ,\ \text{and}\ \mathcal{V}^I[y] := \int_{y^{-1}(I)} |y'|\,dx$$

For the sake of the cases of equality, we redo (3) without regularity assumption. Otherwise, readers may skip to Lemma 3..

REMARK 2. Normalization (2) clearly didn't require the regularity assumption for y, and also (3) can be achieved for general $y \in W^{1,2}[0,1]$ satisfying (2) and $\int y = 0$. Let $x_- := 0$ if $y(0) \leqslant 0$. Otherwise let $x_- := \max\{\xi \mid y > 0 \text{ on } [0,\xi[\}$. Moreover, decompose $y = y_+ - y_-$ with $y_+ := \max\{0, y\}$ and $y_- := \max\{0, -y\}$. We have $y_\pm \in W^{1,2}[0,1]$. With λ denoting the Lebesgue measure, let

$$\begin{aligned}
\mu_-(t) &:= x_- + \lambda\{x \in [x_-, t] \mid y(x) < 0\}, \quad z_-(t) := \textstyle\int_{x_-}^t y_-'(x)\,dx \\
\mu_0(t) &:= \mu_-(1) + \lambda\{x \in [x_-, t] \mid y(x) = 0\} \\
\mu_+(t) &:= \mu_0(1) + \lambda\{x \in [x_-, t] \mid y(x) > 0\}, \quad z_+(t) := \textstyle\int_{x_-}^t y_+'(x)\,dx
\end{aligned} \tag{4}$$

Clearly, $\mu_+(1) = 1$, and we let $\mu_0(1) =: x_+ < 1$. We define the continuous function u on $[0,1]$ piecewise, as follows:

$$\begin{array}{rcll}
u(x) &=& y(x) & \text{for } 0 \leqslant x < x_- \\
u(\mu_-(t)) &=& y(x_-) - z_-(t) & \text{for } x_- \leqslant t \leqslant 1 \quad (x_- \leqslant \mu_-(t) \leqslant \mu_-(1)) \\
u(x) &=& 0 & \text{for } \mu_-(1) \leqslant x \leqslant \mu_0(1) =: x_+ \\
u(\mu_+(t)) &=& z_+(t) & \text{for } x_- \leqslant t \leqslant 1 \quad (\mu_0(1) \leqslant \mu_+(t) \leqslant 1)
\end{array} \tag{5}$$

Proof that u is well-defined by (5) and of other claims in the remark. Clearly, if $x_- > 0$, then $y(x_-) = 0$. Let us see that the second line defines $u(x)$ correctly for $x \in [x_-, \mu_-(1)]$: If $x_- \leqslant t_1 < t_2$ and $\mu_-(t_1) = \mu_-(t_2)$, then $\{x \in]t_1, t_2[\mid y(x) < 0\}$ is an open set with measure zero, hence empty. Therefore $y_- \equiv 0$ on $]t_1, t_2[$, hence $z_-(t_1) = z_-(t_2)$. A similar argument shows that $u(x)$ is well-defined on $[x_+, 1]$ by line 4. It is easy to see that the definitions match at the seams $x = x_-$, $x = \mu_-(1)$, and $x = x_+$. On the open set of those t for which $y(t) < 0$, we have $\mu_-'(t) = 1$ and $u'(\mu_-(t)) = \frac{d}{dt}u(\mu_-(t)) = -z_-'(t) = -y_-'(t) = y'(t)$. The image under μ_- of these points t has full measure in $[x_-, \mu_-(1)]$. Similarly, $u'(\mu_+(t)) = y'(t)$, on the open set $\{t \mid y(t) > 0\}$. This shows that $u'(x)$ is defined almost everywhere on $[x_+, 1]$.

In order to show that u is absolutely continuous, say, on $[x_-, \mu_-(1)]$, we have to verify the fundamental theorem of calculus for u and its almost–everywhere derivative u'. But $\{t \mid y(t) < 0\}$ is open, and is therefore a union of at most countably many open intervals, and their images under μ_- are countably many open intervals with full total measure in $[x_-, \mu_-(1)]$. The Lebesgue integral $\int_{\mu_-(t_1)}^{\mu_-(t_2)} u'(x)\,dx$ can therefore be decomposed into a countable sum, and in each term of this sum, it holds $\mu_-(t) - t = \text{const}$, such that the inversion $u(x) = y(x_-) - z_-(\mu_-^{-1}(x))$, $u'(x) = -z_-'(\mu_-^{-1}(x))$ is well-defined. All of the integrals with the exception of at most two (namely of those over intervals containing the boundary points), contribute 0, and the fundamental theorem for y implies the one for u. A similar argument applies on $[x_+, 1]$. The equations $\mathcal{W}[u] = \mathcal{W}[y]$, $\mathcal{E}[u] = \mathcal{E}[y]$, $\int u = 0$, are verified by a similar resort to σ–additivity.

We have shown that u is well-defined, satisfies the conditions (3) and preserves the ingredients of Theorem 1., and that therefore (3), and (2'), may be assumed without loss of generality in the proof of (1). □

We now come to the basic dichotomy result of our argument:

LEMMA 2. *Let $y \in W^{1,2}[0,1]$ satisfy the normalization conditions (2), (3). Then, if $\mathcal{V}_{[0,x_+]} \geqslant y(1)$, estimate (1) holds, even in the absence of the area constraint. — However, if $\mathcal{V}_{[0,x_+]} < y(1)$, there exists a function z satisfying $\mathcal{E}[z] = \mathcal{E}[y]$, but $\mathcal{W}[z] \geqslant \mathcal{W}[y]$. This z still satisfies the area constraint $\int z = 0$ and the normalization constraints (2'), and, with $x_\pm$ replaced by $x'_- = 0$ and a new $x'_+ \leqslant x_+$, (3). It differs by a constant from y on $[x_+, 1]$, and is increasing on $[0, x_+] \supset [0, x'_+]$.*

Proof. For $0 \leqslant \alpha \leqslant x_+$, let

$$p_\alpha(x) := \begin{cases} |y'(x)| & \text{if } x < \alpha \\ -|y|'(x) & \text{if } \alpha \leqslant x < x_+ \\ y'(x) & \text{if } x_+ \leqslant x \leqslant 1 \end{cases} \qquad \text{and} \qquad Y_\alpha(x) := \int_{x_+}^{x} p_\alpha(t)\,dt$$

In brief words, let $\bar{y} = -y$ for $x < x_-$ and $\bar{y} = y$ for $x \geqslant x_-$. Then $Y_\alpha(x) = \bar{y}(x)$ if $x \geqslant \alpha$ and Y_α starts accumulating the total variation of $\bar{y}$ as x decreases beneath α.

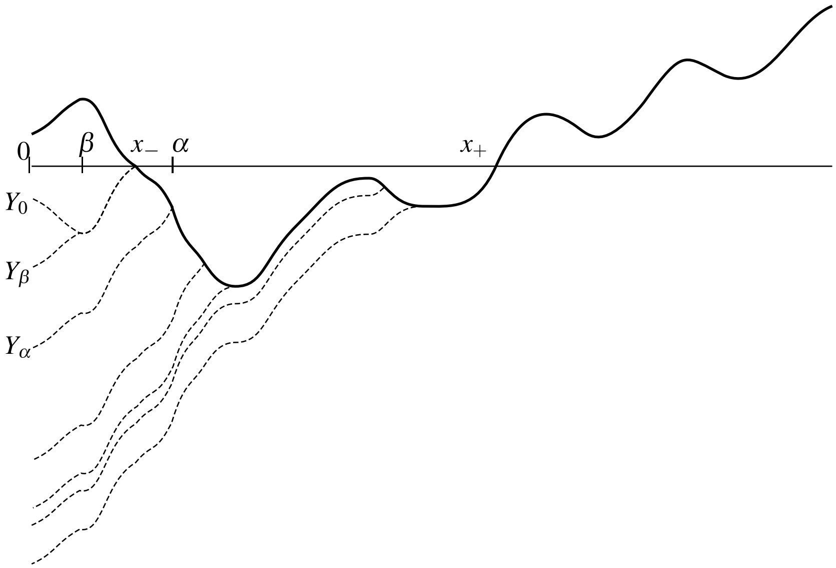

Figure 1: *Getting rid of x_- in (3) by means of the homotopy Y_α defined in the proof of Lemma 2. This violates the area constraint, which must be restored by a subsequent shift.*

Figure 1. shows the construction. It is clear that Y_α is nonincreasing with respect to α, that $Y_\alpha(x) = y(x)$ for $x \geqslant \alpha$, and $Y_\alpha(x) \leqslant -|y(x)| \leqslant 0$ for $x \leqslant \alpha \leqslant x_+$. We immediately conclude $\mathcal{E}[Y_\alpha] = \mathcal{E}[y]$ and $\mathcal{W}[Y_\alpha] \geqslant \mathcal{W}[y]$.

Moreover $Y_0(0) = -|y(0)| > -y(1)$, and $Y_{x_+}(0) = -\mathcal{V}_{[0,x_+]}[y]$. If this latter number is less or equal $-y(1)$, then by the intermediate value theorem, there will exist some $\alpha \in]0,x_+]$ such that $Y_\alpha(0) = -y(1) = -Y_\alpha(1)$. (Actually this Y_α is unique, even though α itself may not be unique.) In this case, we conclude (1), using Lemma 1.c on Y_α:

$$\mathcal{E}[y] = \mathcal{E}[Y_\alpha] \geqslant 4\mathcal{W}[Y_\alpha] \geqslant 4\mathcal{W}[y] .$$

On the other hand, however, if $\mathcal{V}_{[0,x_+]}[y] < y(1)$, we choose $z(x) := Y_{x_+}(x) + \sigma$, where σ is determined such that z satisfies the area constraint. Since $Y_{x_+} \leqslant y$, we have $\sigma \geqslant 0$. The only nontrivial estimate on z is the one for $\mathcal{W}$, when $\sigma > 0$; we write Y for Y_{x_+}, and remember $|Y| \geqslant |y|$, and $|Y'| = |y'|$ almost everywhere:

$$\begin{aligned}\int_0^1 |zz'|\,dx - \int_0^1 |yy'|\,dx \geqslant \int_0^1 |zz'|\,dx - \int_0^1 |YY'|\,dx = \\ = \int_{Y<-\sigma} (|Y+\sigma| - |Y|)|Y'|\,dx + \int_{-\sigma\leqslant Y<0} (Y+\sigma-|Y|)|Y'|\,dx + \int_{Y\geqslant 0} \sigma|Y'|\,dx > \\ > -\sigma \int_{Y<0} |Y'|\,dx + \sigma\int_{Y\geqslant 0} Y'\,dx = \sigma(-\mathcal{V}_{[0,x_+]}[y] + y(1)) > 0 .\end{aligned} \tag{6}$$

This proves the lemma. For later reuse, we will denote by $\mathbf{M}$ the transformation that takes y into Y_{x_+}, because it creates **m**onotonicity for $y < 0$, and by σ any shift operation $y \mapsto y + \sigma$ with some $\sigma \geqslant 0$. $\square$

4. Extending the Range of Monotonicity; Transformations F and P

Based on Lemma 2., we can now assume that y satisfies the area constraint and is increasing on $[0,x_+] = \overline{y^{-1}(]-\infty,0[)}$. We next show that we can even get monotonicity wherever $y < |y(0)|$. Indeed, any interval $[\alpha,\beta]$ such that $0 \leqslant y(\alpha) = y(\beta) \leqslant |y(0)|$ and $y(x) \geqslant y(\alpha)$ for $x \in [\alpha,\beta]$, can be chopped out, and $-y|_{[\alpha,\beta]}$ can be spliced into the graph of y instead of it, at the appropriate place. This procedure (called transformation $\mathbf{F}$ and depicted in Figure 2. violates the area constraint of course, however, the dichotomy of Lemma 2. again applies: either the total variation of the negative part of $\mathbf{F}y$ (larger than the corresponding quantity for y) is at least as large as $y(1) = (\mathbf{F}y)(1)$, or else we make the negative part monotonic by means of operation $\mathbf{M}$ and shift the whole graph upwards to restore the area constraint.

Let us exploit $\mathbf{F}$ as extensively as possible: there may be more than one choice to do so; for definiteness, let us apply the following algorithm: We will call $y(0) =: -a$. Let w_1 be the smallest of all nonnegative local minimum values of y. If $w_1 > a$ or w_1 doesn't exist, then we are done with transformations $\mathbf{F}$ already. Otherwise let x_{1-} and x_{1+} be the smallest and largest numbers in $y^{-1}(\{w_1\})$ and flip y on the interval $[x_{1-},x_{1+}]$, remove it from its original place and attach it at $y^{-1}(\{-w_1\})$. Continue with the next smallest nonnegative local minimum w_2, and proceed inductively until no local minimum values remain in $[0,a]$. This will also have cleared $[0,a]$ of any local maxima, because such a maximum would have to be followed by a local minimum, since $y(1) > a$. Let $\hat{y}$ denote the result of applying all these transformations $\mathbf{F}$ to y. If

$\mathcal{V}^{]-\infty,0]}[\hat{y}] \geqslant \hat{y}(1) = y(1)$, the classical Opial inequality applies according to Lemma 1., otherwise we shift $\mathbf{M}\hat{y}$ to restore the area constraint. The resulting function may have a larger a, and there may be further minima below the new a, which will again all be flipped over. The described procedure $\sigma \circ \mathbf{M} \circ \mathbf{F} \circ \ldots \circ \mathbf{F}$, which we are thus iterating, reduces the number of local extrema and must therefore terminate after finitely many iterations. Let us denote by $\tilde{\mathbf{F}}$ the composition of the maximal possible number of operations $\sigma \circ \mathbf{M} \circ \mathbf{F} \circ \ldots \circ \mathbf{F}$.

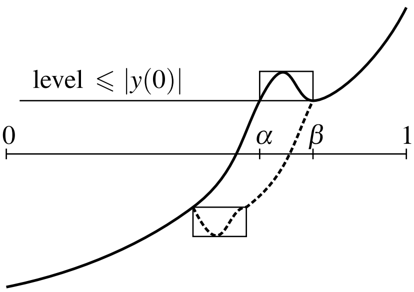

Figure 2: *The transformation* **F***: With the solid graph showing* y*,* $\mathbf{F}y$ *is produced by choosing the dashed graph where it is different from the solid one.* **F** *flips an oscillation to the negative values.*

REMARK 3. Applying single operations $\mathbf{F}$ in a haphazard matter may not decrease the number of local extrema in every step; the special algorithm given here has the sole purpose of ascertaining termination. Another algorithm towards the same end could be to create block–monotonicity as defined below (page 348 and Figure 5.), and then apply $\mathbf{F}$ from left to right. This approach might however require further care with another convergence issue similar to the one in the next step and appears less simple.

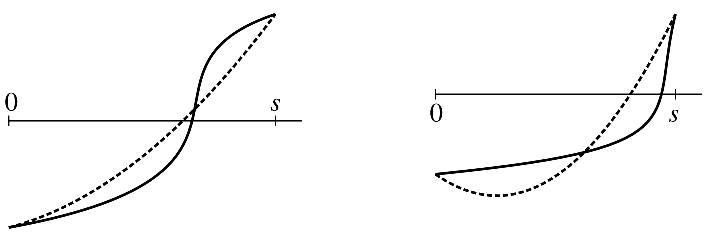

Figure 3: *The transformation* $\mathbf{P}$*. With the solid graph showing* y *on the interval* $[0,s]$*,* $\mathbf{P}y$ *is produced by taking a parabola (dashed) instead, preserving the area constraint. This may or may not preserve monotonicity.*

Let us rename the resulting function $\tilde{\mathbf{F}}y$ into y, with a redefined similarly. There exists s such that $y(s) = a$, because $y(1) > a$. s is unique, because we have exploited $\tilde{\mathbf{F}}$ to its maximal possible extent. In a next step, we minimize $\int_0^s u'^2\,dx$ under the boundary conditions $u(0) = y(0) = -a$, $u(s) = y(s) = a$, and subject to the area constraint $\int_0^s u(x)\,dx = \int_0^s y(x)\,dx$. The minimizer is a quadratic polynomial. We replace y with u on the interval $[0,s]$, but do not change y outside this interval. The resulting function is called $\mathbf{P}y$. The procedure is sketched in Figure 3. $s<1$ since $y(1)>|y(0)|$; hence $\int_0^s(\mathbf{P}y)\,dx=\int_0^s y\,dx<0$, and with $-(\mathbf{P}y)(s)=(\mathbf{P}y)(0)$ we conclude $(\mathbf{P}y)''>0$.

We may or may not lose monotonicity in this step (if we do, then $\mathcal{W}$ has increased), but in any case we do not increase $\mathcal{E}$ (we decrease it, unless $y|_{[0,s]}$ was a quadratic polynomial already). However, if we do lose monotonicity, we take again advantage of Lemma 2. to restore it (or else wrap up the proof with an invocation of Lemma 1.). A transformation $\sigma\circ\mathbf{M}\circ\mathbf{P}$ will be denoted as $\tilde{\mathbf{P}}$. If we did not lose monotonicity under transformation $\mathbf{P}$, then $\tilde{\mathbf{P}}=\mathbf{P}$. Otherwise, the transformation $\tilde{\mathbf{P}}$ increases $a:=|y(0)|$, and the new, larger interval $[-a,a]$ may now again contain oscillations.

We therefore repeat the transformations $\tilde{\mathbf{F}}$ and $\tilde{\mathbf{P}}$ alternatively (assuming that $|y(0)|<y(1)$ remains true after each step, because otherwise the proof is completed with Lemma 1.). This procedure can use at most finitely many instances of $\tilde{\mathbf{F}}$, because $\tilde{\mathbf{P}}$ preserves the number of local extrema, whereas $\tilde{\mathbf{F}}$ decreases them. Therefore, the only way the procedure can fail to terminate after finitely many steps is by eventually having an infinite repetition of steps $\tilde{\mathbf{P}}$, with the interspersed $\tilde{\mathbf{F}}$ all being the identity. In this case, the limiting function y_* can be constructed and all inequalities are preserved under the limit; moreover the limiting function is a monotonic quadratic polynomial on the appropriate limiting interval $[0,s_*]$. Let us prove these latter claims, referring to Figure 4. for illustration.

We have a sequence y_n of functions on $[0,1]$, generated from some y_0 by $y_n:=\tilde{\mathbf{P}}y_{n-1}$. It holds $\mathcal{E}[y_n]\leqslant\mathcal{E}[y_{n-1}]$ and $\mathcal{W}[y_n]>\mathcal{W}[y_{n-1}]$. On an interval $[0,s_n]$, y_n is patched together from two parabolas: $y_n(x)=m_n+k_n(x-x_n)|x-x_n|$, where $k_n>0$; and $x_n>0$ by our assumption that $\mathbf{P}y_{n-1}$ is a non-monotonic segment of a parabola. The shift σ_n used in $y_n(x)=\tilde{\mathbf{P}}y_{n-1}(x)=(\mathbf{M}\circ\mathbf{P})(y_{n-1})(x)+\sigma_n$ can be given explicitly by $\sigma_n=\frac{2}{3}k_nx_n^3$. A fortiori, $\sigma_n<k_nx_n^2$, and therefore $y_n(s_{n-1})<|y_n(0)|$, and hence $s_n>s_{n-1}$. We also conclude $y_n(0)<y_{n-1}(0)<\ldots<y_0(0)<0$. This will bound k_n away from 0 uniformly: Indeed,

$$
\begin{aligned}
2k_n\equiv(\mathbf{P}y_{n-1})''&=\frac{(\mathbf{P}y_{n-1})'(s_{n-1})-(\mathbf{P}y_{n-1})'(0)}{s_{n-1}}>(\mathbf{P}y_{n-1})'(s_{n-1})\\
&=\frac{|y_{n-1}(0)|-(m_n-\sigma_n)}{(s_{n-1}-x_n)/2}>\frac{|y_{n-1}(0)|}{1/2}\geqslant 2|y_0(0)|
\end{aligned}
$$

With $\sum\sigma_j$ being bounded above by, say, a rough estimate $\sum_1^n\sigma_j<y_n(1)\leqslant\sqrt{1}\times(\int y_n'^2)^{1/2}\leqslant(\mathcal{E}[y_0])^{1/2}$, we can now conclude that $\frac{2}{3}k_nx_n^3=\sigma_n\to 0$, and hence $x_n\to 0$.

The boundedness of $\mathcal{E}[y_n]$ permits to extract a subsequence that converges uniformly, and weakly in $W^{1,2}[0,1]$. Let the limit be y_*, and let the limit of the increasing sequence s_n be s_*. All $y_n|_{[s_*,1]}$ differ only by constants, and they form a monotonic sequence. Therefore, $\int_{s_*}^1|y_ny_n'|\to\int_{s_*}^1|y_*y_*'|$, because y_n' doesn't depend on n on the interval of integration, and because of the strong L_2-convergence of y_n. Not only on the interval $[x_n,s_n]$, but on all of $[x_n,s_*]$, it holds $y_n'\geqslant 0$, because we are in the situation where all interspersed operations $\mathbf{F}$ are the identity. Therefore, $|y_n y_n'|(x) = |y_n(x)| y_n'(x)$ on this interval, and again we can conclude by the strong convergence of y_n and the weak convergence of y_n' that $\int_0^{s_*} |y_n y_n'| \to \int_0^{s_*} |y_* y_*'|$. Together with the lower semicontinuity of $\mathcal{E}$ under weak $W^{1,2}$ convergence, this shows that $\mathcal{E}[y_0] - 4\mathcal{W}[y_0] \geqslant \mathcal{E}[y_*] - 4\mathcal{W}[y_*]$. The monotonicity of y_* on $[0, s_*]$ follows from $x_n \to 0$. y_* is a quadratic polynomial there, as a uniform (on each compact subinterval of $]0, s_*[$) limit of quadratic polynomials (and because the estimate on k_n precludes y_* from being linear).

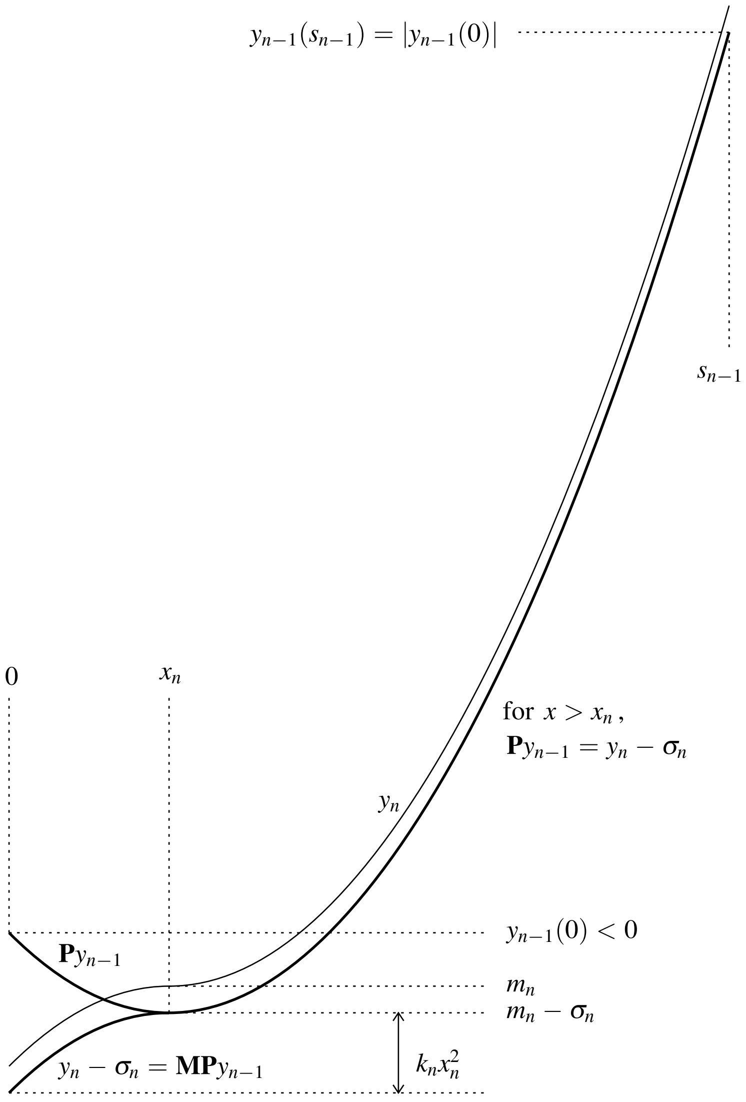

Figure 4: *Iterating the operation* $\tilde{\mathbf{P}} = \sigma \circ \mathbf{M} \circ \mathbf{P}$

We have therefore seen that it is no loss of generality to make the following assumption when proving the inequality (1):

$$\begin{aligned} & y(0) = -a\,,\; y(s) = a \\ & y \text{ increasing and a quadratic polynomial on } [0,s] \\ & y(x) > a \text{ for } x \in\,]s,1] \end{aligned} \tag{7}$$

REMARK 4. [Cases of Equality] For an optimizer in the inequality (1), we may assume $\mu_-(1) = \mu_0(1)$; otherwise we chop out the interval $[\mu_-(1), \mu_0(1)]$ and stretch the remaining intervals uniformly, thus leaving $\mathcal{W}$ and the area 0 unchanged, but decreasing $\mathcal{E}$ strictly.

Returning to the proof of Lemma 2., which did not require the assumption of finitely many local extrema, we note that $\mathcal{W}[Y_\alpha] > \mathcal{W}[y]$, unless $|Y_\alpha| \equiv |y|$. In the case where $\mathcal{V}_{[0,x_+]}[y] \geqslant y(1)$, we are led to Lemma 1.(c), and for an optimizer y satisfying (3), its corresponding function Y_α must be among the cases of equality in Lemma 1.(c). One of them is $Y_\alpha(x) = 2y(1)(x - \frac{1}{2})$; all other cases require $Y_\alpha'(0) < 0$, hence α must be 0. In either case, the only way for a continuous y to satisfy both (3) and $|y| = |Y_\alpha|$ is $y = Y_\alpha$, and among them, only $y(x) = 2y(1)(x - \frac{1}{2})$ satisfies the area constraint.

Furthermore, if a function y satisfying (2) is optimal, then the u constructed in (5) is an optimizer satisfying (3) and therefore $u(x) = 2u(1)(x - \frac{1}{2}) = 2y(1)(x - \frac{1}{2})$. Then $y(0) = u(0) < 0$, so $x_- = 0$. Also, $\{t \mid y(t) = 0\}$ has measure 0. For all other points t, we have seen $2y(1) \equiv u'(\mu_\pm(t)) = y'(t)$; therefore y' is constant a.e., and we conclude further that $y(x) = 2y(1)(x - \frac{1}{2})$.

In the other case, namely, if in Lemma 2., it holds $\mathcal{V}_{[0,x_+]}[y] < y(1)$, then we have a comparison function $z = u + \sigma$ with the strict inequality $\mathcal{W}[z] > \mathcal{W}[u] \geqslant \mathcal{W}[y]$, unless $\sigma = 0$, due to estimate (6). So for a possible optimizer y other than the one found in the previous paragraph, σ must vanish, and its interval rearrangement u according to (5), must be another optimizer satisfying $a := |u(0)| < u(1)$, u nondecreasing on $[0, x_+]$, $u(x_+) = 0$. Clearly such u would have to be monotonic on the preimage of $[-a, a]$, because otherwise the transformation $\mathbf{F}$ could be applied, producing an equally good $\mathbf{F}u$, on which Lemma 3. would strictly improve. u would have to be a quadratic polynomial on $[0,s] = u^{-1}([-a,a])$, because a quadratic polynomial is the unique minimizer of $\int_0^s u'^2$ under an area constraint.

We will show in Remark 7. that such an optimizer cannot exist; alternatively, from the Euler–Lagrange equation according to Brown-Plum [5], one could infer that the quadratic formula for $u|_{[0,s]}$ holds on all of $[0,1]$, and then deal with the finite dimensional optimization problem.

5. Block–Monotonicity; Transformation Ch

A rearrangement like the one used to achieve the normalization (3) will further improve on (7) and produce a transparent monotonicity structure of y on the interval $[s,1]$, too: the preimage of the (finite, by assumption) set of all critical values cuts the interval $[s,1]$ into finitely many sub-intervals that can be rearranged to achieve "block–monotonicity" as defined below:

$$
\begin{array}{l}
y(0) = -a\,,\ y(s) = a \\
y \text{ increasing and a quadratic polynomial on } [0,s] \\
y(x) \text{ block–monotonic increasing on } [s,t] \text{ and } [t,r]\,,\ \text{ where} \\
y(t) = y(1) =: b \\
y(r) = \max y =: c \\
y \text{ decreasing on } [r,1]
\end{array}
\tag{8}
$$

The intervals $]s,t]$, $]t,r]$, and $]r,1]$ may each be empty, and equality may hold in each of $a \leqslant b \leqslant c$.

DEFINITION 1. A function f is called block–monotonic increasing, on an interval $[s,t]$, if there exists a partition into finitely many intervals $I_i = [a_{i-1}, a_i]$, where $s = a_0 < a_1 < a_2 < \ldots < a_n = t$, such that $f(x_i) \leqslant f(x_{i+1})$, whenever $x_i \in I_i$ and $x_{i+1} \in I_{i+1}$, and moreover, each interval I_i is partitioned into an odd number k of intervals $a_{i-1} =: x_{i,0} < x_{i,1} < x_{i,2} < \ldots < x_{i,k} = a_i$, such that $f : [x_{i,j-1}, x_{i,j}] \to f(I_i)$ is bijective, and increasing for odd j and decreasing for even j.

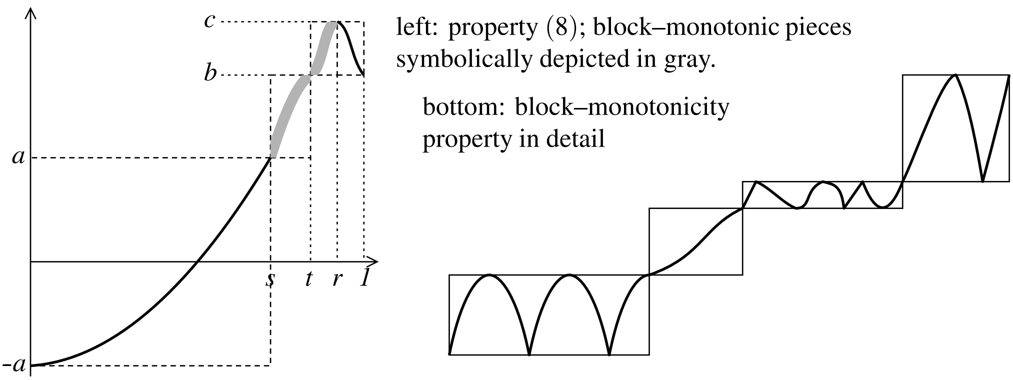

Figure 5: *The normalization (8), and the definition of block monotonicity*

Let us explain in more detail how block-monotonicity can be achieved: The set of local extreme values of y will be called C_Y, its preimage $y^{-1}(C_Y) =: C_X$. Now $[0,1] \setminus C_X$ consists of finitely many open intervals, which will be rearranged based on the order of y. (There may be more than one feasible reordering.)

Each interval in $y([0,1]) \setminus C_Y$ that lies below $y(1)$ has an odd number of preimage intervals; those on which y increases are one more in number than those on which y decreases. Each interval in $y([0,1]) \setminus C_Y$ that lies above $y(1)$ has an even number of preimage intervals, on half of which y increases. In this latter case, we reserve one of the intervals with decreasing y to become part of the interval $[r,1]$. The non-reserved intervals, as well as all those on which $y < y(1)$, are ordered according to increasing y-value, in such a way that those with the same image carry alternately an increasing and a decreasing graph. Finally, the reserved intervals (all with a decreasing graph) follow according to decreasing y-values.

Having achieved the form (8), we can get a brief estimate on s that will be needed later:

LEMMA 3. *If a function with monotonicity properties (8) satisfies* $\int_0^1 y\,dx = 0$, *then* $s > \frac{3}{4}$.

Proof.

$$0 = \int_0^1 y\,dx \geqslant \int_0^s (-a + 2ax^2/s^2)\,dx + (1-s)a = -\frac{1}{3}as + (1-s)a\ .$$

The "$\geqslant$" is strict, unless $s = 1$. The estimate follows immediately. □

We mention briefly that the parameter t as such will not play a role subsequently; it has merely been named, because it is conspicuous and has been spawned by the critical level $b = y(1)$.

Let us interrupt the proof to discuss a few variants that are not needed for the proof, but give extra insight:

REMARK 5. [Variant] A similar block-monotonicity structure could have been achieved from the very onset, before using operation $\mathbf{M}$ already; it would have produced a pattern "decreasing – block–monotonically increasing – decreasing"; either of the decreasing parts could be empty. A reflection in the x-coordinate could be applied to pairs of subintervals of I_i such as to exchange increasing and decreasing y on these subintervals. This could be used to achieve, for instance, that the decreasing pieces at the beginning and end become the steepest possible, or the least steep possible, if such an a-priori estimate should be desirable for further arguments.

REMARK 6. [A relaxed variational problem] On those intervals I_i consisting of $N \geqslant 3$ subintervals, a sequence of Steiner symmetrizations can be applied on adjacent pairs of subintervals $[x_{i,j-1}, x_{i,j+1}]$. In the limit, a y having $N-1$ reflection symmetries on I_i is obtained, as shown in the leftmost interval I_i in Figure 5.. It is possible to change block–monotonicity into genuine monotonicity at the price of relaxing the functional. For instance, on the leftmost interval I_i just mentioned (with $N = 5$ subintervals) y could be replaced by its increasing rearrangement y^*, which, with the symmetry achieved already, is given by $y^*(x) = y(a_{i-1} + (x - a_{i-1})/N)$ for $a_{i-1} \leqslant x \leqslant a_i$. We may therefore assume (genuine, not block–)monotonicity "decreasing – increasing – decreasing" at the price of letting $\mathcal{E}[y,N] := \int N^2(x)y'^2(x)\,dx$ and $\mathcal{W}[y,N] := \int N(x)|y(x)y'(x)|\,dx$, where we introduce another function N whose range is restricted to the odd positive integers. Even though we have not pursued this idea successfully, it may be an option to treat the possible oscillations that prevent a straightforward existence proof for an optimizer by direct methods. By construction, N actually depends on y only.

This idea is in the spirit of Young measures, and if it could be turned into a complete abstract existence argument, this argument would likely generalize to higher space dimensions, possibly subject to more technicalities from measure theory.

With (8), we will do our final piece of "surgery": We chop off the interval $J_0 := [r,1]$, and also chop out the intervals $[x_{i,j}, x_{i,j+2}]$ as depicted in Figure 6. j is chosen even, so these intervals contain first one increasing and then one decreasing segment from the definition of block-monotonicity, and they will be denoted by J_ℓ, where $\ell = 1, 2, \ldots, \ell_{\max}$. On J_ℓ for $\ell \geqslant 1$, the minimum of y (taken on at the boundary of this interval) will be called b_ℓ, and the maximum c_ℓ. The length of J_ℓ will be called d_ℓ. For symmetry, we also define the corresponding quantities $b_0 := b$, $c_0 := c$, $d_0 := 1 - r$ for J_0. Moreover we let $J := \bigcup_{\ell \geqslant 0} J_\ell$ and $d := \sum_{\ell \geqslant 0} d_\ell$.

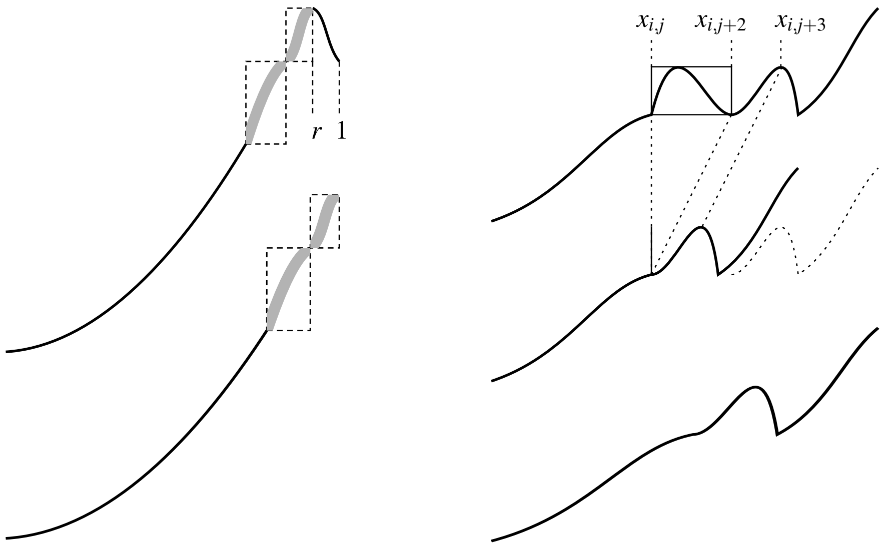

Figure 6: *Operation* **Ch**: *chopping off an interval and stretching the rest to original width. Case 1 (left) chops off the interval* $[r, 1]$, *case 2 (right) chops out an interval* $[x_{i,j}, x_{i,j+2}]$ *from the middle, thus slightly simplifying the block–monotonicity structure.*

We stretch out the remaining set $[0, 1] \setminus J$ uniformly to restore the total length one, and add a constant σ to restore the area constraint. The procedure of chopping and stretching will be called **Ch** and is depicted in Figure 6. Let us call the resulting function z again. This is the only place, where some highly calculational estimates are required to show that the transformation **Ch** still decreases the difference $\mathcal{E} - 4\mathcal{W}$. Namely, in order to show that (1) for y follows from (1) for z, we need to show that $\mathcal{E}[y] - 4\mathcal{W}[y] - \mathcal{E}[z] + 4\mathcal{W}[z] \geqslant 0$, or equivalently,

$$\int_0^1 y'^2\,dx - 4\int_0^1 |yy'|\,dx - \int_0^1 z'^2\,dx + 4\int_0^1 |zz'| \geqslant 0\,. \tag{9}$$

In Lemma 3., the fact that y is a monotonic quadratic polynomial on $[0, s]$, together with the area constraint, has given a lower estimate $s > \frac{3}{4}$, hence $d < \frac{1}{4}$; this comes in handy during the calculations.

The components of the stretched-out $[0, 1] \setminus J$ can be sewn together, and y remains continuous at the seams. This is why we have an estimate

$$\int_{[0,1]\setminus J} y'^2\,dx \geqslant \frac{(a + c)^2}{1 - d}\,, \tag{10}$$

which is obtained by minimizing $\int_0^{1-d} u'^2\,dx$ subject to the boundary conditions $u(0)=-a$, $u(1-d)=c$. We clearly have

$$\int_0^1 z'^2\,dx = (1-d)\int_{[0,1]\setminus J} y'^2\,dx\,. \tag{11}$$

By applying the standard Opial inequality (Lemma 1.a), or rather a scaled version thereof, to $(y-b_\ell)|_{J_\ell}$, we have (*only* for $\ell \geqslant 1$):

$$\int_{J_\ell} y'^2\,dx \geqslant \frac{4}{d_\ell}\int_{J_\ell} y|y'|\,dx - \frac{4b_\ell}{d_\ell}\int_{J_\ell}|y'|\,dx \geqslant \frac{4}{d_\ell}\int_{J_\ell}|yy'|\,dx - \frac{8b_\ell(c_\ell-b_\ell)}{d_\ell}\,, \tag{12}$$

and similarly (with the same argument used on the duplication of J_0)

$$\int_{J_0} y'^2\,dx \geqslant \frac{2}{d_0}\int_{J_0} y|y'|\,dx - \frac{2b_0}{d_0}\int_{J_0}|y'|\,dx \geqslant \frac{2}{d_0}\int_{J_0}|yy'|\,dx - \frac{2b_0(c_0-b_0)}{d_0}\,. \tag{13}$$

Finally, we need

$$\int_0^1 |zz'|\,dx = \int_{[0,1]\setminus J}|(y+\sigma)y'|\,dx = a^2+\sigma^2+\int_{[s,1]\setminus J}|yy'|\,dx + \sigma(c-a)\,. \tag{14}$$

This result is obtained by splitting $[0,1]\setminus J = [0,s]\cup([s,1]\setminus J)$. Combining these estimates, we get

$$\begin{aligned}
D &:= \textstyle\int_0^1 y'^2\,dx - \int_0^1 z'^2\,dx + 4\int_0^1|zz'|\,dx - 4\int_0^1|yy'|\,dx \geqslant \\
\overset{(11),(14)}{}&\geqslant \textstyle\int_J y'^2\,dx + d\int_{[0,1]\setminus J} y'^2\,dx + 4\left(a^2+\sigma^2+\sigma(c-a)\right) - 4\int_J|yy'|\,dx - 4a^2 \\
\overset{(12),(13),(10)}{}&\geqslant \textstyle\left(\frac{2}{d_0}-4\right)\int_{J_0}|yy'|\,dx + \sum_{\ell\geqslant 1}\left(\frac{4}{d_\ell}-4\right)\int_{J_\ell}|yy'|\,dx \\
&\quad \textstyle - 2\frac{b_0(c_0-b_0)}{d_0} - 8\sum_{\ell\geqslant 1}\frac{b_\ell(c_\ell-b_\ell)}{d_\ell} + \frac{d(a+c)^2}{1-d} + 4\left(\sigma_*^2+\sigma_*(c-a)\right)
\end{aligned} \tag{15}$$

where we have used

$$\sigma \geqslant \sigma_* := \frac{1}{1-d}\sum_{\ell\geqslant 0} d_\ell b_\ell\,,$$

due to the area constraint. We minimize the right hand side with respect to a; the worst-case scenario for a is

$$a = a_* := \frac{2\sigma_*(1-d)}{d} - c = \frac{2}{d}\left(\sum_{\ell\geqslant 0} d_\ell b_\ell\right) - c\,.$$

We conclude, using

$$\int_{J_\ell}|yy'|\,dx = (c_\ell^2 - b_\ell^2) = (c_\ell-b_\ell)^2 + 2(c_\ell-b_\ell)b_\ell \quad \text{for} \quad \ell \geqslant 1\,,$$

and similarly $\int_{J_0} |yy'|\,dx = \frac{1}{2}(c_0 - b_0)^2 + (c_0 - b_0)b_0$, that

$$D \geqslant \tfrac{1-2d_0}{d_0}(c_0-b_0)^2 - 4(c_0-b_0)b_0 + 4\textstyle\sum_{\ell\geqslant 1}\tfrac{1-d_\ell}{d_\ell}(c_\ell-b_\ell)^2 - 8\textstyle\sum_{\ell\geqslant 1}(c_\ell-b_\ell)b_\ell - 4\frac{1-2d}{d(1-d)^2}\Big(\sum_{\ell\geqslant 0} d_\ell b_\ell\Big)^2 + \frac{8c}{1-d}\Big(\sum_{\ell\geqslant 0} d_\ell b_\ell\Big) \tag{16}$$

In the last term, we pull c into the sum and use $c \geqslant c_\ell = (c_\ell - b_\ell) + b_\ell$. In the second last, we use the Cauchy–Schwarz inequality: $\sum_\ell d_\ell b_\ell \leqslant (\sum_\ell d_\ell)^{1/2}(\sum_\ell d_\ell b_\ell^2)^{1/2}$. This produces a quadratic form in the variables $(c_\ell - b_\ell)$ and b_ℓ, namely

$$\begin{aligned} D \geqslant\ & \tfrac{1-2d_0}{d_0}(c_0-b_0)^2 - \Big(4 - 8\tfrac{d_0}{1-d}\Big)(c_0-b_0)b_0 + \Big(\tfrac{8}{1-d} - 4\tfrac{(1-2d)}{(1-d)^2}\Big) d_0 b_0^2 \\ & + 4\sum_{\ell\geqslant 1}\frac{1-d_\ell}{d_\ell}(c_\ell-b_\ell)^2 - 8\sum_{\ell\geqslant 1}(c_\ell-b_\ell)b_\ell \\ & - 4\frac{1-2d}{(1-d)^2}\sum_{\ell\geqslant 1} d_\ell b_\ell^2 + \frac{8}{1-d}\sum_{\ell\geqslant 1} d_\ell(c_\ell-b_\ell)b_\ell + \frac{8}{1-d}\sum_{\ell\geqslant 1} d_\ell b_\ell^2 \end{aligned} \tag{17}$$

For each $\ell \geqslant 0$ separately, we can minimize over b_ℓ, with $(c_\ell - b_\ell)$ given: The worst case scenario occurs for

$$b_0 = b_{0*} := \frac{(1-d-2d_0)(1-d)}{2d_0}(c_0 - b_0)\,,$$

$$b_\ell = b_{\ell *} := \frac{(1-d-d_\ell)(1-d)}{d_\ell}(c_\ell - b_\ell)$$

and provides the estimate

$$\begin{aligned} D &\geqslant \big(1 - 2d_0 - (1-d-2d_0)^2\big)\tfrac{(c_0-b_0)^2}{d_0} + 4\textstyle\sum_{\ell\geqslant 1}\big(1-d_\ell-(1-d-d_\ell)^2\big)\tfrac{(c_\ell-b_\ell)^2}{d_\ell} \\ &\geqslant (2d-d^2)\frac{(c_0-b_0)^2}{d_0} + 4(2d-d^2)\sum_{\ell\geqslant 1}\frac{(c_\ell-b_\ell)^2}{d_\ell} \geqslant \frac{7d}{4}\sum_{\ell\geqslant 0}\frac{v_\ell^2}{d_\ell}\,, \end{aligned}$$

where we have used $d, d_0, d_\ell < \frac{1}{4}$ in the estimate of the coefficients, and where $v_\ell = (1 \text{ or } 2)(c_\ell - b_\ell) = \mathcal{V}_{J_\ell}[y]$, for $\ell = 0$ and $\ell \geqslant 1$ respectively. Again invoking the Cauchy–Schwarz inequality, $\sum v_\ell \leqslant (\sum v_\ell^2/d_\ell)^{1/2} \times (\sum d_\ell)^{1/2}$, we conclude

$$D \geqslant \frac{7}{4}\Big(\sum_{\ell\geqslant 0} v_\ell\Big)^2 = \frac{7}{4}\mathcal{V}_J[y]^2\,. \tag{18}$$

We have ended up with an increasing comparison function z (satisfying the area constraint), and it is immediate to prove (1) for such a function: With $c := z(1) \geqslant |z(0)| =: a$, a lower estimate for $\mathcal{E}[z]$ is again given by $\mathcal{E}[\mathbf{P}z]$ where $(\mathbf{P}z)(x) = -a + (c+a)x - 3(c-a)x(1-x)$. We get by straightforward calculation that $\mathcal{E}[\mathbf{P}z] = 4(a^2 + c^2 - ac)$. Therefore

$$\mathcal{E}[z] - 4\mathcal{W}[z] \geqslant 4(a^2 + c^2 - ac) - 2(a^2 + c^2) = 2(c-a)^2 \geqslant 0\,. \tag{19}$$

We have therefore proved Theorem 1..

Effectively, we have shown more: if y has finitely many local extrema and satisfies (7) and the area constraint, then from (18) and (19)

$$\mathcal{E}[y] - 4\mathcal{W}[y] \geqslant \frac{7}{4}\left(\int_0^1 |y'|\,dx - ((\max y) - y(0))\right)^2 + 2(y(1) - |y(0)|)^2 \tag{20}$$

This inequality prevails by approximation, if the constraint of finitely many local extrema is abandoned.

REMARK 7. [Cases of Equality] In order to check the cases of equality, we have to resume at the end of Remark 4. and show that an optimizer u satisfying (7) and the area constraint cannot exist, unless $y(1) = |y(0)|$. This is now immediate from (20).

REMARK 8. The first term in (20) shows that an optimizer must be monotonic. The second term shows that $y(1) = |y(0)|$ is required for an optimizer, which in turn implies $s = 1$ and therefore monotonicity again. Although (20) gives much more than we need, we cannot immediately exploit its full power, because this estimate is still contingent on the normalization (7), which in turn was used for $\sigma \geqslant \sigma_* > 0$ and $d < \frac{1}{4}$. A general quantitative estimate of a type similar to (20) would be interesting.

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