Mathematical Inequalities & Applications
Volume 7, Number 3 (2004), 379–391

INEQUALITIES FOR A POLYNOMIAL AND ITS DERIVATIVE

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(*communicated by Th. Rassias*)

Abstract. In this paper we consider a class of polynomials $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$, $1 \leqslant \mu \leqslant n$, not vanishing in the disk $|z| < K$. For $K \geqslant 1$, we investigate the dependence of $\max_{|z|=1} |P(Rz) - P(z)|$ on $\max_{|z|=1} |P(z)|$ and for $K > 0$ we estimate $\max_{|z|=R} |P'(z)|$ in terms of $\max_{|z|=r} |P(z)|$, $0 \leqslant r \leqslant R \leqslant K$. Our results not only generalize some known polynomial inequalities, but olso a variety of interesting results can be deduced from these by a fairly uniform procedure. We also obtain a generalization of a Theorem of Paul Turan.

1. Introduction and statement of results

If $P(z)$ is a polynomial of degree n and $P'(z)$ its derivative, then

$$\max_{|z|=1} |P'(z)| \leqslant n \max_{|z|=1} |P(z)| \tag{1}$$

$$\max_{|z|=R>1} |P(z)| \leqslant R^n \max_{|z|=1} |P(z)|. \tag{2}$$

Inequality (1) is a well-known result of S. Bernstein (for reference see [6] or [15], whereas inequality (2) is a simple deduction from maximum modulus principle (see [18]). In both (1) and (2) equality holds only when $P(z)$ is a constant multiple of z^n.

If we restrict ourselves to a class of polynomials of degree n having no zeros in $|z| < 1$, then

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{2} \max_{|z|=1} |P(z)| \tag{3}$$

$$\max_{|z|=R>1} |P(z)| \leqslant \frac{R^n + 1}{2} \max_{|z|=1} |P(z)|. \tag{4}$$

Inequality (3) was conjectured by Erdös and latter verified by Lax [13], whereas Anykeny and Rivlin [1] used (3) to prove (4).

As an extension of (3) Malik [14] verified that if $P(z)$ does not vanish in $|z| < K$, $K \geqslant 1$, then

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{1+K} \max_{|z|=1} |P(z)|. \tag{5}$$

Mathematics subject classification (2000): 30A10, 30C10, 30D15.

Key words and phrases: Polynomials, inequalities.

Paper MIA-07-39

Chan and Malik [7] generalised (5) in a different direction and proved that, if $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$, $1 \leqslant \mu \leqslant n$ is polynomial of degree n which does not vanish in $|z| < K$, where $K \geqslant 1$, then

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{1+K^{\mu}} \max_{|z|=1} |P(z)|. \tag{6}$$

Inequality (6) was independently proved by Qazi [17, Lemma 1], who also under the same hypothesis proved that

$$\max_{|z|=1} |P'(z)| \leqslant \frac{1+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right| K^{\mu+1}}{1+K^{\mu+1}+\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|(K^{\mu+1}+K^{2\mu})} \max_{|z|=1} |P(z)|. \tag{7}$$

Recently Frappier, Rahman and Ruscheweyh [11] investigated the dependence of $\max_{|z|=1} |P(Rz) - P(z)|$ on $\max_{|z|=1} |P(z)|$, where $R > 1$ and proved that, if $P(z)$ is a polynomial of degree n, then for all $R > 1$,

$$\max_{|z|=1} |P(Rz) - P(z)| + \Psi(R)|P(o)| \leqslant (R^n - 1) \max_{|z|=1} |P(z)|,$$

where

$$\Psi_n(R) = \frac{(R-1)(R^{n-1}+R^{n-2})\{R^{n-1}+R^n-(n+1)R+(n-1)\}}{R^{n-1}+R^n-(n-1)R+(n-3)}, \; n \geqslant 2$$

and

$$\Psi_1(R) = R - 1.$$

In this paper, we consider the class of polynomials $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$, $1 \leqslant \mu \leqslant n$, not vanishing in the disk $|z| < K$, where $K \geqslant 1$, and investigate the dependence of $\max_{|z|=1} |P(Rz) - P(z)|$ on $\max_{|z|=1} |P(z)|$. We first prove the following more general result which includes not only inequality (7) as a special case, but also leads to a standarad development of interesting generalizations of some well known results.

THEOREM 1. *Let $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ be a polynomial of degree n which does not vanish in $|z| < K$, where $K \geqslant 1$, then for every $R \geqslant 1$ and $|z| = 1$,*

$$|P(Rz) - P(z)| \leqslant (R^n - 1) \frac{1+\left\{\frac{R^\mu - 1}{R^n - 1}\right\}\left|\frac{a_\mu}{a_0}\right| K^{\mu+1}}{1+K^{\mu+1}+\left\{\frac{R^\mu - 1}{R^n - 1}\right\}\left|\frac{a_\mu}{a_0}\right|(K^{\mu+1}+K^{2\mu})} \max_{|z|=1} |P(z)|. \tag{9}$$

REMARK 1. If we divide the two sides of inequality (9) by $R-1$ and make $R \to 1$, we immediately get inequality (7).

If we use the fact that $|P(Rz)| \leqslant |P(Rz) - P(z)| + |P(z)|$, then the following corollary is an immediate consequence of Theorem 1.

COROLLARY 1. *If* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ *is a polynomial of degree* n *which does not vanish in* $|z| < K$, *where* $K \geqslant 1$, *then for every* $R \geqslant 1$,

$$\max_{|z|=R>1} |P(z)| \leqslant \frac{R^n \left\{1 + \frac{R^\mu - 1}{R^n - 1}\left|\frac{a_\mu}{a_0}\right| K^{\mu+1}\right\} + K^{\mu+1} + \frac{R^\mu - 1}{R^n - 1}\left|\frac{a_\mu}{a_0}\right| K^{2\mu}}{1 + K^{\mu+1} + \left\{\frac{R^\mu - 1}{R^n - 1}\right\}\left|\frac{a_\mu}{a_0}\right|(K^{\mu+1} + K^{2\mu})} \max_{|z|=1} |P(z)|.$$

The inequality

$$\frac{R^\mu - 1}{R^n - 1} \leqslant \frac{\mu}{n} \tag{11}$$

holds for all $R \geqslant 1$ and $1 \leqslant \mu \leqslant n$. To prove this inequality, we observe that it is trivial for $R = 1$, and for every $R \geqslant 1$ it easily follows when $\mu = n$. Hence to establish (11), it suffices to consider the case $1 \leqslant \mu \leqslant n-1$ and $R > 1$. Now, if $R > 1$ and $1 \leqslant \mu \leqslant n-1$, then we have

$$\begin{aligned}
\mu R^n - nR^\mu + (n-\mu) &= \mu R^\mu (R^{n-\mu} - 1) - (n-\mu)(R^\mu - 1) \\
&= (R-1)\{\mu R^\mu (R^{n-\mu-1} + R^{n-\mu-2} + \cdots + 1) \\
&\quad - (n-\mu)(R^{\mu-1} + \cdots + R + 1)\} \\
&\geqslant (R-1)\{\mu(n-\mu)R^\mu - (n-\mu)\mu R^{\mu-1}\} \\
&= \mu(n-\mu)(R-1)^2 R^{\mu-1} \\
&> 0.
\end{aligned}$$

This implies $\mu(R^n - 1) \geqslant n(R^\mu - 1)$, for all values of $R > 1$ and $1 \leqslant \mu \leqslant n-1$, which is equivalent to (11).

With the help of inequality (11), a simple direct calculation yields,

$$\begin{aligned}
&\frac{R^n \left\{1 + \frac{R^\mu - 1}{R^n - 1}\left|\frac{a_\mu}{a_0}\right| K^{\mu+1}\right\} + K^{\mu+1} + \frac{R^\mu - 1}{R^n - 1}\left|\frac{a_\mu}{a_0}\right| K^{2\mu}}{1 + K^{\mu+1} + \frac{R^\mu - 1}{R^n - 1}\left|\frac{a_\mu}{a_0}\right|(K^{\mu+1} + K^{2\mu})} \\
&\leqslant \frac{R^n \left\{1 + \frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right| K^{\mu+1}\right\} + K^{\mu+1} + \frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right| K^{2\mu}}{1 + K^{\mu+1} + \frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|(K^{\mu+1} + K^{2\mu})}.
\end{aligned} \tag{12}$$

Hence from Theorem 1, we easily deduce the following:

COROLLARY 2. *If* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ *is a polynomial of degree* n, *which does not vanish in the disk* $|z| < K$, *where* $K \geqslant 1$, *then for every* $R > 1$,

$$\max_{|z|=R>1} |P(z)| \leqslant \frac{R^n \left\{1 + \frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right| K^{\mu+1}\right\} + \frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right| K^{2\mu}}{1 + K^{\mu+1} + \frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|(K^{\mu+1} + K^{2\mu})} \max_{|z|=1} |P(z)|. \tag{13}$$

Inequality (10) provides a refinement of a result due to Govil and Dewan [9, Treorem 1.9] which is also a special case of inequality (13) when $\mu = 1$.

Next, if we take $\mu = 1$ in Theorem 1, we get the following:

COROLLARY 3. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n*, which does not vanish in the disk* $|z| < K$*,* $K \geqslant 1$*, then for every* $R > 1$*,*

$$|P(Rz) - P(z)| \leqslant (R^n - 1)\frac{1 + \dfrac{R-1}{R^n-1}\left|\dfrac{a_1}{a_0}\right| K^2}{1 + K^2 + 2\dfrac{R-1}{R^n-1}\left|\dfrac{a_1}{a_0}\right| K^2} \max_{|z|=1} |P(z)|. \tag{14}$$

REMARK 2. Dividing the two sides of inequality (14) by $R-1$ and making $R \to 1$, it follows that, if $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n such that $P(z) \neq 0$ in $|z| < K$, $K \geqslant 1$, then

$$|P'(z)| \leqslant n\frac{n|a_0| + K^2|a_1|}{n(1+K^2)|a_0| + 2K^2|a_1|} \max_{|z|=1} |P(z)|. \tag{15}$$

Inequality (15) is a refinement of inequality (5) and was also independently proved by Govil, Rahman and Schmeisser [12].

Now, it is known (for reference see [17, Remark 1]), that

$$\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right| K^\mu \leqslant 1. \tag{16}$$

Using this fact and the inequality (11), it is easy to verify that

$$\frac{1 + \dfrac{R^\mu - 1}{R^n - 1}\left|\dfrac{a_\mu}{a_0}\right| K^{\mu+1}}{1 + K^{\mu+1} + \dfrac{R^\mu - 1}{R^n - 1}\left|\dfrac{a_\mu}{a_0}\right| (K^{\mu+1} + K^{2\mu})} \leqslant \frac{1}{1 + K^\mu}. \tag{17}$$

By using these observations, the following result is an immediate consequence of Theorem 1.

COROLLARY 4. *If* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ *is a polynomial of degree* n*, which does not vanish in the disk* $|z| < K$*, where* $K \geqslant 1$*, then for every* $R > 1$*,*

$$|P(Rz) - P(z)| \leqslant \frac{R^n - 1}{1 + K^\mu} \max_{|z|=1} |P(z)| \tag{18}$$

and in fortiori

$$\max_{|z|=R} |P(z)| \leqslant \frac{R^n + K^\mu}{1 + K^\mu} \max_{|z|=1} |P(z)|. \tag{19}$$

Inequality (19) is a generalization of a result due to Govil and Dutt [8, Theorem 1.6] and inequality (10) is an improvement over this bound. Also for $K = \mu = 1$, inequality (19) reduces to inequality (4) due to Ankeny and Rivlin.

Next we prove the following theorem, which is an improvement as well a generalization of a result proved by Bidkham and Dewan [10].

THEOREM 2. *If* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ *is a polynomial of degree* n *having no zeros in the disk* $|z| < K$, $K \geqslant 0$, *then for* $0 \leqslant r \leqslant R \leqslant K$,

$$\max_{|z|=R} |P'(z)| \leqslant \frac{nR^{\mu-1}(R^{\mu}+K^{\mu})^{\frac{n}{\mu}-1}}{(r^{\mu}+K^{\mu})^{n/\mu}} \left\{ \max_{|z|=r} |P(z)| - \min_{|z|=K} |P(z)| \right\}. \tag{20}$$

The result is best possible and equality holds for the polynomial $P(z) = (z^{\mu} + K^{\mu})^{n/\mu}$, *where* n *is a multiple of* μ.

If we take $\mu = 1 = r$ in Theorem 2, we get the following:

COROLLARY 5. *If* $P(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *having no zeros in the disk* $|z| < K$, *where* $K \geqslant 1$, *then for* $1 \leqslant R \leqslant K$,

$$\max_{|z|=R} |P'(z)| \leqslant \frac{n(R+K)^{n-1}}{(1+K)^n} \left\{ \max_{|z|=1} |P(z)| - \min_{|z|=K} |P(z)| \right\}. \tag{21}$$

The result is sharp and equality holds for $P(z) = (z+K)^n$.

If we take $R = K = 1$ in Theorem 2, we get the following generalization of result due to Aziz and Dawood [3].

COROLLARY 6. *If* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ *is a polynomial of degree* n, *not vanishing in the disk* $|z| < 1$, *then for* $0 < r \leqslant 1$,

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{2} \left(\frac{2}{1+r^{\mu}} \right)^{n-1} \left\{ \max_{|z|=r} |P(z)| - \min_{|z|=1} |P(z)| \right\}. \tag{22}$$

The result is best possible and equality holds for the polynomial $P(z) = (z^{\mu} + 1)^{n/\mu}$, *where* n *is a multiple of* μ.

Lastly in this paper we prove the following result which is a generalization of a theorem due to Paul Turan [19].

THEOREM 3. *If* $P(z) = \sum_{j=s}^{n} a_j z^j$ *is a polynomial of degree* n *having all its zeros in the disk* $|z| \leqslant K \leqslant 1$ *with* s*-fold zeros at origin, then for* $|z| = 1$,

$$\max_{|z|=1} |P'(z)| \geqslant \frac{n+Ks}{1+K} \max_{|z|=1} |P(z)|. \tag{23}$$

The result is sharp and extremal polynomial is

$$P(z) = z^s(z+K)^{n-s}, \quad 0 < s \leqslant n.$$

The result proved by Turan [19] is a special case of Theorem 3, when $s = 0$ and $K = 1$. Also for $s = 0$, it reduces to inequality (5) due to Malik [14].

2. Lemmas

For the proofs of these theorems we need the following lemmas.

LEMMA 1. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leqslant 1$, then for $R > 1$*

$$|P(Rz)| > |P(z)| \quad \textit{for} \quad |z| = 1. \tag{24}$$

The lemma is a special case of result due to Aziz and Rather [4, Lemma 2], when $K = 1$.

LEMMA 2. *If $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ is a polynomial of degree n having no zeros in the disk $|z| \leqslant K$ where $K \geqslant 1$, then for $|z| = 1$ and $R > 1$,*

$$|P(Rz) - P(z)| \leqslant \frac{1}{K^{\mu+1}} \left\{ \frac{1 + \dfrac{R^\mu - 1}{R^n - 1}\left|\dfrac{a_\mu}{a_0}\right| K^{\mu+1}}{\dfrac{R^\mu - 1}{R^n - 1}\left|\dfrac{a_\mu}{a_0}\right| K^{\mu-1} + 1} \right\} |Q(Rz) - Q(z)|. \tag{25}$$

Proof of Lemma 2. The result is trivial if $R = 1$, so we suppose that $R > 1$. Since $P(z)$ has all zeros in $|z| \geqslant K$ where $K \geqslant 1$, the polynomial $F(z) = P(Kz)$ has all its zeros in $|z| \geqslant 1$, so that the polynomial $G(z) = z^n\overline{F(1/\overline{z})}$ has all its zeros in $|z| \leqslant 1$ and $|F(z)| = |G(z)|$ for $|z| = 1$. Hence the function $G(z)/F(z)$ is analytic in $|z| \leqslant 1$ and

$$\left|\frac{G(z)}{F(z)}\right| = \left|\frac{G(z)}{F(z)}\right| = 1.$$

A direct application of the maximum modules principle shows that

$$|G(z)| \leqslant |F(z)| \quad \text{for} \quad |z| \leqslant 1. \tag{26}$$

We now show that all the zeros of $f(z) = F(z) - \beta G(z)$ lie in $|z| \leqslant 1$, for every β with $|\beta| > 1$. First suppose that $F(z)$ has all its zeros on $|z| = 1$. If $z_1, z_2, \ldots, z_n$ are zeros of $F(z)$, then $|z_j| = 1$ for all $j = 1, 2, \ldots, n$ and we have $F(z) = c\Pi_{j=1}^{n}(z - z_j)$, so that

$$\begin{aligned} G(z) &= Z^n\overline{F(1/\overline{z})} = \overline{c}\,\Pi_{j=1}^n(1 - z\overline{z}_j) \\ &= \overline{c}\,\Pi_{j=1}^n\,(1 - z/z_j) \\ &= \overline{c}\,\Pi_{j=1}^n(-1)^n\left\{\frac{z - z_j}{z_j}\right\} \\ &= \left\{\overline{c}(-1)^n\Pi_{j=1}^n\frac{1}{z_j}\right\}\Pi_{j=1}^n(z - z_j) \\ &= uF(z), \end{aligned}$$

where

$$|u| = \left|\frac{\overline{c}}{c}(-1)^n\Pi_{j=1}^n\frac{1}{z_j}\right| = 1.$$

Hence all the zeros of $f(z) = F(z) - \beta G(z) = (1 - \beta u)F(z)$ also lie on $|z| = 1$ and therefore, in $|z| \leqslant 1$. Now suppose that $F(z)$ has at least one zero in $|z| < 1$, then obviously $F(z)/G(z)$ is not a constant and hence from (26), it follows that

$$|G(z)| < |F(z)| \quad \text{for} \quad |z| < 1. \tag{27}$$

Replacing z by $\frac{1}{\overline{z}}$ in (27), we obtain

$$|F(z)| < |G(z)| \quad \text{for} \quad |z| > 1.$$

Using Rouche's theorem, we conclude that polynomial $f(z) = F(z) - \beta G(z)$ has all its zeros in $|z| \leqslant 1$. Thus in any case the polynomial $f(z)$ has all its zeros in $|z| \leqslant 1$, for every β, with $|\beta|>1$. Applying Lemma 1 to the polynomial $f(z)$, we get

$$|f(z)| < |f(Rz)| \quad \text{for} \quad |z| = 1 \quad \text{and} \quad R > 1.$$

Since all zeros of $f(Rz)$ lie in $|z| \leqslant 1/R < 1$, again Rouche's theorem shows that the polynomial

$$g(z) = f(Rz) - f(z) = (F(Rz) - F(z)) - \beta(G(Rz) - G(z)) \tag{28}$$

has all its in $|z| < 1$, for every complex number β with $|\beta| > 1$ and $R > 1$. This implies

$$|F(Rz) - F(z)| \leqslant |G(Rz) - G(z)| \tag{29}$$

for $|z| \geqslant 1$ and $R > 1$. If inequality (29) is not true, then there is a point $z = z_0$ with $|z_0| \geqslant 1$ such that

$$|F(Rz_0) - F(z_0)| > |G(Rz_0) - G(z_0)|.$$

Since $G(z)$ has all its zeros in $|z| \leqslant 1$, it follows that all the zeros of $G(Rz) - G(z)$ lie in $|z| < 1$ for every $R > 1$. Hence

$$G(Rz_0) - G(z_0) \neq 0 \quad \text{with} \quad |z_0| \geqslant 1.$$

We take

$$\beta = \frac{F(Rz_0) - F(z_0)}{G(Rz_0) - G(z_0)},$$

so that $|\beta| > 1$ and with this choise of β, from (28), we get $g(z_0) = 0$, where $|z_0| \geqslant 1$. This contradicts the fact that all the zeros of $g(z)$ lie in $|z| < 1$. Thus

$$|F(Rz) - F(z)| \leqslant |G(Rz) - G(z)| \quad \text{for} \quad |z| \geqslant 1 \quad \text{and} \quad R > 1.$$

Replacing $F(z)$ by $P(Kz)$ and $G(z)$ by $z^n\overline{P(K/\overline{z})}$, we get

$$\begin{aligned}|P(RKz - P(Kz))| &\leqslant \left|R^n z^n\overline{P(K/R\overline{z})} - z^n\overline{P(K/\overline{z})}\right| \\ &= |R^n P(Kz/R) - P(Kz)| \quad \text{for} \quad |z| = 1, \quad R > 1.\end{aligned}$$

Since the polynomial $R^n P(Kz/R) - P(Kz)$ does not vanish in $|z| \leqslant 1$, therefore

$$H(z) = \frac{P(RKz) - P(Kz)}{R^n P(Kz/R) - P(Kz)}$$

is analytic in $|z| \leqslant 1$ and by the maximum modules principles, we have

$$|H(z)| \leqslant 1 \quad \text{for} \quad |z| \leqslant 1.$$

Also, it can be easily seen that

$$H(o) = H'(o) = \cdots = H^{(\mu-1)}(o) = 0$$

and

$$H^{\mu}(o) = \frac{R^{\mu} - 1}{R^n - 1}\left(a_{\mu}/a_0\right) K^{\mu}.$$

Hence by a generalized form of Schewarz's Lemma

$$|H(z)| \leqslant |z|^{\mu} \frac{|z| + \dfrac{R^{\mu} - 1}{R^n - 1}\left|\dfrac{a_{\mu}}{a_0}\right| K^{\mu}}{\dfrac{R^{\mu} - 1}{R^n - 1}\left|\dfrac{a_{\mu}}{a_0}\right| K^{\mu}|z| + 1} \quad \text{for} \quad |z| \leqslant 1.$$

Equivalently

$$\left|\frac{P(RKz) - P(Kz)}{R^n P(Kz/R) - P(Kz)}\right| \leqslant |z|^{\mu} \frac{|z| + \dfrac{R^{\mu} - 1}{R^n - 1}\left|\dfrac{a_{\mu}}{a_0}\right| K^{\mu}}{\dfrac{R^{\mu} - 1}{R^n - 1}\left|\dfrac{a_{\mu}}{a_0}\right| K^{\mu}|z| + 1} \quad \text{for} \quad |z| \leqslant 1.$$

We take $z = e^{i\theta}/K$, $0 \leqslant \theta < 2\pi$, so that $|z| = 1/K$ and we get

$$\frac{|P(Re^{i\theta}) - P(e^{i\theta})|}{|R^n P(e^{i\theta}/R) - P(e^{i\theta})|} \leqslant \frac{1}{K^{\mu+1}} \frac{1 + \dfrac{R^{\mu} - 1}{R^n - 1}\left|\dfrac{a_{\mu}}{a_0}\right| K^{\mu+1}}{\dfrac{R^{\mu} - 1}{R^n - 1}\left|\dfrac{a_{\mu}}{a_0}\right| K^{\mu-1} + 1}.$$

This implies

$$\frac{|P(Rz) - P(z)|}{|R^n P(z/R) - P(z)|} \leqslant \frac{1}{K^{\mu+1}} \frac{1 + \dfrac{R^{\mu} - 1}{R^n - 1}\left|\dfrac{a_{\mu}}{a_0}\right| K^{\mu+1}}{\dfrac{R^{\mu} - 1}{R^n - 1}\left|\dfrac{a_{\mu}}{a_0}\right| K^{\mu-1} + 1} \quad \text{for} \quad |z| \leqslant 1. \tag{30}$$

From this (25) follows and this completes the proof of Lemma 2.

LEMMA 3. *If $P(z)$ is a polynomial of degree n, then for every $R > 1$*

$$|P(Rz) - P(z)| + |Q(Rz) - Q(z)| \leqslant (R^n - 1) \max_{|z|=1} |P(z)|. \tag{31}$$

The above lemma was proved by Aziz [2] (see also [11]).

LEMMA 4. *If $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ is a polynomial of degree n which does not vanish in $|z| < K$, $K \geqslant 1$, then*

$$\max_{|z|=1} |P'(z)| \leqslant \frac{1}{1+K^\mu} \left\{ \max_{|z|=1} |P(z)| - \min_{|z|=K} |P(z)| \right\}. \tag{32}$$

This lemma was proved by Dewan and Pukhta [16, Theorem 1.4], (see also [5]).

Next we use Lemma 4 to prove the following:

LEMMA 5. *Let $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ be a polynomial of degree n such that*

$$M(P,r) = \max_{|z|=r} |P(z)| \quad \textit{and} \quad m(P,r) = \min_{|z|=r} |P(z)|.$$

If $P(z)$ has no zeros in $|z| < K$, $K > 0$, then for $0 \leqslant r \leqslant R \leqslant K$,

$$M(P,r) \geqslant \left\{ \frac{r^\mu + K^\mu}{R^\mu + K^\mu} \right\}^{n/\mu} M(P,R) + \left[1 - \left\{ \frac{r^\mu + K^\mu}{R^\mu + K^\mu} \right\}^{n/\mu} \right] m(P,K). \tag{33}$$

The result is sharp and equality holds for the polynomial $P(z) = (z^\mu + K^\mu)^{n/\mu}$, where n is a multiple of μ.

Proof of Lemma 5. By hypothesis $P(z)$ has no zeros in $|z| < K$, therefore, the polynomial $F(z) = P(sz)$ has no zeros in $|z| < \frac{K}{s}$, $\frac{K}{s} \geqslant 1$ where $0 \leqslant s \leqslant K$. Since $\frac{K}{s} \geqslant 1$, by Lemma 4, it follows that

$$\max_{|z|=1} |F'(z)| \leqslant \frac{n}{1 + \frac{K^\mu}{s^\mu}} \left\{ \max_{|z|=1} |F(z)| - \min_{|z|=K/s} |F(z)| \right\}.$$

This gives

$$\max_{|z|=s} |P'(z)| \leqslant \frac{ns^{\mu-1}}{s^\mu + K^\mu} \left\{ \max_{|z|=s} |P(z)| - \min_{|z|=K} |P(z)| \right\}. \tag{34}$$

Now, for $0 \leqslant r \leqslant R \leqslant K$, and $0 \leqslant \theta \leqslant 2\pi$, we have

$$\begin{aligned} |P(Re^{i\theta}) - P(re^{i\theta})| &= \left| \int_r^R e^{i\theta} P'(se^{i\theta}) ds \right| \\ &\leqslant \int_r^R |P'(se^{2\theta})| ds. \end{aligned}$$

This gives

$$|P(Re^{i\theta})| \leqslant |P(re^{2\theta})| + \int_r^R |P'(se^{2\theta})|ds,$$

from which it follows that

$$M(P,R) \leqslant M(P,r) + \int_r^R M(P',s)ds. \tag{35}$$

Using (34) in (35), we obtain

$$M(P,R) \leqslant M(P,r) + n\left[\int_r^R \frac{s^{\mu-1}}{s^\mu + K^\mu}M(P,s)ds - \int_r^R \frac{s^{\mu-1}}{s^\mu + K^\mu}m(P,K)ds\right]. \tag{36}$$

If

$$\phi(R) = M(P,r) + n\left[\int_r^R \frac{s^{\mu-1}}{s^\mu + K^\mu}M(P,s)ds - \int_r^R \frac{s^{\mu-1}}{s^\mu + K^\mu}M(P,K)ds\right],$$

then

$$\phi'(R) = \frac{nR^{\mu-1}}{R^\mu + K^\mu}M(P,R) - \frac{nR^{\mu-1}}{R^\mu + K^\mu}m(P,K). \tag{37}$$

From (37) with the help of (36), we conclude that

$$\phi'(R) - \frac{nR^{\mu-1}}{R^\mu + K^\mu}\{\phi(R) - m(P,K)\} \leqslant 0. \tag{38}$$

Multiplying the two sides of (38) by $(R^\mu + K^\mu)^{-n/\mu}$, we get

$$\phi'(R)(R^\mu + K^\mu)^{-n/\mu} - n(\phi(R) - m(P,K))(R^\mu + K^\mu)^{-n/\mu^{-1}}R^{\mu-1} \leqslant 0,$$

which implies

$$\frac{d}{dR}\left\{(\phi(R) - m(P,K))(R^\mu + K^\mu)^{-n/\mu}\right\} \leqslant 0. \tag{39}$$

From (39) we conclude that the function

$$\{\phi(R) - m(P,K)\}(R^\mu + K^\mu)^{-n/\mu}$$

is a non increasing function of R in $(0,K)$. Hence for $0 \leqslant r \leqslant R \leqslant K$,

$$\phi(r) \geqslant \left[\frac{K^\mu + r^\mu}{K^\mu + R^\mu}\right]^{n/\mu}\phi(R) + \left\{1 - \left[\frac{K^\mu + r^\mu}{K^\mu + R^\mu}\right]^{n/\mu}\right\}m(P,K). \tag{40}$$

Since $\phi(R) \geqslant M(P,R)$ and $\phi(r) = M(P,r)$ it follows from (40) that

$$M(P,r) \geqslant \left[\frac{K^\mu + r^\mu}{K^\mu + R^\mu}\right]^{n/\mu}M(P,R) + \left\{1 - \left[\frac{K^\mu + r^\mu}{K^\mu + R^\mu}\right]^{n/\mu}\right\}m(P,K).$$

This proves the lemma completely.

3. Proofs of Theorems

Proof of Theorem 1. Since $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ does not vanish in $|z| < K$, $K \geqslant 1$, by Lemma 2, we have

$$\frac{K^{\mu+1}\left\{\dfrac{R^\mu - 1}{R^n - 1}\left|\dfrac{a_\mu}{a_0}\right| K^{\mu-1} + 1\right\}}{1 + \dfrac{R^\mu - 1}{R^n - 1}\left|\dfrac{a_\mu}{a_0}\right| K^{\mu+1}} |P(Rz) - P(z)| \leqslant |Q(Rz) - Q(z)|. \tag{41}$$

Inequality (41) implies with the help of Lemma 3 that

$$\left\{1 + \frac{K^{\mu+1}\left(\dfrac{R^\mu - 1}{R^n - 1}\left|\dfrac{a_\mu}{a_0}\right| K^{\mu-1} + 1\right)}{1 + \dfrac{R^\mu - 1}{R^n - 1}\left|\dfrac{a_\mu}{a_0}\right| K^{\mu+1}}\right\} |P(Rz) - P(z)| \leqslant |P(Rz) - P(z)| + |Q(Rz) - Q(z)|$$

$$\leqslant (R^n - 1) \max_{|z|=1} |P(z)|.$$

This gives

$$|P(Rz) - P(z)| \leqslant (R^n - 1)\left\{\frac{1 + \dfrac{R^\mu - 1}{R^n - 1}\left|\dfrac{a_\mu}{a_0}\right| K^{\mu+1}}{1 + K^{\mu+1} + \dfrac{R^\mu - 1}{R^n - 1}\left|\dfrac{a_\mu}{a_0}\right| (K^{2\mu} + K^{\mu+1}}\right\} \max_{|z|=1} |P(z)|,$$

which is (9) and this proves Theorem 1 completely.

Proof of Theorem 2. By hypothesis $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ does not vanish in $|z| < K$, therefore the polynomial $F(z) = P(Rz)$ has no zero in $|z| < K/R$, $K/R \geqslant 1$. Applying Lemma 4 to the polynomial $F(z)$, we get

$$\max_{|z|=1} |F'(z)| \leqslant \frac{n}{1 + \dfrac{K^\mu}{R^\mu}} \left\{\max_{|z|=1} |F(z)| - \min_{|z|=K/R} |F(z)|\right\},$$

which gives

$$\max_{|z|=R} |P'(z)| \leqslant \frac{nR^{\mu-1}}{R^\mu + K^\mu} \left\{\max_{|z|=R} |P(z)| - \min_{|z|=K} |P(z)|\right\}. \tag{42}$$

Now if $0 \leqslant r \leqslant R \leqslant K$, then by Lemma 5, we have

$$\max_{|z|=R} |P(z)| \leqslant \left(\frac{R^\mu + K^\mu}{r^\mu + k^\mu}\right)^{n/\mu} \max_{|z|=r} |P(z)| + \left\{1 - \left(\frac{R^\mu + K^\mu}{r^\mu + k^\mu}\right)^{n/\mu}\right\} \min_{|z|=K} |P(z)|. \tag{43}$$

From (42) and (43), it follows that

$$\max_{|z|=R}|P'(z)| \leqslant \frac{nR^{\mu-1}}{R^{\mu}+K^{\mu}}\left\{\left(\frac{R^{\mu}+K^{\mu}}{r^{\mu}+K^{\mu}}\right)^{n/\mu}\max_{|z|=r}|P(z)| - \left(\frac{R^{\mu}+K^{\mu}}{r^{\mu}+K^{\mu}}\right)^{n/\mu}\min_{|z|=K}|P(z)|\right\}$$
$$= \frac{nR^{\mu-1}}{R^{\mu}+K^{\mu}}\left(\frac{R^{\mu}+K^{\mu}}{r^{\mu}+K^{\mu}}\right)^{n/\mu}\left\{\max_{|z|=r}|P(z)| - \min_{|z|=K}|P(z)|\right\},$$

which is equivalent to (20) and this completes the proof of Theorem 2.

Proof of Theorem 3. Since $P(z)$ has all its zeros in $|z| \leqslant K$, $K \leqslant 1$, whit s-fold zeros at origin, we write

$$P(z) = z^s h(z), \tag{44}$$

where $h(z)$ is a polynomial of degree $n-s$ having all its zeros in $|z| \leqslant K$, $K \leqslant 1$. From (44) we get

$$\frac{zP'(z)}{P(z)} = s + \frac{zh'(z)}{h(z)}. \tag{45}$$

If $z_1, z_2, \dots, z_{n-s}$ are the zeros of $h(z)$, then $|z_j| \leqslant K \leqslant 1$, and from (45), we have

$$\begin{aligned} Re\left\{\frac{e^{i\theta}P'(e^{i\theta})}{P(e^{i\theta})}\right\} &= s + Re\left\{\frac{e^{i\theta}h'(e^{i\theta})}{h(e^{i\theta})}\right\} \\ &= s + Re\sum_{j=1}^{n-s}\left(\frac{e^{i\theta}}{e^{i\theta}-zj}\right) \\ &= s + \sum_{j=1}^{n-s} Re\left(\frac{1}{1-z_je^{-i\theta}}\right) \end{aligned} \tag{46}$$

for points $e^{i\theta}$, $0 \leqslant \theta < 2\pi$, which are not the zeros of $h(z)$. Now, if $|w| \leqslant K \leqslant 1$, then it can be easily verified that

$$Re\left(\frac{1}{1-w}\right) \geqslant \frac{1}{1+K}.$$

Using this fact in (46), we get

$$\begin{aligned} \left|\frac{P'(e^{i\theta})}{P(e^{i\theta})}\right| \geqslant Re\left(\frac{e^{i\theta}P'(e^{i\theta})}{P(e^{i\theta})}\right) &= s + \sum_{j=1}^{n-s} Re\left(\frac{1}{1-z_je^{-i\theta}}\right) \\ \geqslant s + \frac{n-s}{1+K} \end{aligned}$$

which gives

$$|P'(e^{i\theta})| \geqslant \frac{n+sK}{1+K}|P(e^{i\theta})| \tag{47}$$

for points $e^{i\theta}$, which are not the zeros of $P(z)$. Since inequality (47) is trivially true for points $e^{i\theta}$, which are the zeros of $P(z)$, it follows that

$$|P'(z)| \geqslant \frac{n+sK}{1+K}|P(z)| \quad \text{for} \quad |z| = 1.$$

This immediately leads to

$$\max_{|z|=1} |P'(z)| \geqslant \frac{n+sK}{1+K} \max_{|z|=1} |P(z)|$$

which completes the proof of Theorem 3.

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(Received January 19, 2002)

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