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UNIFORM NON–SQUARENESS OF ψ–DIRECT SUMS OF BANACH SPACES $X \oplus_\psi Y$

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Dedicated to Professor Hiroshi Sato
on the occasion of his retirement
from Kyushu University

(*communicated by S. Saitoh*)

Abstract. We characterize the uniform non-squareness of the ψ-direct sum $X \oplus_\psi Y$ of Banach spaces X and Y, where ψ is a convex function on the unit interval satisfying certain conditions. As a corollary the uniform non-squareness of an $\ell_{p,q}$-sum $X \oplus_{p,q} Y$ is characterized. In the course of doing this a monotonicity property of absolute norms on $\mathbb{C}^2$ is shown.

1. Introduction and preliminaries

It is well known (Bonsall-Duncan [2]) that for any absolute normalized norm on $\mathbb{C}^2$, that is,

$$\|(z,w)\| = \|(|z|,|w|)\| \quad \text{and} \quad \|(1,0)\| = \|(0,1)\| = 1, \tag{1}$$

there corresponds a unique convex (continuous) function ψ on the unit interval with some conditions. That is, for any such norm $\|\cdot\|$ on $\mathbb{C}^2$ let

$$\psi(t) = \|(1-t,t)\| \quad (0 \leqslant t \leqslant 1). \tag{2}$$

Then the function ψ is convex and satisfies

$$\psi(0) = \psi(1) = 1 \quad \text{and} \quad \max\{1-t,t\} \leqslant \psi(t) \leqslant 1 \quad (0 \leqslant t \leqslant 1). \tag{3}$$

Conversely for any convex function ψ on $[0,\ 1]$ satisfying (3), define

$$\|(z,w)\|_\psi = \begin{cases} (|z|+|w|)\psi\left(\frac{|w|}{|z|+|w|}\right) & \text{if } (z,w) \neq (0,0), \\ 0 & \text{if } (z,w) = (0,0). \end{cases} \tag{4}$$

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Then $\|\cdot\|_\psi$ is an absolute normalized norm on $\mathbb{C}^2$ and satisfies (2). Thus the set N_a of all absolute normalized norms on $\mathbb{C}^2$ corresponds to the family Ψ of all convex functions on $[0, 1]$ satisfying (3) in a one-to-one fashion under the equation (2). The ℓ_p-norms $\|\cdot\|_p$ are such examples and for any norm $\|\cdot\|$ in N_a we have

$$\|\cdot\|_\infty \leqslant \|\cdot\| \leqslant \|\cdot\|_1 \tag{5}$$

([2]). To the ℓ_p-norms correspond the convex functions

$$\psi_p(t) = \begin{cases} \{(1-t)^p + t^p\}^{1/p} & \text{if } 1 \leqslant p < \infty, \\ \max\{1-t, t\} & \text{if } p = \infty. \end{cases} \tag{6}$$

Recently Takahashi, Kato and Saito [13] used absolute norms to introduce the ψ-direct sum $X \oplus_\psi Y$ of Banach spaces X and Y as their direct sum $X \oplus Y$ equipped with the norm

$$\|(x, y)\|_\psi = \|(\|x\|, \|y\|)\|_\psi. \tag{7}$$

This extends the notion of the ℓ_p-sum $X \oplus_p Y$ and provides a plenty of concrete (non ℓ_p-type) norms on $X \oplus Y$. Thus it is quite natural to ask how various properties of $X \oplus_\psi Y$ are described with those of X and Y and with the convex function ψ. In [13] and [10] the strict and uniform convexity of $X \oplus_\psi Y$ are characterized (see Theorem A below); these results were extended to ψ-direct sums of an arbitrary finite number of Banach spaces in a recent paper of the present authors [7]. On the other hand, Saito-Kato-Takahashi [11] showed that all absolute normalized norms on $\mathbb{C}^2$ are uniformly non-square except the ℓ_1- and ℓ_∞-norms (see Theorem B below).

The aim of this paper is to characterize the uniform non-squareness of the space $X \oplus_\psi Y$. We shall show that $X \oplus_\psi Y$ is uniformly non-square if and only if X and Y are uniformly non-square and neither $\psi = \psi_1$ nor $\psi = \psi_\infty$ (in (6)). To do this we shall present a monotonicity property of absolute norms on $\mathbb{C}^2$. As a corollary it is obtained that an $\ell_{p,q}$-sum $X \oplus_{p,q} Y$ is uniformly non-square if and only if X and Y are uniformly non-square, where $\ell_{p,q}$ is the Lorentz sequence space, $1 \leqslant q \leqslant p \leqslant \infty$ and neither $p = q = 1$ nor $p = q = \infty$. This includes the well-known result for ℓ_p-sums $X \oplus_p Y$ as the case $p = q$. Finally several examples will be given. In particular those of uniformly non-square spaces which are not uniformly convex, resp. not strictly convex, are easily constructed.

Recall that a Banach space X is called *uniformly non-square* ([5]; cf. [1, 8]) provided there exists a δ $(0 < \delta < 1)$ such that, whenever $\|(x-y)/2\| > 1-\delta$, $\|x\| = \|y\| = 1$, one has $\|(x+y)/2\| \leqslant 1-\delta$. X is called *strictly convex* provided, if $\|x\| = \|y\| = 1$, $x \neq y$, then $\|(x+y)/2\| < 1$. X is called *uniformly convex* if for any $\epsilon > 0$ there is a δ $(0 < \delta < 1)$ such that, whenever $\|x-y\| \geqslant \epsilon$, $\|x\| \leqslant 1$, $\|y\| \leqslant 1$, one has $\|(x+y)/2\| < 1-\delta$. As is well known, the notion of uniform non-squareness lies between uniform convexity and super-reflexivity and there is no implication between uniform non-squareness and strict convexity in general (cf. [1]). A function ψ on $[0, 1]$ is called *strictly convex* if, for any $s, t \in [0, 1]$, $s \neq t$, and for any c $(0 < c < 1)$, one has $\psi((1-c)s + ct) < (1-c)\psi(s) + c\psi(t)$. We have the following.

THEOREM A. ([13, 10]). *Let X and Y be Banach spaces and let $\psi \in \Psi$. Then* (i) *$X \oplus_\psi Y$ is strictly convex if and only if X and Y are strictly convex, and ψ is strictly convex* ([13, Theorem 1]).

(ii) *$X \oplus_\psi Y$ is uniformly convex if and only if X and Y are uniformly convex, and ψ is strictly convex* ([10, Theorem 1]).

Saito-Kato-Takahashi [11] characterized those absolute norms on $\mathbb{C}^2$ which are uniformly non-square as follows.

THEOREM B. ([11]). *Let $\psi \in \Psi$. Then $(\mathbb{C}^2, \|\cdot\|_\psi)$ is uniformly non-square if and only if $\psi \neq \psi_1$ and $\psi \neq \psi_\infty$.*

2. Monotonicity properties of absolute norms

We discuss monotonicity properties of absolute norms on $\mathbb{C}^2$ for later use. The following facts are fundamental.

LEMMA 1. ([2, p.36, Lemma 2]). *Let $\|\cdot\| \in N_a$.*
(i) *If $|p| \leqslant |r|$ and $|q| \leqslant |s|$, then $\|(p,q)\| \leqslant \|(r,s)\|$.*
(ii) *If $|p| < |r|$ and $|q| < |s|$, then $\|(p,q)\| < \|(r,s)\|$.*

The next assertion for a norm $\|\cdot\|$ in N_a is not true in general:

$$\textit{Let } |p| \leqslant |r| \textit{ and } |q| \leqslant |s|. \textit{ If } |p| < |r| \textit{ or } |q| < |s|, \textit{ then } \|(p,q)\| < \|(r,s)\|. \tag{8}$$

Indeed (8) is not valid for the ℓ_∞-norm. The norms in N_a satisfying (8) are characterized as follows.

PROPOSITION 1. (Takahashi, Kato and Saito [13]). *Let $\psi \in \Psi$. Then the following assertions are equivalent.*
(i) *If $|z| \leqslant |u|$ and $|w| < |v|$, or $|z| < |u|$ and $|w| \leqslant |v|$, then $\|(z,w)\|_\psi < \|(u,v)\|_\psi$.*
(ii) *$\psi(t) > \psi_\infty(t)$ for all $t \in (0,1)$.*
In particular if ψ is strictly convex, the assertion (i) *holds true.*

A more precise (component-wise) result is given in [13]. Now, take an arbitrary $\psi \in \Psi$. We give a condition for which the foregoing assertion (i) is valid component-wise for given (z,w) and (u,v).

PROPOSITION 2. *Let $\psi \in \Psi$ and let $(z,w), (u,v) \in \mathbb{C}^2$.*
(i) *Let $|z| < |u|$ and $|w| = |v|$. Then $\|(z,w)\|_\psi = \|(u,v)\|_\psi$ if and only if $\|(u,v)\|_\psi = |v|$.*
(ii) *Let $|z| = |u|$ and $|w| < |v|$. Then $\|(z,w)\|_\psi = \|(u,v)\|_\psi$ if and only if $\|(u,v)\|_\psi = |u|$.*

Proof. (i) Assume $\|(z,w)\|_\psi = \|(u,v)\|_\psi$. Put

$$s = \frac{|w|}{|z|+|w|},\ t = \frac{|w|}{|u|+|w|}.$$

Then $0 < t < s < 1$ and $\psi(s)/s = \|(z,w)\|_\psi/|w|$ and $\psi(t)/t = \|(u,v)\|_\psi/|v|$. Therefore

$$\begin{aligned} 0 = \frac{\psi(t)}{t} - \frac{\psi(s)}{s} &\geqslant \frac{\psi(t)}{t} - \frac{1}{s}\cdot\frac{(1-s)\psi(t)+(s-t)}{1-t} \\ &= (\psi(t)-t)\frac{s-t}{st(1-t)} \geqslant 0, \end{aligned}$$

and hence $\psi(t) = t$. Consequently we have

$$\|(u,v)\|_\psi = \frac{\psi(t)}{t}|v| = |v|.$$

The converse assertion is direct from

$$|w| \leqslant \|(z,w)\|_\psi \leqslant \|(u,v)\|_\psi = |v| = |w|.$$

(ii) Let $|w| < |v|$. Put $\tilde{x} = (w,z)$, $\tilde{y} = (v,u)$ and let $\tilde{\psi}(t) = \psi(1-t)$. Then by applying the above argument to $\|\cdot\|_{\tilde{\psi}}$, we have the conclusion. □

3. Uniform non-squareness of $X \oplus_\psi Y$

We need the following lemma.

LEMMA 2. *Let $\{x_n\}$ and $\{y_n\}$ be non-zero sequences in a Banach space X such that $\{\|x_n\|\}$ and $\{\|y_n\|\}$ converge to non-zero limits, respectively. Then the following are equivalent.*

(i) $\lim_{n\to\infty}\|x_n + y_n\| = \lim_{n\to\infty}(\|x_n\| + \|y_n\|)$.

(ii) $\displaystyle\lim_{n\to\infty}\left\|\frac{x_n}{\|x_n\|} + \frac{y_n}{\|y_n\|}\right\| = 2$.

Proof. Let $\lim_{n\to\infty}\|x_n\| = a$, $\lim_{n\to\infty}\|y_n\| = b$. We may assume that $0 < a \leqslant b$. Suppose (i) to be true. Then

$$\begin{aligned} 2 \geqslant \left\|\frac{x_n}{\|x_n\|} + \frac{y_n}{\|y_n\|}\right\| &= \frac{1}{\|x_n\|\|y_n\|}\Big\|\|y_n\|x_n + \|x_n\|y_n\Big\| \\ &= \frac{1}{\|x_n\|\|y_n\|}\Big\|\|y_n\|(x_n+y_n) - (\|y_n\|-\|x_n\|)y_n\Big\| \\ &\geqslant \frac{1}{\|x_n\|\|y_n\|}\Big|\|y_n\|\|x_n+y_n\| - \Big|\|y_n\|-\|x_n\|\Big|\|y_n\|\Big| \\ &= \frac{1}{\|x_n\|}\Big|\|x_n+y_n\| - \Big|\|y_n\|-\|x_n\|\Big|\Big| \\ &\to \frac{1}{a}(a+b-|b-a|) = 2 \end{aligned}$$

as $n \to \infty$, which asserts (ii). Conversely assume that (ii) is true. Then

$$\begin{aligned} \|x_n\| + \|y_n\| &\geqslant \|x_n + y_n\| \\ &\geqslant \|y_n\| \left\| \frac{x_n}{\|y_n\|} + \frac{y_n}{\|y_n\|} \right\| \\ &= \|y_n\| \left\| \left(\frac{x_n}{\|x_n\|} + \frac{y_n}{\|y_n\|} \right) - \left(\frac{x_n}{\|x_n\|} - \frac{x_n}{\|y_n\|} \right) \right\| \\ &\geqslant \|y_n\| \left(\left\| \frac{x_n}{\|x_n\|} + \frac{y_n}{\|y_n\|} \right\| - \left| \frac{1}{\|x_n\|} - \frac{1}{\|y_n\|} \right| \|x_n\| \right). \end{aligned}$$

Since the first and the last terms tend to $a+b$ as $n \to \infty$, we have the conclusion. □

Now we present the main result.

THEOREM 1. *Let X and Y be Banach spaces and $\psi \in \Psi$. Then the following are equivalent.*

(i) *$X \oplus_\psi Y$ is uniformly non-square.*

(ii) *X and Y are uniformly non-square and $\psi \neq \psi_1, \psi_\infty$.*

Proof. If $X \oplus_\psi Y$ is uniformly non-square, then clearly X, Y and $(\mathbb{C}^2, \|\cdot\|_\psi)$ are uniformly non-square (embed them into $X \oplus_\psi Y$). Then by Theorem B, $\psi \neq \psi_1$ and $\psi \neq \psi_\infty$. Let us show that the assertion (ii) implies (i). Assume that $X \oplus_\psi Y$ is not uniformly non-square. Then we have a couple of sequences $\{(x_n, y_n)\}, \{(u_n, v_n)\} \in X \oplus_\psi Y$ such that

$$\|(x_n, y_n)\|_\psi = \|(u_n, v_n)\|_\psi = 1 \tag{9}$$

for all $n \in \mathbb{N}$ and

$$\|(x_n + u_n,\ y_n + v_n)\|_\psi \to 2, \tag{10}$$

$$\|(x_n - u_n,\ y_n - v_n)\|_\psi \to 2 \tag{11}$$

as $n \to \infty$. Since $\|x_n\| = \|(x_n, 0)\|_\psi \leqslant \|(x_n, y_n)\|_\psi = 1$, the sequence $\{\|x_n\|\}$ is bounded. So $\{\|x_n\|\}$ has a convergent subsequence. For simplicity we assume that $\{\|x_n\|\}$ itself converges; the same argument works for the other sequences. Thus we may assume that

$$\|x_n\| \to a,\ \|y_n\| \to b, \tag{12}$$

$$\|u_n\| \to c,\ \|v_n\| \to d, \tag{13}$$

and

$$\|x_n + u_n\| \to \alpha,\ \|y_n + v_n\| \to \beta, \tag{14}$$

$$\|x_n - u_n\| \to \gamma,\ \|y_n - v_n\| \to \delta \tag{15}$$

as $n \to \infty$. Then clearly we have

$$\|(a, b)\|_\psi = \|(c, d)\|_\psi = 1 \tag{16}$$

and

$$\|(\alpha,\ \beta)\|_\psi = \|(\gamma,\ \delta)\|_\psi = 2. \tag{17}$$

Hence

$$\begin{aligned} 2 = \|(\alpha,\ \beta)\|_\psi &= \lim_{n\to\infty} \|(\|x_n+u_n\|,\ \|y_n+v_n\|)\|_\psi \\ &\leqslant \lim_{n\to\infty} \|(\|x_n\|+\|u_n\|,\ \|y_n\|+\|v_n\|)\|_\psi \\ &= \|(a+c, b+d)\|_\psi \\ &\leqslant \|(a,b)\|_\psi + \|(c,d)\|_\psi = 2, \end{aligned}$$

and thus we have

$$\|(\alpha,\ \beta)\|_\psi = \|(a+c, b+d)\|_\psi = 2. \tag{18}$$

In the same way

$$\|(\gamma,\ \delta)\|_\psi = \|(a+c, b+d)\|_\psi = 2. \tag{19}$$

Also it is obvious that

$$\alpha \leqslant a+c, \beta \leqslant b+d \quad \text{and} \quad \gamma \leqslant a+c,\ \delta \leqslant b+d.$$

Therefore, in view of Lemma 1, we have neither

$$\alpha < a+c \quad \text{and} \quad \beta < b+d$$

nor

$$\gamma < a+c \quad \text{and} \quad \delta < b+d.$$

Case 1. Let $\alpha = a+c,\ \beta = b+d$ and $\gamma = a+c,\ \delta = b+d$. Note that $(a,b) \neq (0,0)$ and $(c,d) \neq (0,0)$ by (16). Assume first that $a,c > 0$. Then since

$$\lim_{n\to\infty} \|x_n+u_n\| = \alpha = a+c = \lim_{n\to\infty} \|x_n\| + \lim_{n\to\infty} \|u_n\|,$$

we have

$$\left\| \frac{x_n}{\|x_n\|} + \frac{u_n}{\|u_n\|} \right\| \to 2 \quad \text{as } n \to \infty$$

by Lemma 2. In the same way $\lim_{n\to\infty} \|x_n - u_n\| = \gamma = a+c = \lim_{n\to\infty} \|x_n\| + \lim_{n\to\infty} \|u_n\|$ and we have

$$\left\| \frac{x_n}{\|x_n\|} - \frac{u_n}{\|u_n\|} \right\| \to 2 \quad \text{as } n \to \infty.$$

This implies that X is not uniformly non-square, which contradicts our assumption. In case of $b,d > 0$, by a parallel argument we have that Y is not uniformly non-square.

Next let $a,d > 0$. We may assume that $b = c = 0$ (the other cases have been treated above). Then we have

$$\alpha = \beta = 1. \tag{20}$$

Indeed, $\alpha = a = \|(a,0)\|_\psi = \|(a,b)\|_\psi = 1$. Therefore by (18) we have

$$2 = \|(\alpha,\beta)\|_\psi = \|(1,1)\|_\psi = 2\psi(\frac{1}{2}),$$

or $\psi(1/2) = 1$, from which it follows that $\psi = \psi_1$, a contradiction. In case of $b,c > 0$, we have (20) and hence $\psi = \psi_1$ in the same way.

Case 2a. Let $\alpha = a + c,\ \beta < b + d$ and $\gamma = a + c,\ \delta < b + d$. Then since

$$\|(\alpha,\ \beta)\|_{\psi} = \|(a + c,\ b + d)\|_{\psi} = 2, \tag{18}$$

we obtain $\alpha = 2$ by Proposition 2 and in the same way $\gamma = 2$. Hence $a = c = 1$ as $0 \leqslant a,\ c \leqslant 1$. Therefore

$$\|x_n\| \to 1, \ \ \|u_n\| \to 1$$

and

$$\|x_n \pm u_n\| \to 2.$$

Consequently, by Lemma 2, we have

$$\left\| \frac{x_n}{\|x_n\|} \pm \frac{u_n}{\|u_n\|} \right\| \to 2 \quad \text{as } n \to \infty,$$

which asserts that X is not uniformly non-square.

Case 2b. When $\alpha < a + c,\ \beta = b + d$ and $\gamma < a + c,\ \delta = b + d$, by a parallel argument to Case 2a, Y is not uniformly non-square.

Case 3a. Let $\alpha = a + c,\ \beta < b + d$ and $\gamma < a + c,\ \delta = b + d$. Then as in Case 2a we have

$$\alpha = a + c = 2 \quad \text{and} \quad \delta = b + d = 2$$

and hence $a = c = b = d = 1$. Therefore

$$1 = \|(a, b)\|_{\psi} = \|(1, 1)\|_{\psi} = 2\psi(\frac{1}{2}),$$

or $\psi(1/2) = 1/2$, by which we have $\psi = \psi_{\infty}$, a contradiction.

Case 3b. In the case $\alpha < a + c,\ \beta = b + d$ and $\gamma = a + c,\ \delta < b + d$, we have $\psi = \psi_{\infty}$ in the same way. This completes the proof. □

Now consider the Lorentz $\ell_{p,q}$-norm $\|\cdot\|_{p,q}$, $1 \leqslant q \leqslant p \leqslant \infty,\ q < \infty$:

$$\|(z_1,\ z_2)\|_{p,q} = \left\{ z_1^{*q} + 2^{(q/p)-1} z_2^{*q} \right\}^{1/q},$$

where $\{z_1^*,\ z_2^*\}$ is the non-increasing rearrangement of $\{|z_1|,\ |z_2|\}$. (Note that in case of $1 \leqslant p < q \leqslant \infty$, $\|\cdot\|_{p,q}$ is not a norm but a quasi-norm (cf. [6], [14, p.126]). Clearly $\|\cdot\|_{p,q}$ is an absolute normalized norm and the corresponding convex function $\psi_{p,q}$ is given by

$$\psi_{p,q}(t) \begin{cases} \{(1-t)^q + 2^{q/p-1} t^q\}^{1/q} & \text{if } 0 \leqslant t \leqslant 1/2, \\ \{t^q + 2^{q/p-1}(1-t)^q\}^{1/q} & \text{if } 1/2 \leqslant t \leqslant 1. \end{cases} \tag{21}$$

Then $\psi_{p,q}$ yields the $\ell_{p,q}$-sum $X \oplus_{p,q} Y$:

$$\|(x, y)\|_{p,q} = \left\{ \max(\|x\|^q, \|y\|^q) + 2^{(q/p)-1} \min(\|x\|^q, \|y\|^q) \right\}^{1/q} \tag{22}$$

COROLLARY 1. *Let* $1 \leqslant q \leqslant p \leqslant \infty$ *and not* $p = q = 1, \infty$. *Then* $\ell_{p,q}$*-sum* $X_1 \oplus_{p,q} X_2$ *is uniformly non-square if and only if* X_1 *and* X_2 *are uniformly non-square. In particular* ℓ_p*-sum* $X_1 \oplus_p X_2$, $1 < p < \infty$, *is uniformly non-square if and only if* X_1 *and* X_2 *are uniformly non-square.*

EXAMPLE 1. (cf. [10, 11]). Let $1/2 \leqslant \alpha \leqslant 1$. Let

$$\psi_\alpha(t) = \begin{cases} \frac{\alpha-1}{\alpha} t + 1 & \text{if } 0 \leqslant t \leqslant \alpha, \\ t & \text{if } \alpha \leqslant t \leqslant 1. \end{cases} \tag{23}$$

Then $\psi_\alpha \in \Psi$ and the norm of $X \oplus_{\psi_\alpha} Y$ is given by

$$\|(x,y)\|_{\psi_\alpha} = \max\{\|x\| + (2 - \frac{1}{\alpha})\|y\|,\ \|y\|\}. \tag{24}$$

In particular we have

$$\|(x,y)\|_{\psi_\alpha} = \begin{cases} \|x\| + \|y\| & \text{if } \alpha = 1, \\ \max\{\|x\|, \|y\|\} & \text{if } \alpha = 1/2. \end{cases} \tag{25}$$

Thus $\|\cdot\|_{\psi_\alpha}$ are non ℓ_p-type norms 'combining' the ℓ_1- and ℓ_∞-sum norms as α varies from 1 to $1/2$. Indeed, we see (24) as follows.

$$\begin{aligned} \|(x,y)\|_{\psi_\alpha} &= \begin{cases} (\|x\| + \|y\|)\left\{\frac{\alpha-1}{\alpha}\frac{\|y\|}{\|x\|+\|y\|} + 1\right\} & \text{if } \frac{\|y\|}{\|x\|+\|y\|} \leqslant \alpha, \\ (\|x\| + \|y\|)\frac{\|y\|}{\|x\|+\|y\|} & \text{if } \frac{\|y\|}{\|x\|+\|y\|} \geqslant \alpha \end{cases} \\ &= \begin{cases} \|x\| + \left(2 - \frac{1}{\alpha}\right)\|y\| & \text{if } \frac{\|y\|}{\|x\|+\|y\|} \leqslant \alpha, \\ \|y\| & \text{if } \frac{\|y\|}{\|x\|+\|y\|} \geqslant \alpha. \end{cases} \end{aligned}$$

Noting that $\|x\| + \left(2 - \frac{1}{\alpha}\right)\|y\| \geqslant \|y\|$ if and only if $\|y\|/(\|x\| + \|y\|) \leqslant \alpha$, we have (24).

Now, according to Theorem 1, $X \oplus_{\psi_\alpha} Y$ *is uniformly non-square if and only if* X *and* Y *are uniformly non-square and neither* $\alpha = 1$ *nor* $\alpha = 1/2$. Assume next that X and Y are uniformly convex. Then, since ψ_α is not strictly convex, $X \oplus_{\psi_\alpha} Y$ is not uniformly convex by Theorem A (ii). Thus, if $1/2 < \alpha < 1$, then $X \oplus_{\psi_\alpha} Y$ is an example of uniformly non-square spaces which are not uniformly convex. In the same way, if X and Y are uniformly non-square and $1/2 < \alpha < 1$, then $X \oplus_{\psi_\alpha} Y$ is uniformly non-square but not strictly convex by Theorem 1 and Theorem A (i).

EXAMPLE 2. (cf. [10]). Let $1 \leqslant q < p \leqslant \infty$ and $2^{1/p-1/q} < \lambda < 1$. Let $\psi_{p,q,\lambda} = \max\{\psi_p, \lambda\psi_q\}$, where ψ_p is as in (6). Then $\psi_{p,q,\lambda} \in \Psi$ and, as is easily seen, the norm of $X \oplus_{\psi_{p,q,\lambda}} Y$ is given by

$$\|(x,y)\|_{\psi_{p,q,\lambda}} = \max\{\|(x,y)\|_p,\ \lambda\|(x,y)\|_q\}. \tag{26}$$

Indeed, ψ_p and $\lambda\psi_q$ meet in $(0,1)$ (note that $\psi_p < \psi_q$, and ψ_p and ψ_q have their minimums $2^{1/p-1}$ and $2^{1/q-1}$ respectively), and $\psi_{p,q,\lambda}$ is convex, so $\psi_{p,q,\lambda} \in \Psi$.

According to Theorem 1, $X \oplus_{\psi_{p,q,\lambda}} Y$ *is uniformly non-square if and only if* X *and* Y *are uniformly non-square.*

PROBLEM. For ψ-direct sums of an arbitrary finite number of Banach spaces $X_1, X_2, \cdots, X_n$ the situation seems not simple. Characterize the uniform non-squareness of $(X_1 \oplus X_2 \oplus \cdots \oplus X_n)_\psi$ (cf. [7, 12]).

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