Mathematical
Inequalities
& Applications
Volume 7, Number 3 (2004), 405–418

ON THE MAXIMUM PRINCIPLE FOR ELLIPTIC OPERATORS

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(*communicated by J. Pečarić*)

Abstract. In this paper we obtain some estimates for solutions of second order elliptic equations whose leading coefficients are functions of vanishing mean oscillation.

1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^n$, $n \geqslant 3$, and

$$L_0 = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$$

an uniformly elliptic operator whose coefficients a_{ij} are measurable in Ω. Moreover, let u be a solution of the Dirichlet problem

$$(1.1) \qquad \begin{cases} u \in W^{2,p}(\Omega) \cap C^0(\overline{\Omega}), \\ L_0 u = f \in L^p(\Omega), \\ u_{|\partial\Omega} = 0, \end{cases}$$

with $p > \dfrac{n}{2}$.

It is well known that if the coefficients a_{ij} satisfy some regularity hypotheses, then u verifies the bound

$$(1.2) \qquad \sup_{\Omega} |u| \leqslant c \|f\|_{L^p(\Omega)},$$

where $c \in \mathbb{R}_+$ depends on Ω, p, on the ellipticity constant and on the regularity of a_{ij}. Actually, the estimate (1.2) has been proved in [6] when the coefficients a_{ij} are Hölder continuous, in [7] when the a_{ij} are continuous and in [8] if the a_{ij} belong to $W^{1,n}(\Omega)$.

For $p = n$, the bound (1.2) is the classical Aleksandrov-Pucci estimate, and it holds with no regularity assumptions on the a_{ij} and with the constant c depending only on Ω and on the ellipticity constant (see for instance [10] and [1]).

Mathematics subject classification (2000): 35J25, 35B50.
Key words and phrases: Elliptic equations, a priori bounds, discontinuous coefficients.

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Paper MIA-07-41

For a more recent treatment of such theory, see [5] Chapter 9. Note also that some interesting generalizations of (1.2) in the case $p = n$ can be found in [2] and [3].

On the other hand, an example in [9] shows that, when the ellipticity constant is small enough, the estimate (1.2), with $p < n$ and c depending only on Ω and on the ellipticity constant, does not hold. Therefore, it is of interest the study of the problem (1.1) with $p < n$ and with the coefficients a_{ij} in a space wider than those already considered in the literature. Observe that both the hypotheses a_{ij} uniformly continuous and $a_{ij} \in W^{1,n}$ imply $a_{ij} \in VMO$ (see [4]).

In this paper we fix an arbitrary open (bounded or not) subset Ω of $\mathbb{R}^n$, $n \geqslant 3$, a real number $p > \frac{n}{2}$, and we consider the second order uniformly elliptic differential operator

$$L = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} + a,$$

where the coefficients a_{ij} are bounded and locally VMO. We also suppose that the coefficients a_i and a satisfy suitable local summability conditions and that a is negative. In this situation, we will prove that if u is a solution of the problem

$$\begin{cases} Lu \geqslant f \in L^p_{loc}(\Omega), \\ u \in W^{2,p}_{loc}(\Omega) \cap C^0(\overline{\Omega}), \quad u_{|\partial\Omega} \leqslant 0, \\ \limsup_{|x| \longrightarrow +\infty} u(x) \leqslant 0 \quad \text{if } \Omega \text{ is unbounded}, \end{cases} \tag{1.3}$$

then there exist an open ball $B \subset\subset \Omega$ and a positive constant c_0 such that

$$\sup_{\Omega} u \leqslant c_0 \left(\fint_B |f^-|^p \right)^{1/p}, \tag{1.4}$$

where f^- is the negative part of f,

$$\fint_B |f^-|^p = |B|^{-1} \int_B |f^-|^p$$

and c_0 depends on n, p, on the ellipticity constant and on the regularity of the coefficients of L.

2. Some notation

Let Ω be an open subset of $\mathbb{R}^n$. If $p \in [1, +\infty[$, we shall denote by $M^p(\Omega)$ the set of all functions $g \in L^p_{loc}(\overline{\Omega})$ such that

$$||g||_{M^p(\Omega)} = \sup_{x \in \Omega} ||g||_{L^p(\Omega(x,1))} < +\infty. \tag{2.1}$$

Here, for each positive real number r, $\Omega(x,r) = \Omega \cap B(x,r)$, where $B(x,r)$ is the open ball of $\mathbb{R}^n$ of radius r centered at x. The position (2.1) defines a norm on $M^p(\Omega)$. Moreover, $\tilde{M}^p(\Omega)$ will denote the closure of $L^\infty(\Omega)$ in $M^p(\Omega)$.

Let $\Sigma(\Omega)$ be the collection of all Lebesgue measurable subsets of Ω; for each $E \in \Sigma(\Omega)$, we denote by $|E|$ and χ_E the Lebesgue measure and the characteristic function of E, respectively. It can be observed that a function g (in $M^p(\Omega)$) belongs to $\tilde{M}^p(\Omega)$ if and only if

$$\lim_{t \to 0^+} \tau_g(t) = 0\,,$$

where

$$\tau_g(t) = \sup_{\substack{E\in\Sigma(\Omega)\\ \sup_{x\in\Omega}|E(x,1)|\leqslant t}} ||\chi_E g||_{M^p(\Omega)} \quad \forall t \in \mathbb{R}_+\,.$$

Then a *modulus of continuity* of g in $\tilde{M}^p(\Omega)$ is a map $\tilde{\sigma}_p[g,\Omega] : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that

$$\tau_g(t) \leqslant \tilde{\sigma}_p[g,\Omega](t) \quad \forall t \in \mathbb{R}_+\,, \quad \lim_{t\to 0^+} \tilde{\sigma}_p[g,\Omega](t) = 0\,.$$

Moreover, $\tilde{M}^p_{loc}(\Omega)$ will denote the set of all functions $g : \Omega \longrightarrow \mathbb{R}$ such that $\zeta g \in \tilde{M}^p(\Omega)$ for every $\zeta \in C^\infty_o(\Omega)$. Note that

$$L^p(\Omega) \subseteq \tilde{M}^p(\Omega) \subseteq M^p(\Omega)\,, \quad \tilde{M}^p_{loc}(\Omega) = L^p_{loc}(\Omega)\,,$$

in particular, for any bounded open set Ω, we have

$$L^p(\Omega) = \tilde{M}^p(\Omega) = M^p(\Omega)\,.$$

If $g \in L^p(\Omega)$, we put

$$|g|_{p,\Omega} = ||g||_{L^p(\Omega)}\,,\ \omega_p[g,\Omega](t) = \sup_{\substack{E\in\Sigma(\Omega)\\ |E|\leqslant t}} |g|_{p,E}\,,\ t \in R_+\,;$$

clearly, $\lim_{t\to 0^+} \omega_p[g,\Omega](t) = 0$ and the function $\omega_p[g,\Omega] : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a modulus of continuity of g in $L^p(\Omega)$.

If Ω has the property

(2.2) $$|\Omega(x,r)| \geqslant A\, r^n \quad \forall x \in \Omega\,, \forall r \in]0,1],$$

where A is some positive constant independent of x and r, one can consider the space $BMO(\Omega,t)$, $t \in \mathbb{R}_+$, consisting of all functions g in $L^1_{loc}(\overline{\Omega})$ such that

$$[g]_{BMO(\Omega,t)} = \sup_{\substack{x\in\Omega\\ r\in]0,t]}} \fint_{\Omega(x,r)} \left| g - \fint_{\Omega(x,r)} g \right| < +\infty\,.$$

If $g \in BMO(\Omega) = BMO(\Omega, t_A)$, with

$$t_A = \sup_{t\in\mathbb{R}_+} \left(\sup_{\substack{x\in\Omega\\ r\in]0,t]}} \frac{r^n}{|\Omega(x,r)|} \leqslant \frac{1}{A} \right),$$

we shall say that g is in $VMO(\Omega)$ when $[g]_{BMO(\Omega,t)} \to 0$ as $t \to 0^+$. Moreover, a function $\eta[g,\Omega] : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is called a *modulus of continuity* of g in $VMO(\Omega)$ if

$$[g]_{BMO(\Omega,t)} \leqslant \eta[g,\Omega](t) \quad \forall t \in \mathbb{R}_+\,, \quad \lim_{t\to 0^+} \eta[g,\Omega](t) = 0\,.$$

We will also say that $g \in VMO_{loc}(\Omega)$ if ζg belongs to $VMO(\Omega)$ for each $\zeta \in C^\infty_o(\Omega)$.

A more detailed account of properties of the above defined function spaces can be found in [12], [13] and [14].

3. Preliminary results

Let B be an open ball of $\mathbb{R}^n$, $n \geqslant 3$, with radius $d \in \mathbb{R}_+$ and $p \in \left]\frac{n}{2}, +\infty\right[$. Consider in B the operator

$$L = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} + a \tag{3.1}$$

and the following condition on the coefficients of L:

$$(h_B) \qquad \begin{cases} a_{ij} = a_{ji} \in L^\infty(B) \cap VMO(B)\,, \quad i,j = 1,\dots,n, \\ \exists\, \mu \in \mathbb{R}_+ \quad : \quad \sum_{i,j=1}^{n} a_{ij}\, \xi_i\, \xi_j \,\geqslant\, \mu|\xi|^2 \quad \text{a.e. in } B\,,\ \forall\, \xi \in \mathbb{R}^n, \\ a_i \in L^r(B),\ i = 1,\dots,n, \quad r > n \text{ if } p \leqslant n, \text{ and } r = p \text{ if } p > n, \\ a \in L^p(B), \quad a \leqslant 0 \quad \text{a.e. in } B\,. \end{cases}$$

Note that under the assumption (h_B) the operator L from $W^{2,p}(B)$ into $L^p(B)$, is bounded, and we have the following estimate

$$|Lu|_{p,B} \leqslant c_1 ||u||_{W^{2,p}(B)} \quad \forall u \in W^{2,p}(B), \tag{3.2}$$

where $c_1 \in \mathbb{R}_+$ depends on $n, p, d, |a_{ij}|_{\infty,B}, |a_i|_{r,B}, |a|_{p,B}$.

We prove the following “maximum principle”, that was already known in the special case in which the function h is non-negative in the whole ball, $a = 0$ and the coefficients a_{ij} belong to $VMO(\mathbb{R}^n)$ (see [15]).

LEMMA 3.1. *Suppose that condition (h_B) holds, and let h, w be elements of $W^{2,p}(B)$ such that $h_{|\partial B} \geqslant 0$ and w is a solution of the problem*

$$\begin{cases} Lw = 0 \quad \text{in } B, \\ w_{|\partial B} = h_{|\partial B}. \end{cases} \tag{3.3}$$

Then $w \geqslant 0$ in B.

Proof. It can be assumed without loss of generality that $p \in \left]\frac{n}{2}, n\right[$. Observe that (3.3) is equivalent to the problem

$$\begin{cases} L(w-h) = -Lh \ \text{ in } B, \\ w - h \in W^{2,p}(B) \cap \overset{\circ}{W}{}^{1,p}(B). \end{cases} \tag{3.4}$$

Application of Theorem 5.1 of [14] (see also the proof of that result) yields that there exist extensions $p(a_{ij})$ of a_{ij} to $\mathbb{R}^n$ ($i,j = 1,\dots,n$) such that

(3.5) $p(a_{ij}) \in L^\infty(\mathbb{R}^n) \cap VMO(\mathbb{R}^n)$,

(3.6) $\operatorname{supp} p(a_{ij})$ is compact, (3.7) there exist $\delta \in \mathbb{R}_+, N \in \mathbb{N}$ and an open covering $(U_k)_{k=1,\dots,N}$ of ∂B such that, if $B_\delta = \{x \in \mathbb{R}^n : \mathrm{dist}(x,B) < \delta\}$ and x belongs to $B_\delta \setminus \overline{B}$, there are elements $x_1 \in U_{k_1} \cap B, \dots, x_l \in U_{k_l} \cap B$, with l in $\{1,\dots,N\}$, for which $p(a_{ij})(x) = \theta_{k_1}(x)a_{ij}(x_1) + \dots + \theta_{k_l}(x)a_{ij}(x_l)$, where $\theta_{k_1},\dots,\theta_{k_l} \in C_o^\infty(\mathbb{R}^n)$ and $\theta_{k_1}(x) + \dots + \theta_{k_l}(x) = 1$.

It follows from Theorem 2.1 of [15] and from (3.5) that $w-h$ is the unique solution of the problem (3.4), and we have the estimate

$$||w-h||_{W^{2,p}(B)} \leqslant c_2 |Lh|_{p,B}\,, \tag{3.8}$$

where c_2 depends on n, p, d, μ, $|a_{ij}|_{\infty,B}$, $[p(a_{ij})]_{BMO(\mathbb{R}^n,\cdot)}$, $|a_i|_{r,B}$, $|a|_{p,B}$, $\omega_r[a_i,B]$ and $\omega_p[a,B]$. Therefore it follows from (3.2) and (3.8) that for $w-h$ we also have the bound

$$||w-h||_{W^{2,p}(B)} \leqslant c_3 ||h||_{W^{2,p}(B)}\,, \tag{3.9}$$

with $c_3 \in \mathbb{R}_+$ depending on the same parameters on which c_2 does.

Denote now by $\dot{a}_i$, $\dot{a}$ the extensions of a_i and a, respectively, with zero values out of B, and let $(J_k)_{k\in\mathbb{N}}$ be a sequence of mollifiers; for each $k \in \mathbb{N}$ we put

$$a_{ij}^k = J_k * p(a_{ij})\,, \quad a_i^k = J_k * \dot{a}_i\,, \quad a^k = J_k * \dot{a}\,.$$

By (3.6) we have that $p(a_{ij}) \in L^q(\mathbb{R}^n)$ for each $q \in [1,+\infty[$, and hence

$$a_{ij}^k \longrightarrow p(a_{ij}) \ \text{ in } L^q(\mathbb{R}^n) \quad \forall q \in [1,+\infty[\,; \tag{3.10}$$

moreover,

$$[a_{ij}^k]_{BMO(\mathbb{R}^n,\cdot)} \leqslant [p(a_{ij})]_{BMO(\mathbb{R}^n,\cdot)} \quad \forall k \in \mathbb{N} \tag{3.11}$$

(see for instance [11]). Similarly, the sequences $(a_i^k)_{k\in\mathbb{N}}$ and $(a^k)_{k\in\mathbb{N}}$ satisfy the following conditions:

$$a_i^k \longrightarrow a_i \ \text{ in } L^r(B)\,, \quad a^k \longrightarrow a \ \text{ in } L^p(B)\,, \tag{3.12}$$

$$\begin{cases} |a_i^k|_{r,B} \leqslant |a_i|_{r,B},\ |a^k|_{p,B} \leqslant |a|_{p,B}, \\ \omega_r[a_i^k,B] \leqslant \omega_r[a_i,B],\ \omega_p[a^k,B] \leqslant \omega_p[a,B],\ \forall k \in \mathbb{N}, \end{cases} \tag{3.13}$$

$$a^k \leqslant 0 \ \text{ in } B\,, \quad \forall k \in \mathbb{N}\,. \tag{3.14}$$

For each positive integer k consider now the Dirichlet problem

$$\begin{cases} L^k v^k = -L^k h \quad \text{in } B, \\ v^k \in W^{2,p}(B) \cap \overset{\circ}{W}{}^{1,p}(B), \end{cases} \tag{3.15}$$

where

$$L^k = \sum_{i,j=1}^n a_{ij}^k \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n a_i^k \frac{\partial}{\partial x_i} + a^k\,.$$

Applying (3.7) we obtain that

$$|p(a_{ij})|_{\infty,B_\delta} \leqslant |a_{ij}|_{\infty,B}\,, \quad i,j=1,\ldots,n\,, \sum_{i,j=1}^{n} p(a_{ij})\xi_i\xi_j \geqslant \mu|\xi|^2 \text{ a.e. in } B_\delta\,, \forall \xi \in \mathbb{R}^n\,,$$

so that there exists $k_o = k_o(\delta) \in \mathbb{N}$ such that for every $k \geqslant k_o$ the following holds:

$$|a^k_{ij}|_{\infty,B} \leqslant |p(a_{ij})|_{\infty,B_\delta}\,, \quad i,j=1,\ldots,n\,, \quad \sum_{i,j=1}^{n} a^k_{ij}\xi_i\xi_j \geqslant \mu|\xi|^2 \text{ in } B\,, \quad \forall \xi \in \mathbb{R}^n\,.$$

Without loss of generality we shall denote the subsequence $(a^k_{ij})_{k\geqslant k_o}$ again by $(a^k_{ij})_{k\in\mathbb{N}}$. It follows from the regularity of the coefficients of L^k, from (3.14) and from Theorem 2.1 of [15] that for each positive integer k there exists a unique solution v^k of the problem (3.15), and for such solution the relations (3.11) and (3.13) provide the bound

$$||v^k||_{W^{2,p}(B)} \leqslant c_4||h||_{W^{2,p}(B)}\,, \tag{3.16}$$

where $c_4 \in \mathbb{R}_+$ depends only on $n,\, p,\, d,\, \mu,\, |a_{ij}|_{\infty,B},\, [p(a_{ij})]_{BMO(\mathbb{R}^n,\cdot)}\,,\, |a_i|_{r,B}, |a|_{p,B}\,,$ $\omega_r[a_i,B], \omega_p[a,B]$. Clearly, $w^k = v^k + h$ is a solution of the problem

$$\begin{cases} L^k w^k = 0 \text{ in } B, \\ w^k \in W^{2,p}(B), \\ w^k_{|\partial B} = h_{|\partial B} \geqslant 0. \end{cases}$$

Moreover, it follows from well known results that $w^k \in C^2(B)\cap C^0(\overline{B})$, so that the weak maximum principle yields that

$$\inf_B w^k \;\geqslant\; \inf_{\partial B}(w^k)^-,$$

and hence

$$w^k \;\geqslant\; 0 \quad \text{in } B\,. \tag{3.17}$$

By (3.16) and by the definition of w^k we have that $(w^k)_{k\in\mathbb{N}}$ is a bounded sequence in $W^{2,p}(B)$; thus there exists a subsequence, that we shall denote again by $(w^k)_{k\in\mathbb{N}}$, such that

$$\begin{cases} w^k \rightharpoonup w' \quad \text{in } W^{2,p}(B), \\ w^k \longrightarrow w' \quad \text{in } W^{1,q}(B),\ 1 < q < \dfrac{np}{n-p}, \\ w^k \longrightarrow w' \quad \text{in } C^0(\overline{B}), \end{cases} \tag{3.18}$$

with $w' \in W^{2,p}(B)\cap W^{1,q}(B)$. Moreover, it follows from (3.17) that $w' \geqslant 0$ in B.

We claim that the sequence $(L^k w^k)_{k\in\mathbb{N}}$ weakly converges to Lw' in $L^p(B)$. In fact, for each $\varphi \in L^{p'}(B)$, where $\frac{1}{p}+\frac{1}{p'}=1$, we have

$$\begin{aligned}\int_B |(L^k w^k - Lw')\varphi|dx \leqslant & \sum_{i,j=1}^n |a_{ij}|_{\infty,B}\int_B |(w^k_{x_ix_j}-w'_{x_ix_j})\varphi|dx \\ & + \sum_{i=1}^n |a_i|_{r,B}|w^k_{x_i}-w'_{x_i}|_{q,B}|\varphi|_{p',B} + \sup_{\overline{B}}|w^k-w'|\,|a|_{p,B}|\varphi|_{p',B} \\ & + \sum_{i,j=1}^n |(a^k_{ij}-a_{ij})\varphi|_{p',B}|w^k_{x_ix_j}|_{p,B} \\ & + \sum_{i=1}^n |a^k_i - a_i|_{r,B}|w^k_{x_i}|_{q,B}|\varphi|_{p',B} + |a^k-a|_{p,B}\sup_{\overline{B}}|w^k|\,|\varphi|_{p',B},\end{aligned}$$

where $q=\dfrac{rp}{r-p}$. The weak convergence of $(L^k w^k)_{k\in\mathbb{N}}$ to Lw' in $L^p(B)$ follows now from (3.10), (3.12) and (3.18). Therefore $Lw'=0$ a.e. in B. On the other hand, $w'_{|\partial B}=h_{|\partial B}$, and hence the uniqueness of the solution of problem (3.3) yields that $w=w'$, so that $w\geqslant 0$ in B. $\square$

Lemma 3.1 can be used in the proof of our next result. It should be noted that the constant in the estimate (e_B) below depends explicitely on the radius d of the ball B, and this fact will be crucial in the proof of Theorem 4.1.

LEMMA 3.2. *Suppose that B has radius $d<1$ and that condition (h_B) holds. If u is a solution of the problem*

$$(p_B) \qquad \begin{cases} u\in W^{2,p}(B), \\ Lu \geqslant f \in L^p(B), \\ u_{|\partial B}\leqslant 0, \end{cases}$$

then there exists $c\in\mathbb{R}_+$ such that

$$(e_B) \qquad \sup_B u \leqslant c\, d^{2-\frac{n}{p}}|f^-|_{p,B},$$

with c dependent on $n, p, \mu, |a_{ij}|_{\infty,B}, [p(a_{ij})]_{BMO(\mathbb{R}^n,\cdot)}, |a_i|_{r,B}, |a|_{p,B}, \omega_r[a_i,B], \omega_p[a,B]$, and where $p(a_{ij})$ is an extension of a_{ij} to $\mathbb{R}^n$ satisfying (3.5).

Proof. Let $B=B(x_0,d)$, where x_0 is the centre of B, and put $B^*=B(x_0,1)$. Consider the function $T: B\longrightarrow B^*$ defined by the position

$$T(x)=x_0+\frac{x-x_0}{d},$$

and observe that

$$z=T(x) \iff x=x_0+d(z-x_0)=T^{-1}(z).$$

For each function g defined on B, put $g^* = g \circ T^{-1}$. Then

$$(Lu)^* = \sum_{i,j=1}^{n} a_{ij}^*(u_{x_i x_j})^* + \sum_{i=1}^{n} a_i^*(u_{x_i})^* + a^* u^* = d^{-2} \sum_{i,j=1}^{n} a_{ij}^* u_{z_i z_j}^* + d^{-1} \sum_{i=1}^{n} a_i^* u_{z_i}^* + a^* u^*,$$

and hence

$$L^* u^* = d^2 (Lu)^*,$$

where

$$L^* = \sum_{i,j=1}^{n} a_{ij}^* \frac{\partial^2}{\partial z_i \partial z_j} + d \sum_{i=1}^{n} a_i^* \frac{\partial}{\partial z_i} + d^2 a^*.$$

Denote by $p(a_{ij})$ an extension of a_{ij} to $\mathbb{R}^n$ satisfying (3.5) ($i,j = 1,\dots,n$), and put

$$p(a_{ij})^*(z) = p(a_{ij})(x_0 + d(z - x_0)), \quad z \in \mathbb{R}^n.$$

Since

$$p(a_{ij})^* \in L^\infty(\mathbb{R}^n) \cap VMO(\mathbb{R}^n), \; p(a_{ij})^*_{|B^*} = a_{ij}^*, \tag{3.19}$$

we have also

$$a_{ij}^* \in L^\infty(B^*) \cap VMO(B^*) \tag{3.20}$$

(see [14]). Moreover, the hypothesis (h_B) yields that

$$\begin{cases} a_{ij}^* = a_{ji}^*, \quad i,j = 1,\dots,n, \\ \sum_{i,j=1}^{n} a_{ij}^* \xi_i \xi_j \geqslant \mu |\xi|^2 \;\text{ a.e. in } B^*, \quad \forall \xi \in \mathbb{R}^n, \\ a_i^* \in L^r(B^*), \; i = 1,\dots,n, \; a^* \in L^p(B^*), \; a^* \leqslant 0 \text{ a.e. in } B^*. \end{cases} \tag{3.21}$$

Consider now the following problem:

$$\begin{cases} L^* v = h \in L^p(B^*), \\ v \in W^{2,p}(B^*) \cap \overset{\circ}{W}{}^{1,p}(B^*). \end{cases} \tag{3.22}$$

It follows from (3.19), (3.21) and from Theorem 2.1 of [15] that there exists a unique solution v of (3.22) satisfying the estimate

$$||v||_{W^{2,p}(B^*)} \leqslant K |h|_{p,B^*}, \tag{3.23}$$

where K depends on n, p, μ, $|a_{ij}^*|_{\infty,B^*}$, $[p(a_{ij})^*]_{BMO(\mathbb{R}^n,\cdot)}$, $|da_i^*|_{r,B^*}$, $|d^2 a^*|_{p,B^*}$, $\omega_r[da_i^*, B^*]$, $\omega_p[d^2 a^*, B^*]$. Thus by (3.23) we obtain that there is $K_1 \in \mathbb{R}_+$, depending on the same parameters on which K does, such that

$$\max_{\overline{B}^*} |v| \leqslant K_1 |h|_{p,B^*}, \tag{3.24}$$

and hence for each $z \in B^*$ there is $g(z,\cdot) \in L^{p'}(B^*)$ $\left(\dfrac{1}{p} + \dfrac{1}{p'} = 1\right)$ for which

$$v(z) = -\int_{B^*} g(z,y)h(y)dy. \tag{3.25}$$

The map $g(z,\cdot)$ is the Green function for the operator L^* in B^*, and it has the following properties:

$$\int_{B^*} g(z,y)\tilde{h}(y)dy \geqslant 0 \quad \forall \tilde{h} \in L^p(B^*)\,,\ \tilde{h} \geqslant 0\,, \tag{3.26}$$

$$|g(z,\cdot)|_{p',B^*} \leqslant K_1. \tag{3.27}$$

Setting $h = L^*u^* = d^2(Lu)^*$ in (3.22), we have that $v - u^*(\in W^{2,p}(B^*))$ is a solution of the problem

$$\begin{cases} L^*(v - u^*) = 0 \quad \text{in } B^*, \\ (v - u^*)_{|\partial B^*} = -u^*_{|\partial B^*} \geqslant 0, \end{cases}$$

and so it follows from (3.20), (3.21) and from Lemma 3.1 that $v - u^* \geqslant 0$ in B^*. Thus, applying (3.25) with $h = L^*u^*$, it follows from (3.26) and (3.27) that

$$\begin{aligned} u^*(z) &\leqslant -\int_{B^*} g(z,y)d^2(Lu)^*(y)dy \leqslant -d^2\int_{B^*} g(z,y)f^*(y)dy \\ &\leqslant -2d^2\int_{B^*} g(z,y)(f^*)^-(y)dy \leqslant 2d^2|g(z,\cdot)|_{p',B^*}|(f^*)^-|_{p,B^*} \\ &\leqslant 2d^2K_1|(f^-)^*|_{p,B^*} \quad \forall z \in B^*. \end{aligned} \tag{3.28}$$

It is now easy to deduce the statement from (3.28). □

4. Main results

Let Ω be an open subset of $\mathbb{R}^n$, $n \geqslant 3$. For our purposes we need to introduce a sequence of functions of class $C_o^\infty(\Omega)$. It is well known that there exists a function $\tilde{\alpha} \in C^\infty(\Omega) \cap C^{0,1}(\overline{\Omega})$ which is equivalent to $\operatorname{dist}(\cdot,\partial\Omega)$ (see for instance [17]). Consider a function $\tilde{\varphi}$, which is a restriction to $[0,+\infty[$ of a map in $C_o^\infty(\mathbb{R})$, satisfying the condition

$$0 \leqslant \tilde{\varphi} \leqslant 1, \quad \tilde{\varphi}(t) = 1 \ \text{ if } \ t \leqslant \frac{1}{2}, \quad \tilde{\varphi}(t) = 0 \ \text{ if } \ t \geqslant 1\,,$$

and for every positive integer k define the function

$$\psi_k : x \in \overline{\Omega} \longrightarrow \big(1 - \tilde{\varphi}(k\tilde{\alpha}(x))\big)\,\tilde{\varphi}\Big(\frac{|x|}{2k}\Big).$$

It is easy to show that each ψ_k belongs to $C_o^\infty(\Omega)$ and

$$0 \leqslant \psi_k \leqslant 1, \quad \operatorname{supp}\psi_k \subseteq \overline{\Omega}_{2k}, \quad \psi_{k|\overline{\Omega}_k} = 1,$$

where

$$\Omega_k = \left\{ x \in \Omega \ : \ |x| < k, \ \tilde{\alpha}(x) > \frac{1}{k} \right\}.$$

Suppose now that Ω has the property (2.2) and $p \in \left]\frac{n}{2}, +\infty\right[$. Consider in Ω the operator

$$L = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} + a\,, \tag{4.1}$$

and put

$$L_0 = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}\,.$$

We will make the following assumptions on the coefficients of L:

$$(h) \qquad \begin{cases} a_{ij} = a_{ji} \in L^{\infty}(\Omega) \cap VMO_{loc}(\Omega), \quad i,j = 1, \ldots, n, \\ \exists \mu \in \mathbb{R}_+ : \displaystyle\sum_{i,j=1}^{n} a_{ij} \xi_i \xi_j \geqslant \mu |\xi|^2 \ \text{ a.e. in } \Omega, \quad \forall \xi \in \mathbb{R}^n, \\ a_i \in \tilde{M}^r_{loc}(\Omega), i = 1, \ldots, n, \ \text{where } r > n \text{ if } p \leqslant n, r = p \text{ if } p > n, \\ a \in \tilde{M}^p_{loc}(\Omega), \exists a_0 \in \mathbb{R}_+ : a \leqslant -a_0 \text{ a.e. in } \Omega. \end{cases}$$

Our aim here is to prove the following result.

THEOREM 4.1. *Suppose that (h) is satisfied, and let* u *be a solution of the problem*

$$(p) \qquad \begin{cases} Lu \geqslant f \in L^p_{loc}(\Omega), \\ u \in W^{2,p}_{loc}(\Omega) \cap C^0(\overline{\Omega}), \\ u_{|\partial\Omega} \leqslant 0, \\ \limsup_{|x| \longrightarrow +\infty} u(x) \leqslant 0 \quad \textit{if } \Omega \textit{ is unbounded.} \end{cases}$$

Then there exist an open ball $B \subset\subset \Omega$ *and a positive constant* c_0 *such that*

$$(e) \qquad \sup_{\Omega} u \leqslant c_0 \left(\fint_B |f^-|^p \right)^{1/p},$$

where c_0 *depends only on* n, p, μ, $|a_{ij}|_{\infty,\Omega}$, $\eta[\psi_k a_{ij}, \Omega]$, $||\psi_k a_i||_{M^r(\Omega)}$, $||\psi_k a||_{M^p(\Omega)}$, $\tilde{\sigma}_r[\psi_k a_i, \Omega]$, $\tilde{\sigma}_p[\psi_k a, \Omega]$ $(k \in \mathbb{N})$ *and* a_0.

Proof. Without loss of generality it can be assumed that $\sup_{\Omega} u > 0$. It follows from the last two conditions of (p) that there is $y \in \Omega$ such that $\sup_{\Omega} u = u(y)$; moreover, there exists $\mu_0 \in]0, \min(1, \text{dist}(y, \partial\Omega))[$ such that $B(y, \mu_0) \subset\subset \Omega$ and $u(x) > 0$ in $B(y, \mu_0)$. Write $\mu_0 = \alpha \lambda_0$ with $\alpha \geqslant 1$; we will later choose α suitable for our pourposes. Moreover, put

$$\varphi(x) = \varphi_{\lambda}(x) = 1 + \lambda^2 - \frac{|x-y|^2}{\alpha^2}, \tag{4.2}$$

$$x \in \overline{B(y, \alpha\lambda)} \text{ and } \lambda \in]0, \lambda_0],$$

and note that

$$1 \leqslant \varphi \leqslant 1 + \lambda^2 \tag{4.3}$$

(the function φ has also been used in [16] in a special case). Consider now the map v defined by the position

$$v(x) = \varphi(x)u(x) - u(y)\,, \quad x \in \overline{B(y, \alpha\lambda)}\,. \tag{4.4}$$

As $\varphi_{|\partial B(y,\alpha\lambda)} = 1$, v satisfies the following condition:

$$v_{|\partial B(y,\alpha\lambda)} = u_{|\partial B(y,\alpha\lambda)} - u(y) \leqslant 0\,. \tag{4.5}$$

It follows from $Lu \geqslant f$ that

$$\varphi Lu \geqslant \varphi f \quad \text{in } B(y, \alpha\lambda) = B\,. \tag{4.6}$$

From this latter relation by easy computations we obtain that

$$L_0(\varphi u) - uL_0\varphi - 2\sum_{i,j=1}^{n} a_{ij}\varphi_{x_j}u_{x_i} + \sum_{i=1}^{n} a_i(\varphi u)_{x_i} - u\sum_{i=1}^{n} a_i\varphi_{x_i} + a\varphi u \geqslant \varphi f \text{ in } B. \tag{4.7}$$

Since

$$u_{x_i}\varphi_{x_j} = \frac{\varphi_{x_j}}{\varphi}(\varphi u)_{x_i} - \frac{\varphi_{x_i}\varphi_{x_j}}{\varphi^2}(\varphi u)\,,$$

from (4.7) it follows that

$$L_0(\varphi u) + \sum_{i=1}^{n} b_i(\varphi u)_{x_i} + c\varphi u \geqslant \varphi f + u\sum_{i=1}^{n} a_i\varphi_{x_i} \quad \text{in } B\,, \tag{4.8}$$

where

$$b_i = a_i - 2\sum_{j=1}^{n} a_{ij}\frac{\varphi_{x_j}}{\varphi}\,, \quad i = 1, \ldots, n\,, \tag{4.9}$$

and

$$c = a - \sum_{i,j=1}^{n} a_{ij}\frac{\varphi_{x_i x_j}}{\varphi} + 2\sum_{i,j=1}^{n} a_{ij}\frac{\varphi_{x_i}\varphi_{x_j}}{\varphi^2}\,. \tag{4.10}$$

Therefore (4.8) yields that

$$L_0 v + \sum_{i=1}^{n} b_i v_{x_i} + cv \geqslant \varphi f + u\sum_{i=1}^{n} a_i\varphi_{x_i} - cu(y) \quad \text{in } B\,. \tag{4.11}$$

We can now choose α in such a way that $c \leqslant 0$ in B. In fact, as $\varphi_{x_i} = -\frac{2(x_i-y_i)}{\alpha^2}$, $\varphi_{x_ix_j} = 0$ if $i \neq j$, $\varphi_{x_ix_j} = -\dfrac{2}{\alpha^2}$ if $i = j$, we have

$$\begin{aligned} c &\leqslant -a_0 + 2\frac{\sum_{i=1}^n a_{ii}}{\varphi} \cdot \frac{1}{\alpha^2} + 8\lambda^2 \frac{\sum_{i,j=1}^n a_{ij}}{\varphi^2} \cdot \frac{1}{\alpha^2} \\ &\leqslant -a_0 + (2\sum_{i=1}^n |a_{ii}|_{\infty,\Omega} + 8\sum_{i,j=1}^n |a_{ij}|_{\infty,\Omega}) \cdot \frac{1}{\alpha^2}, \end{aligned}$$

and hence, fixed α such that

$$\alpha^2 \geqslant \frac{2\sum_{i=1}^n |a_{ii}|_{\infty,\Omega} + 8\sum_{i,j=1}^n |a_{ij}|_{\infty,\Omega}}{a_0}, \tag{4.12}$$

it follows that $c \leqslant 0$ in B. Therefore by (4.4), (4.5), (4.9), (4.10), (4.11) and by Lemma 3.2 we obtain:

$$v(x) \leqslant c_1\lambda^{2-\frac{n}{p}} |(\varphi f + u\sum_{i=1}^n a_i\varphi_{x_i} - cu(y))^-|_{p,B} \quad \forall x \in B, \tag{4.13}$$

with $c_1 \in \mathbb{R}_+$ depending on n, p, μ, a_0, $|a_{ij}|_{\infty,\Omega}$, $[p(a_{ij_{|B}})]_{BMO(\mathbb{R}^n,\cdot)}$, $|a_i|_{r,B}$, $|a|_{p,B}, \omega_r[a_i,B]$, $\omega_p[a,B]$. Here we are choosing

$$p(a_{ij_{|B}}) = (\psi_{k_1}a_{ij})_o,$$

where k_1 is a positive integer such that $\psi_{k_1|B} = 1$ and $(\psi_{k_1}a_{ij})_o$ is the extension of $\psi_{k_1}a_{ij}$ to $\mathbb{R}^n$ with zero values out of Ω. Since $\psi_{k_1}a_{ij}$ belongs to $VMO(\Omega)$ and its support is a compact subset of Ω, by Lemma 4.2 of [14] we deduce that $(\psi_{k_1}a_{ij})_o$ is in $L^\infty(\mathbb{R}^n) \cap VMO(\mathbb{R}^n)$ and

$$[(\psi_{k_1}a_{ij})_o]_{BMO(\mathbb{R}^n,t)} \leqslant [\psi_{k_1}a_{ij}]_{BMO(\Omega,t)} \tag{4.14}$$

for t small enough. Moreover, it is easy to show that

$$\begin{cases} |a_i|_{r,B} \leqslant ||\psi_{k_1}a_i||_{M^r(\Omega)}, \ |a|_{p,B} \leqslant ||\psi_{k_1}a||_{M^p(\Omega)}, \\ \omega_r[a_i,B] \leqslant \tilde{\sigma}_r[\psi_{k_1}a_i,\Omega], \ \omega_p[a,B] \leqslant \tilde{\sigma}_p[\psi_{k_1}a,\Omega]. \end{cases} \tag{4.15}$$

Thus (4.14) and (4.15) yield that the constant c_1 depends on $n, p, \mu, |a_{ij}|_{\infty,\Omega}$, $[\psi_{k_1}a_{ij}]_{BMO(\Omega,\cdot)}, ||\psi_{k_1}a_i||_{M^r(\Omega)}, ||\psi_{k_1}a||_{M^p(\Omega)}, \tilde{\sigma}_r[\psi_{k_1}a_i,\Omega], \tilde{\sigma}_p[\psi_{k_1}a,\Omega]$ and a_0. It follows now from (4.13) that

$$v(x) \leqslant c_1\lambda^{2-\frac{n}{p}}(|\varphi f^-|_{p,B} + u(y)|\sum_{i=1}^n a_i\varphi_{x_i}|_{p,B}) \quad \forall x \in B; \tag{4.16}$$

this latter relation for $x = y$ gives:

$$u(y) \leqslant c_1\lambda^{-\frac{n}{p}}(|\varphi f^-|_{p,B} + u(y)|\sum_{i=1}^n a_i\varphi_{x_i}|_{p,B}). \tag{4.17}$$

Since $|\varphi| \leqslant 1+\lambda^2$ and $|\varphi_{x_i}| \leqslant \dfrac{2}{\alpha}\cdot\lambda$, applying the Hölder inequality if $p \leqslant n$, from (4.17) it follows that

$$(4.18) \qquad u(y) \leqslant c_2\Big((\lambda^{-\frac{n}{p}}+\lambda^{2-\frac{n}{p}})|f^-|_{p,B}+u(y)\lambda^{1-\frac{n}{r}}\sum_{i=1}^{n}|a_i|_{r,B}\Big)$$

$$\leqslant c_2\Big((\lambda^{-\frac{n}{p}}+1)|f^-|_{p,B}+u(y)\sum_{i=1}^{n}|a_i|_{r,B}\Big),$$

with $c_2 \in \mathbb{R}_+$ depending on the same parameters as c_1.

Moreover,

$$|a_i|_{r,B(y,\alpha\lambda)} = |\chi_B\psi_{k_1}a_i|_{r,B(y,1)\cap\Omega} \leqslant ||\chi_B\psi_{k_1}a_i||_{M^r(\Omega)},$$

and this last relation shows that it is possible to choose λ small enough, independent on y, such that $\sum_{i=1}^{n}|a_i|_{r,B} \leqslant \dfrac{1}{2c_2}$. Therefore (4.18) provides the estimate

$$u(y) \leqslant 2c_2(\lambda^{-\frac{n}{p}}+1)|f^-|_{p,B}.$$

The theorem follows now easily. □

COROLLARY 4.2. *Suppose that (h) is satisfied, and let u be a solution of the problem*

$$(p') \qquad \begin{cases} Lu = f \in L^\infty(\Omega), \\ u \in W^{2,p}_{loc}(\Omega)\cap C^0(\overline{\Omega}), \\ u_{|\partial\Omega} = 0, \\ \lim\limits_{|x|\longrightarrow+\infty} u(x) = 0 \quad \textit{if } \Omega \textit{ is unbounded.} \end{cases}$$

Then

$$(e') \qquad \sup_{\Omega}|u| \leqslant c_0|f|_{\infty,\Omega},$$

where c_0 is the constant of the statement of Theorem 4.1.

Proof. The statement can be easily obtained applying Theorem 4.1 to the functions u and $-u$. □

The following uniqueness result is an obvious consequence of Corollary 4.2.

COROLLARY 4.3. *If (h) is satisfied, the problem*

$$(p'') \qquad \begin{cases} Lu = 0, \\ u \in W^{2,p}_{loc}(\Omega)\cap C^0(\overline{\Omega}), \\ u_{|\partial\Omega} = 0, \\ \lim\limits_{|x|\longrightarrow+\infty} u(x) = 0 \quad \textit{if } \Omega \textit{ is unbounded} \end{cases}$$

has only the zero solution.

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(Received June 23, 2003)

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