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SOME COMPACT GENERALIZATIONS OF BERNSTEIN–TYPE INEQUALITIES FOR POLYNOMIALS

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Abstract. Let $P(z)$ be a polynomial of degree $n \geqslant 1$. In this paper we consider a more general problem of investigating the dependence of maximum of

$$\left|P(Rz) - \alpha P(z) + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}P(z)\right|, \qquad R \geqslant 1,$$

on the maximum of $|P(z)|$ on $|z| = 1$ where α, β are arbitrary complex numbers with $|\alpha| \leqslant 1$, $|\beta| \leqslant 1$ and obtain certain sharp compact generalizations of well-known Bernstein-type polynomial inequalities.

1. Introduction and statement of results

Let $P(z)$ be a polynomial of degree at most n, then according to a famous result known as Bernstein's inequality (for reference, see [11, p. 531] or [14]),

$$\max_{|z|=1} |P'(z)| \leqslant n \max_{|z|=1} |P(z)| \tag{1}$$

whereas concerning the maximum modulus of $P(z)$ on a large circle $|z| = R > 1$, we have

$$\max_{|z|=R} |P(z)| \leqslant R^n \max_{|z|=1} |P(z)| \tag{2}$$

(for reference see [11, p. 442] or [12, vol. I, p. 137]).

If we restrict ourselves to the class of polynomials having no zero in $|z| < 1$, then inequalities (1) and (2) can be sharpened. In fact, if $P(z) \neq 0$ in $|z| < 1$, then (1) and (2) can be respectively replaced by

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{2} \max_{|z|=1} |P(z)| \tag{3}$$

and

$$\max_{|z|=R} |P(z)| \leqslant \frac{R^n + 1}{2} \max_{|z|=1} |P(z)|, \qquad R > 1. \tag{4}$$

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Inequality (3) was conjectured by P. Erdös and later verified by P. D. Lax [8] (see also [4]). Ankeny and Rivlin [1] used (3) to prove inequality (4).

Recently both the inequalities (3) and (4) were generalized by Jain [7] who proved that if $P(z) \neq 0$ in $|z| < 1$, then for every real or complex number β with $|\beta| \leqslant 1$,

$$\left|zP'(z) + \frac{n\beta}{2}P(z)\right| \leqslant \frac{n}{2}\left\{\left|1 + \frac{\beta}{2}\right| + \left|\frac{\beta}{2}\right|\right\} \max_{|z|=1} |P(z)| \quad \text{for} \quad |z| = 1 \tag{5}$$

and

$$\left|P(Rz) + \beta\left(\frac{R+1}{2}\right)^n P(z)\right| \leqslant \frac{1}{2}\left\{\left|R^n + \beta\left(\frac{R+1}{2}\right)^2\right| + \left|1 + \beta\left(\frac{R+1}{2}\right)^n\right|\right\} \max_{|z|=1} |P(z)| \tag{6}$$

for $|z| = 1$ and $R \geqslant 1$.

More recently the authors [5] have investigated the dependence of

$$\max_{|z|=1} |P(Rz) - \alpha P(z)| \quad \text{on} \quad \max_{|z|=1} |P(z)|$$

for every real or complex number α with $|\alpha| \leqslant 1$ and $R \geqslant 1$. As a compact generalization of inequalities (1) and (2), they have shown that if $P(z)$ is a polynomial of degree n, then for every real or complex number α with $|\alpha| \leqslant 1$ and $R \geqslant 1$,

$$|P(Rz) - \alpha P(z)| \leqslant |R^n - \alpha||z|^n \max_{|z|=1} |P(z)| \quad \text{for} \quad |z| \geqslant 1. \tag{7}$$

The results is best possible and equality in (7) holds for $P(z) = \lambda z^n$, $\lambda \neq 0$. Inequality (1) can be obtained from inequality (7) by dividing the two sides of (7) by $R - 1$ and taking limit $R \to 1$ with $\alpha = 1$. For $\alpha = 0$, inequality (7) reduces to (2).

As a corresponding compact generalization of inequalities (3) and (4), the authors [5] have also shown that if $P(z) \neq 0$ in $|z| < 1$, then for every real or complex number α with $|\alpha| \leqslant 1$ and $R \geqslant 1$,

$$|P(Rz) - \alpha P(z)| \leqslant \frac{1}{2}\left\{|R^n - \alpha||z|^n + |1 - \alpha|\right\} \max_{|z|=1} |P(z)| \quad \text{for} \quad |z| \geqslant 1. \tag{8}$$

The result is sharp and equality in (8) holds for $P(z) = z^n + 1$.

In this paper we consider a more general problem of investigating the dependence of

$$\left|P(Rz) - \alpha P(z) + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}P(z)\right|, \qquad R \geqslant 1,$$

on the maximum of $|P(z)|$ on $|z| = 1$ for all real of complex numbers α, β with $|\alpha| \leqslant 1$, $\beta \leqslant 1$ and develop a unified method for arriving at these results. We first prove the following interesting result which is a compact generalization of the inequalities (1), (2) and (7):

THEOREM 1. *If $P(z)$ is a polynomial of degree n, then for all real or complex numbers α, β with $|\alpha| \leqslant 1$, $|\beta| \leqslant 1$ and $R \geqslant 1$,*

$$\left|P(Rz) - \alpha P(z) + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}P(z)\right| \leqslant \left|(R^n - \alpha) + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}\right||z|^n \max_{|z|=1} |P(z)| \quad \textit{for} \quad |z| \geqslant 1. \tag{9}$$

The result is sharp and equality in (9) holds for $P(z) = \lambda z^n$, $\lambda \neq 0$.

REMARK 1. Theorem 1 includes a result due to Jain [7, Lemma 5] as a special case for $\alpha = 0$ whereas inequality (9) reduces to inequality (7) for $\beta = 0$.

Several other interesting results follow from Theorem 1. Here we mention the following result which easily follows by taking $\alpha = 1$ in the Theorem 1.

COROLLARY 1. *If* $P(z)$ *is a polynomial of degree* n, *then for every real or complex number* β *with* $|\beta| \leqslant 1$ *and* $R \geqslant 1$,

$$\left|P(Rz) - P(z) + \beta\Big\{\Big(\frac{R+1}{2}\Big)^n - 1\Big\}P(z)\right| \leqslant \left|(R^n - 1) + \beta\Big\{\Big(\frac{R+1}{2}\Big)^n - 1\Big\}\right| |z|^n \max_{|z|=1} |P(z)| \qquad \text{for} \quad |z| \geqslant 1. \tag{10}$$

The result is sharp and equality in (10) holds for $P(z) = \lambda z^n$, $\lambda \neq 0$.

Dividing the two sides of (10) by $R - 1$ and letting $R \to 1$, we obtain

$$\left|zP'(z) + \frac{n}{2}\beta P(z)\right| \leqslant n\left|1 + \frac{\beta}{2}\right| |z|^n \max_{|z|=1} \quad \text{for} \quad |z| \geqslant 1,$$

which in particular includes a result due to Jain [7, Lemma 2] as a special case.

Next we use Theorem 1 to prove the following interesting result.

THEOREM 2. *If* $P(z)$ *is a polynomial of degree* n, *then for all real or complex numbers* α, β *with* $|\alpha| \leqslant 1$, $|\beta| \leqslant 1$ *and* $R \geqslant 1$,

$$\begin{aligned}\left|P(Rz) - \alpha P(z) + \beta\Big\{\Big(\frac{R+1}{2}\Big)^n - |\alpha|\Big\}P(z)\right| &\\ + \left|Q(Rz) - \alpha Q(z) + \beta\Big\{\Big(\frac{R+1}{2}\Big)^n - |\alpha|\Big\}Q(z)\right| &\\ \leqslant \Big[\left|R^n - \alpha + \beta\Big\{\Big(\frac{R+1}{2}\Big)^n - |\alpha|\Big\}\right| |z|^n &\\ + \left|1 - \alpha + \beta\Big\{\Big(\frac{R+1}{2}\Big)^n - |\alpha|\Big\}\right|\Big] \max_{|z|=1} |P(z)| &\end{aligned} \tag{11}$$

for $|z| \geqslant 1$, *where* $Q(z) = z^n\overline{P(1/\overline{z})}$.

The result is sharp and equality in (11) holds for $P(z) = \lambda z^n$, $\lambda \neq 0$.

REMARK 2. Theorem 2 includes some well-known polynomial inequalities as special cases. For example, inequality (11) reduces to a result due to A. Aziz [2, Lemma 2] for $\alpha = 1$ and $\beta = 0$ whereas for $\alpha = 0$, Theorem 2 reduces to a result due to Jain [6, Theorem 1]. If we take $\beta = 0$ inequality (11), we obtain Theorem 3 of [5] due to A. Aziz and N. A. Rather.

The following corollary immediately follows from Theorem 2 by taking $\alpha = 1$.

COROLLARY 2. *If $P(z)$ is a polynomial of degree n, then for every real or complex number β with $|\beta| \leqslant 1$ and $R \geqslant 1$,*

$$\begin{aligned}&\left|P(Rz) - P(z) + \beta\Big\{\Big(\frac{R+1}{2}\Big)^n - 1\Big\}P(z)\right| \\ &\qquad + \left|Q(Rz) - Q(z) + \beta\Big\{\Big(\frac{R+1}{2}\Big)^n - 1\Big\}Q(z)\right| \\ &\leqslant \Big[\Big|R^n - 1 + \beta\Big\{\Big(\frac{R+1}{2}\Big)^n - 1\Big\}\Big||z|^n \\ &\qquad + |\beta|\Big\{\Big(\frac{R+1}{2}\Big)^n - 1\Big\}\Big] \max_{|z|=1}|P(z)| \qquad (12)\end{aligned}$$

for $|z| \geqslant 1$, where $Q(z) = z^n\overline{P(1/\overline{z})}$.

Inequality (12) becomes equality for $P(z) = \lambda z^n$, $\lambda \neq 0$.

REMARK 3. Corollary 2 includes as special case a result due to Jain [7, Lemma 3] which is obtained by dividing the two sides of (12) by $R-1$ and letting $R \to 1$.

Theorem 1 can be sharpened if we restrict ourselves to the class of polynomials having no zeros in $|z| < 1$. In this direction, we next prove the following interesting result which is a compact generalization of inequalities (3), (4), (5), (6) and (8).

THEOREM 3. *If $P(z)$ is a polynomial of degree n which does not vanish in $|z| < 1$, then for all real or complex numbers α, β with $|\alpha| \leqslant 1$, $|\beta| \leqslant 1$ and $R \geqslant 1$*

$$\begin{aligned}&\left|P(Rz) - \alpha P(z) + \beta\Big\{\Big(\frac{R+1}{2}\Big)^n - |\alpha|\Big\}P(z)\right| \\ &\leqslant \frac{1}{2}\Big[\Big|R^n - \alpha + \beta\Big\{\Big(\frac{R+1}{2}\Big)^n - |\alpha|\Big\}\Big||z|^n \\ &\qquad + \Big|1 - \alpha + \beta\Big\{\Big(\frac{R+1}{2}\Big)^n - |\alpha|\Big\}\Big|\Big] \max_{|z|=1}|P(z)| \qquad (13)\end{aligned}$$

for $|z| \geqslant 1$. The result is best possible and equality in (13) holds for $P(z) = z^n + 1$.

REMARK 4. If we take $\alpha = 0$ in Theorem 3, we get inequality (6) whereas inequality (8) follows by taking $\beta = 0$ in inequality (13). For $\alpha = \beta = 0$, inequality (13) reduces to inequality (4).

The next corollary which is obtained by taking $\alpha = 1$ in Theorem 3, is a refinement of Corollary 1, for polynomials not vanishing in the unit disk.

COROLLARY 3. *If $P(z)$ is a polynomial of degree n which does not vanish in $|z| < 1$, then for every real or complex number β with $|\beta| \leqslant 1$ and $R \geqslant 1$,*

$$\begin{aligned}&\left|P(Rz) - P(z) + \beta\Big\{\Big(\frac{R+1}{2}\Big)^n - 1\Big\}P(z)\right| \\ &\leqslant \frac{1}{2}\Big[\Big|(R^n - 1) + \beta\Big\{\Big(\frac{R+1}{2}\Big)^n - 1\Big\}\Big||z|^n + |\beta|\Big\{\Big(\frac{R+1}{2}\Big)^n - 1\Big\}\Big] \max_{|z|=1}|P(z)|\end{aligned}$$

for $|z| \geqslant 1.$ (14)

The result is best possible and equality in (13) holds for $P(z) = z^n + 1$.

REMARK 5. Dividing the two sides of (14) by $R-1$ and letting $R \to 1$, we obtain, in particular, inequality (3).

2. Lemmas

For the proofs of these theorems we need following lemmas.

LEMMA 1. *If* $P(z)$ *is a polynomial of degree* n *having all its zeros in* $|z| \leqslant k$, *where* $k \leqslant 1$, *then for every* $R > 1$

$$|P(Rz) \geqslant \left(\frac{R+k}{1+k}\right)^n |P(z)| \quad for \quad |z| = 1.$$

Lemma 1 follows by using the argument similar to the proof of Theorem 1 [3]. Here we use Lemma 1 to prove:

LEMMA 2. *If* $F(z)$ *is a polynomial of degree* n *having all its zeros in* $|z| \leqslant 1$ *and* $P(z)$ *is a polynomial of degree atmost* n *such that*

$$|P(z)| \leqslant |F(z)| \quad for \quad |z| = 1,$$

then for all real or complex numbers α, β *with* $|\alpha| \leqslant 1$, $|\beta| \leqslant 1$ *and* $R \geqslant 1$,

$$\left|P(Rz) - \alpha P(z) + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}P(z)\right| \leqslant \left|F(Rz) - \alpha F(z) + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}F(z)\right| \quad for \quad |z| \geqslant 1. \tag{15}$$

REMARK 6. Dividing the two sides of (15) by $R-1$ and letting $R \to 1$ with $\alpha = 1$, we obtain a result due to Malik and Vong [10].

Proof of Lemma 2. In case $R = 1$, we have nothing to prove. Henceforth we assume $R > 1$. By hypothesis $F(z)$ is a polynomial of degree n having all its zeros in $|z| \leqslant 1$ and $P(z)$ is a polynomial of degree atmost n such that

$$|P(z)| \leqslant |F(z)| \quad \text{for} \quad |z| = 1, \tag{16}$$

therefore, if $F(z)$ has a zero of multiplicity s at $z = e^{i\theta_0}$, then $P(z)$ must also have a zero of multiplicity atleast s at $z = e^{i\theta_0}$. If $P(z)/F(z)$ is a constant, then the inequality (15) is obvious. We now assume that $P(z)/F(z)$ is not a constant so that by the maximum modulus principle, it follows that

$$|P(z)| < |F(z)| \quad \text{for} \quad |z| > 1.$$

Suppose $F(z)$ has m zeros on $|z| = 1$ where $0 \leqslant m < n$ so that we write

$$F(z) = F_1(z)F_2(z)$$

where $F_1(z)$ is a polynomial of degree m whose all zeros lie on $|z|=1$ and $F_2(z)$ is a polynomial of degree exactly $n-m$ having all its zeros in $|z|<1$. This implies with the help of inequality (16) that

$$P(z)=P_1(z)F_1(z)$$

where $P_1(z)$ is a polynomial of degree atmost $n-m$. Now, from inequality (16), we get

$$|P_1(z)|\leqslant|F_2(z)| \quad \text{for} \quad |z|=1$$

where $F_2(z)\neq 0$ for $|z|=1$. Therefore, for every real or complex number λ with $|\lambda|>1$, a direct application of Rouche's theorem shows that all the zeros of the polynomial $P_1(z)-\lambda F_2(z)$ of degree $n-m\geqslant 1$ lie in $|z|<1$. Hence the polynomial

$$G(z)=F_1(z)\Big(P_1(z)-\lambda F_2(z)\Big)=P(z)-\lambda F(z)$$

has all its zeros in $|z|\leqslant 1$ with atleast one zero in $|z|<1$, so that we can write

$$G(z)=\left(z-re^{i\delta}\right)H(z)$$

where $r<1$ and $H(z)$ is a polynomial of degree $n-1$ having all its zeros in $|z|\leqslant 1$. Hence with the help of Lemma 1 with $k=1$, for every $R>1$, $0\leqslant\theta<2\pi$,

$$\begin{aligned}|G(Re^{i\theta})| &= |Re^{i\theta}-re^{i\delta}||H(Re^{i\theta})|\\ &\geqslant |Re^{i\theta}-re^{i\delta}|\Big(\frac{R+1}{2}\Big)^{n-1}|H(e^{i\theta})|\\ &= \Big(\frac{R+1}{2}\Big)^{n-1}\left|\frac{Re^{i\theta}-re^{i\delta}}{e^{i\theta}-re^{i\delta}}\right||(e^{i\theta}-re^{i\delta})H(e^{i\theta})|\\ &\geqslant \Big(\frac{R+1}{2}\Big)^{n-1}\Big(\frac{R+1}{1+r}\Big)|G(e^{i\theta})|.\end{aligned}$$

This implies

$$\Big(\frac{1+r}{R+r}\Big)|G(Re^{i\theta})|\geqslant\Big(\frac{R+1}{2}\Big)^{n-1}|G(e^{i\theta})|, \quad R>1 \quad \text{and} \quad 0\leqslant\theta<2\pi. \tag{17}$$

Since $R>1>r$ so that $G(Re^{i\theta})\neq 0$, $0\leqslant\theta<2\pi$ and $\frac{2}{R+1}>\frac{1+r}{R+r}$, from inequality (17), we obtain

$$|G(Re^{i\theta})|>\Big(\frac{R+1}{2}\Big)^{n}|G(e^{i\theta})|, \quad R>1 \quad \text{and} \quad 0\leqslant\theta<2\pi. \tag{18}$$

Equivalently,

$$|G(Rz)|>\Big(\frac{R+1}{2}\Big)^{n}|G(z)| \quad \text{for} \quad |z|=1 \quad \text{and} \quad R>1.$$

Hence for every real or complex number α with $|\alpha|\leqslant 1$, we have

$$\begin{aligned}|G(Rz)-\alpha G(z)| &\geqslant |G(Rz)|-|\alpha||G(z)|\\ &> \Big\{\Big(\frac{R+1}{2}\Big)^{n}-|\alpha|\Big\}|G(z)| \quad \text{for} \quad |z|=1 \quad \text{and} \quad R>1.\end{aligned} \tag{19}$$

Inequality (18) can be written in the form

$$|G(e^{i\theta})| < \left(\frac{2}{R+1}\right)^n |G(Re^{i\theta})| \tag{20}$$

for every $R > 1$ and $0 \leqslant \theta < 2\pi$. Since $G(Re^{i\theta}) \neq 0$ and $\left(\frac{2}{R+1}\right)^n < 1$, from inequality (20), we obtain

$$|G(e^{i\theta})| < |G(Re^{i\theta})|$$

for every $R > 1$ and $0 \leqslant \theta < 2\pi$. Equivalently,

$$|G(z)| < |G(Rz)| \quad \text{for} \quad |z| = 1 \quad \text{and} \quad R > 1.$$

Since all the zeros of $G(Rz)$ lie in $|z| \leqslant 1/R < 1$, a direct application of Rouche's theorem shows that the polynomial $G(Rz) - \alpha G(z)$ has all its zeros in $|z| < 1$ for every real or complex number α with $|\alpha| \leqslant 1$. Applying Rouche's theorem again, it follows from (19) that for every real or complex number β with $|\beta| \leqslant 1$ and $R > 1$, all the zeros of the polynomial

$$T(z) = G(Rz) - \alpha G(z) + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}G(z)$$

lie in $|z| < 1$. Replacing $G(z)$ by $P(z) - \lambda F(z)$, we conclude that all the zeros of

$$\begin{aligned} T(z) = &\left[P(Rz) - \alpha P(z) + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}P(z)\right] \\ &- \lambda\left[F(Rz) - \alpha F(z) + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}F(z)\right] \end{aligned} \tag{21}$$

lie in $|z| < 1$ for every $R > 1$, $|\alpha| \leqslant 1$, $|\beta| \leqslant 1$ and $|\lambda| > 1$. This implies

$$\begin{aligned} &\left|P(Rz) - \alpha P(z) + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}P(z)\right| \\ &\qquad \leqslant \left|F(Rz) - \alpha F(z) + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}F(z)\right| \end{aligned} \tag{22}$$

for $|z| \geqslant 1$ and $R > 1$. If inequality (22) is not true, then there is a point $z = z_0$ with $|z_0| \geqslant 1$ such that

$$\begin{aligned} &\left|P(Rz_0) - \alpha P(z_0) + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}P(z_0)\right| \\ &\qquad > \left|F(Rz_0) - \alpha F(z_0) + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}F(z_0)\right|. \end{aligned}$$

Since all the zeros of $F(z)$ lie in $|z| \leqslant 1$, it follows (as in the case of $G(z)$) that all the zeros of

$$F(Rz) - \alpha F(z) + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}F(z)$$

lie in $|z| < 1$ for all real or complex numbers α, β with $|\alpha| \leqslant 1$, $|\beta| \leqslant 1$ and $R > 1$. Hence

$$F(Rz_0) - \alpha F(z_0) + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}F(z) \neq 0$$

with $|z_0| \geqslant 1$. We choose

$$\lambda = \frac{P(Rz_0) - \alpha P(z_0) + \beta\Big\{\left(\frac{R+1}{2}\right)^n - |\alpha|\Big\}P(z_0)}{F(Rz_0) - \alpha F(z_0) + \beta\Big\{\left(\frac{R+1}{2}\right)^n - |\alpha|\Big\}F(z_0)}$$

so that λ is well defined real or complex number with $|\lambda| > 1$, and with this choice of λ, from (21) we get

$$T(z_0) = 0 \quad \text{with} \quad |z_0| \geqslant 1.$$

This is clearly a contradiction to the fact that all the zeros of $T(z)$ lie in $|z| < 1$. Thus for all real or complex number α, β with $|\alpha| \leqslant 1$, $|\beta| \leqslant 1$ and $R > 1$,

$$\begin{aligned}\Big|P(Rz) - \alpha P(z) + \beta\Big\{\Big(\frac{R+1}{2}\Big)^n - |\alpha|\Big\}P(z)\Big| \\ \leqslant \Big|F(Rz) - \alpha F(z) + \beta\Big\{\Big(\frac{R+1}{2}\Big)^n - |\alpha|\Big\}F(z)\Big|\end{aligned}$$

for $|z| \geqslant 1$. This proves Lemma 2.

We also need:

LEMMA 3. *If $P(z)$ is a polynomial of degree n which does not vanish in $|z| < 1$, then for all real or complex numbers α, β with $|\alpha| \leqslant 1$, $|\beta| \leqslant 1$ and $R \geqslant 1$,*

$$\begin{aligned}\Big|P(Rz) - \alpha P(z) + \beta\Big\{\Big(\frac{R+1}{2}\Big)^n - |\alpha|\Big\}P(z)\Big| \\ \leqslant \Big|Q(Rz) - \alpha Q(z) + \beta\Big\{\Big(\frac{R+1}{2}\Big)^n - |\alpha|\Big\}Q(z)\Big|\end{aligned}$$

for $|z| \geqslant 1$ where $Q(z) = z^n\overline{P(1/\overline{z})}$.

Proof of Lemma 3. By hypothesis, the polynomial $P(z)$ has all its zeros in $|z| \geqslant 1$, therefore, all the zeros of the polynomial $Q(z) = z^n\overline{P(1/\overline{z})}$ lie in $|z| \leqslant 1$ and $|P(z)| = |Q(z)|$ for $|z| = 1$. Applying Lemma 2 with $F(z)$ replaced by $Q(z)$, it follows that

$$\begin{aligned}\Big|P(Rz) - \alpha P(z) + \beta\Big\{\Big(\frac{R+1}{2}\Big)^n - |\alpha|\Big\}P(z)\Big| \\ \leqslant \Big|Q(Rz) - \alpha Q(z) + \beta\Big\{\Big(\frac{R+1}{2}\Big)^n - |\alpha|\Big\}Q(z)\Big|\end{aligned}$$

for $|z| \geqslant 1$, $R \geqslant 1$, $|\alpha| \geqslant 1$ and $|\beta| \leqslant 1$. This proves Lemma 3.

3. Proofs of the theorems

Proof of Theorem 1. Taking $F(z) = Mz^n$ where $M = \max_{|z|=1}|P(z)|$, in Lemma 2 we get the conclusion of Theorem 1.

Proof of Theorem 2. Let $M = \max_{|z|=1} |P(z)|$. In order to prove inequality (11) for $R = 1$, it suffices to show that

$$|P(z)| + |Q(z)| \leqslant (|z|^n + 1)M \quad \text{for} \quad |z| \geqslant 1. \tag{23}$$

The inequality (23) is implicit in [13] but for the sake of completeness here we deduce it from a result of Aziz [2, Lemma 2], according to which is $P(z)$ is a polynomial of degree n, then for every $\rho \geqslant 1$,

$$|P(\rho z) - P(z)| + |Q(\rho z) - Q(z)| \leqslant (\rho^n - 1)M, \quad \text{for} \quad |z| = 1 \tag{24}$$

where $Q(z) = z^n\overline{P(1/\overline{z})}$. Since $|P(z)| = |Q(z)|$ for $|z| = 1$, from inequality (24), we get for $|z| = 1$,

$$\begin{aligned} |P(\rho z)| + |Q(\rho z)| &\leqslant 2|P(z)| + (\rho^n - 1)M \\ &\leqslant 2M + (\rho^n - 1)M = (\rho^n + 1)M \end{aligned}$$

for every $\rho \geqslant 1$, which is clearly equivalent to (23). Henceforth we assume $R > 1$. Since $|P(z)| \leqslant M$ for $|z| = 1$, it follows by Rouche's theorem that for every real or complex number λ with $|\lambda| > 1$, the polynomial $H(z) = P(z) + \lambda M$ does not vanish in $|z| < 1$. Applying Lemma 3 to the polynomial $H(z)$, we get for $|z| \geqslant 1$ and $R > 1$,

$$\begin{aligned} &\left|P(Rz) - \alpha P(z) + \beta\Big\{\Big(\frac{R+1}{2}\Big)^n - |\alpha|\Big\}P(z) + \lambda\Big[1 - \alpha + \beta\Big\{\Big(\frac{R+1}{2}\Big)^n - |\alpha|\Big\}\Big]M\right| \\ &\qquad \leqslant \left|Q(Rz) - \alpha Q(z) + \beta\Big\{\Big(\frac{R+1}{2}\Big)^n - |\alpha|\Big\}Q(z)\right. \\ &\qquad\qquad \left. + \overline{\lambda}\Big[R^n - \alpha + \beta\Big\{\Big(\frac{R+1}{2}\Big)^n - |\alpha|\Big\}\Big]Mz^n\right| \end{aligned} \tag{25}$$

where $|\alpha| \leqslant 1$, $|\beta| \leqslant 1$ and $Q(z) = z^n\overline{P(1/\overline{z})}$. Choosing argument of λ in the right hand side of (25) such that

$$\begin{aligned} &\left|Q(Rz) - \alpha Q(z) + \beta\Big\{\Big(\frac{R+1}{2}\Big)^n - |\alpha|\Big\}Q(z) + \overline{\lambda}\Big[R^n - \alpha + \beta\Big\{\Big(\frac{R+1}{2}\Big)^n - |\alpha|\Big\}\Big]Mz^n\right| \\ &\qquad = |\lambda|\left|R^n - \alpha + \beta\Big\{\Big(\frac{R+1}{2}\Big)^n - |\alpha|\Big\}\right|M|z|^n \\ &\qquad\qquad - \left|Q(Rz) - \alpha Q(z) + \beta\Big\{\Big(\frac{R+1}{2}\Big)^n - |\alpha|\Big\}Q(z)\right| \end{aligned}$$

(which is possible by Theorem 1), we obtain

$$\begin{aligned} &\left|P(Rz) - \alpha P(z) + \beta\Big\{\Big(\frac{R+1}{2}\Big)^n - \lambda\Big\}P(z)\right| - |\lambda|\left|1 - \alpha + \beta\Big\{\Big(\frac{R+1}{2}\Big)^n - |\alpha|\Big\}\right|M \\ &\qquad \leqslant |\lambda|\left|R^n - \alpha + \beta\Big\{\Big(\frac{R+1}{2}\Big)^n - |\alpha|\Big\}\right|M|z|^n \\ &\qquad\qquad - \left|Q(Rz) - \alpha Q(z) + \beta\Big\{\Big(\frac{R+1}{2}\Big)^n - |\alpha|\Big\}Q(z)\right| \end{aligned}$$

for $|z| \geqslant 1$, $|\alpha| \geqslant 1$, $|\beta| \leqslant 1$ and $R > 1$. Equivalently

$$\begin{aligned}&\left|P(Rz) - \alpha P(z) + \beta\Big\{\Big(\frac{R+1}{2}\Big)^n - |\alpha|\Big\}P(z)\right| \\ &\qquad + \left|Q(Rz) - \alpha Q(z) + \beta\Big\{\Big(\frac{R+1}{2}\Big)^n - |\alpha|\Big\}Q(z)\right| \\ &\leqslant |\lambda|\Big[\Big|R^n - \alpha + \beta\Big\{\Big(\frac{R+1}{2}\Big)^n - |\alpha|\Big\}\Big||z|^n \\ &\qquad + \Big|1 - \alpha + \beta\Big\{\Big(\frac{R+1}{2}\Big)^n - |\alpha|\Big\}\Big|\Big]M\end{aligned}$$

for $|z| \geqslant 1$, $|\alpha| \leqslant 1$, $|\beta| \leqslant 1$ and $R > 1$. Finally letting $|\lambda| \to 1$, we get the desired result and this completes the proof of Theorem 2.

Proof of Theorem 3. Since $P(z)$ does not vanish $|z| < 1$, by Lemma 3 we have

$$\begin{aligned}&\left|P(Rz) - \alpha P(z) + \beta\Big\{\Big(\frac{R+1}{2}\Big)^n - |\alpha|\Big\}P(z)\right| \\ &\qquad \leqslant \left|Q(Rz) - \alpha Q(z) + \beta\Big\{\Big(\frac{R+1}{2}\Big)^n - |\alpha|\Big\}Q(z)\right|\end{aligned}$$

for $|z| \geqslant 1$ where $Q(z) = z^n\overline{P(1/\overline{z})}$, $|\alpha| \leqslant 1$, $|\beta| \leqslant 1$ and $R \geqslant 1$. Using this in (11) we get for all α, β with $|\alpha| \leqslant 1$, $|\beta| \leqslant 1$ and $R \geqslant 1$

$$\begin{aligned}2&\left|P(Rz) - \alpha P(z) + \beta\Big\{\Big(\frac{R+1}{2}\Big)^n - |\alpha|\Big\}P(z)\right| \\ &\leqslant \left|P(Rz) - \alpha P(z) + \beta\Big\{\Big(\frac{R+1}{2}\Big)^n - |\alpha|\Big\}P(z)\right| \\ &\qquad + \left|Q(Rz) - \alpha Q(z) + \beta\Big\{\Big(\frac{R+1}{2}\Big)^n - |\alpha|\Big\}Q(z)\right| \\ &\leqslant \frac{1}{2}\Big[\Big|R^n - \alpha + \beta\Big\{\Big(\frac{R+1}{2}\Big)^n - |\alpha|\Big\}\Big||z|^n \\ &\qquad + \Big|1 - \alpha + \beta\Big\{\Big(\frac{R+1}{2}\Big)^n - |\alpha|\Big\}\Big|\Big]\max_{|z|=1}|P(z)|\end{aligned}$$

for $|z| \geqslant 1$, which is equivalent to (13). This completes the proof of Theorem 3.

REMARK 7. A polynomial $P(z)$ of degree n is said to self-inversive if $P(z) = Q(z)$ where $Q(z) = z^n\overline{P(1/\overline{z})}$. It can be now easily seen that Theorem 3 equally holds for self-inversive polynomials as well.

REMARK 8. If $P(z) = c\prod_{j=1}^{n}(z - z_j)$ is a polynomial of degree n having all its zeros in $|z| \leqslant k$, $k \leqslant 1$, then clearly for $|z| = 1$ and $|\beta| \leqslant 1$, we have

$$\begin{aligned}\mathrm{Re}\Big\{\frac{zP'(z)}{P(z)} + \frac{n\beta}{1+k}\Big\} &= \mathrm{Re}\Big(\frac{zP'(z)}{P(z)}\Big) + \frac{n\mathrm{Re}\,(\beta)}{1+k} \\ &= \mathrm{Re}\sum_{j=1}^{n}\frac{z}{z - z_j} + \frac{n\mathrm{Re}\,(\beta)}{1+k} \geqslant \frac{n}{1+k} + \frac{n\mathrm{Re}\,(\beta)}{1+k}.\end{aligned}$$

This gives

$$\left|\frac{zP'(z)}{P(z)} + \frac{n\beta}{1+k}\right| \geqslant \frac{n}{1+k}\Big\{1 + \mathrm{Re}\,(\beta)\Big\} \quad \text{for} \quad |z| = 1,$$

which implies

$$\max_{|z|=1}\left|zP'(z) + \frac{n\beta}{1+k}P(z)\right| \geqslant \frac{n}{1+k}\Big\{1 + \mathrm{Re}\,(\beta)\Big\}\max_{|z|=1}|P(z)|. \tag{26}$$

Inequality (26) is a generalization of Malik's inequality [9] for polynomials having all its zeros in $|z| \leqslant k$, $k \leqslant 1$. For $k = 1$, this reduces to Remark 2 of [7].

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