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VARIATIONAL INEQUALITIES AND OPTIMAL EQUILIBRIUM DISTRIBUTIONS IN TRANSPORTATION NETWORKS

LAURA SCRIMALI

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Abstract. We deal with equilibrium problems in transportation networks within a variational framework. In particular, we present some paradoxical results which show how some changes in input data can be successfully exploited by the traffic manager in order to optimize the traffic distribution thus minimizing the total cost.

1. Introduction

This paper aims to analyze carefully some paradoxical phenomena in transportation networks, see [5, 6]. In fact, it is observed that the policy interventions of the traffic manager are often influenced by the study of the stability and sensitivity of the traffic network equilibria. Therefore, our interest is devoted to seeing how some changes in input data can affect the equilibrium assignment, especially in the presence of route-capacity constraints. Starting from famous Fisk's paradox (see [7]), we give the conditions under which we obtain the paradoxical result for which the increase in the travel demand leads to a decrease in the total travel cost (see [4, 11]). Thus, we deduce that capacity restrictions can be successfully exploited by the traffic manager in order to optimize the traffic distribution and, overall to keep the travel cost below a certain threshold.

In section 2, by using the variational inequality theory, we deal with a basic model where no dependence on time is required. We also give a complete characterization of equilibrium patterns in the presence of capacity constraints on routes. Finally, a sufficient condition under which the paradoxical decrease in the travel cost occurs is presented (see Theorem 1).

In section 3 we consider the time-dependent case and, after introducing the time-dependent variational inequality expressing the problem, we give a sufficient condition which allows us to have the paradox (see Theorem 2).

In section 4 we propose an example which is interesting from a theoretical point of view, since it clarifies how the paradoxical phenomenon can be obtained in a particular network.

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In section 5 we present the paradox in a time-dependent and elastic model of transportation networks, *i.e.* when the travel demand depends on time as well as on the equilibrium pattern (see Theorem 3), while in the last section, we provide an example.

2. The static model

Let us consider a traffic network, where N is the set of nodes and W is the set of the Origin/Destination pairs $w_j,\ j=1,\dots,l$. Let us denote by $\mathcal{R}=\{R_r, r=1,\dots,m\}$ the set of all the routes of the network and by $\mathcal{R}_j$ the set of the routes which connect a given O/D pair w_j. Let us introduce the pair-route incidence matrix $\Phi=(\varphi_{jr})_{j=1,\dots,l\ r=1,\dots,m}$, where $\varphi_{jr}=1$ if $R_r\in\mathcal{R}_j$ and $\varphi_{jr}=0$ otherwise. We consider the flow vectors $F\in\mathbb{R}^m_+$, where $F_r, r=1,...,m$, denotes the flow on route R_r. Moreover, we assume that route flows have to satisfy some capacity restrictions, $\lambda_r\leqslant F_r\leqslant\mu_r,\quad r=1,\dots,m$, as well as the usual demand requirements, $\sum_{r=1}^m\varphi_{jr}F_r=\rho_j,\ \ j=1,\dots,l$, where $\lambda,\mu\in\mathbb{R}^m_+$ and $\rho\in\mathbb{R}^l_+$. Thus the set of feasible flows is given by:

$$\mathbb{K}=\Big\{F\in\mathbb{R}^m_+:\quad \lambda_r\leqslant F_r\leqslant\mu_r,\quad r=1,\dots,m;\quad \sum_{r=1}^m\varphi_{jr}F_r=\rho_j,\quad j=1,\dots,l\Big\},$$

under condition that $\Phi\lambda\leqslant\Phi F\leqslant\Phi\mu$.

Let us assign a cost function $C:\mathbb{K}\to\mathbb{R}^m_+$, then it results that (see [12]):

DEFINITION. $H\in\mathbb{R}^m_+$ is an equilibrium flow if and only if it solves the following variational inequality:

$$H\in\mathbb{K},\quad C(H)(F-H)\geqslant 0,\quad \forall F\in\mathbb{K}.$$

It is well-known that unconstrained equilibrium flows satisfy Wardrop's principle (see [13]), which we formulate as follows:

DEFINITION. $H\in\mathbb{K}$ is an equilibrium flow if $\forall\omega_j\in W$ it results that:

$$C_r(H)=\min_{R_r\in\mathcal{R}_j}C_r(H)\quad \text{if}\quad H_r>0,$$

$$C_r(H)>\min_{R_r\in\mathcal{R}_j}C_r(H)\quad \text{if}\quad H_r=0.$$

Nevertheless, in the presence of capacity constraints, we can refer to a generalized form of the above principle (see [9, 10]).

DEFINITION. A flow $H\in\mathbb{K}$ is an equilibrium flow if and only if

$$\begin{gathered}\forall w_j\in W,\forall R_q,R_s\in\mathcal{R}_j,\\ C_q(H)<C_s(H)\Rightarrow H_q=\mu_q\text{or}H_s=\lambda_s.\end{gathered}\tag{1}$$

Now, let us set:

$$C^j(H) = \max_{R_r \in B_j} C_r(H),$$

where

$$B_j = \{R_r : R_r \in \mathcal{R}_j, H_r > \lambda_r\},$$

and let us introduce the generalized costs:

$$\widetilde{C}_r(H) = \begin{cases} C_r(H) + L_r^j = C^j(H) & \text{if } R_r \in B_j\,, \\ C_r(H) & \text{if } R_r \notin B_j,\ R_r \in \mathcal{R}_j\,, \end{cases}$$

where

$$L_r^j = C^j(H) - C_r(H), \qquad R_r \in B_j.$$

Hence, the generalized principle (1) becomes:

DEFINITION. $H \in \mathbb{K}$ is an equilibrium vector if

$$\begin{gathered} \forall w_j \in W \forall R_q, R_s \in \mathcal{R}_j, \\ \widetilde{C}_r(H) > \widetilde{C}_s(H) \Rightarrow H_r = \lambda_r. \end{gathered} \tag{2}$$

It is worth noting that (2) allows us to neglect the upper bounds of route flows. Moreover (2) coincides with Wardrop's classic principle if we shift the flows, setting $\widetilde{H}_r = H_r - \lambda_r$ and $\widetilde{\rho}_j = \rho_j - \sum_{j=1}^{l} \varphi_{jr}\lambda_r$. Therefore, we can confine our study to the case $\lambda_r = 0$ and apply the classic principle directly.

Now, let us suppose we increase the travel demands for the O/D pairs w_{ν_h}, $h = 1, \dots, k$, with $k < l$. Thus, we have to deal with another model of network, whose set of feasible flows is:

$$\begin{aligned} \mathbb{K}^* = \quad & \Big\{F^* \in \mathbb{R}_+^m : \quad \lambda_r \leqslant F_r^* \leqslant \mu_r, \quad r = 1, \dots, m; \ \sum_{r=1}^{m} \varphi_{jr} F_r^* = \rho_j, \\ & j = 1, \dots, l, j \neq \nu_1, \dots, \nu_k, \ \sum_{r=1}^{m} \varphi_{\nu_h r} F_r^* = \rho_{\nu_h}^*, \quad h = 1, \dots, k\Big\} \end{aligned}$$

where $\rho_{\nu_h}^* = \rho_{\nu_h} + d_h^*$, $d_h^* > 0$, $h = 1, \dots, k$ and $\Phi\lambda \leqslant \Phi F^* \leqslant \Phi\mu$.

The variational inequality which expresses the equilibrium problem is:

$$H^* \in \mathbb{K}^*, \quad C^*(H^*)(F^* - H^*) \geqslant 0, \quad \forall F^* \in \mathbb{K}^*.$$

Let us construct the generalized costs $\widetilde{C}_r^*(H^*)$ even for the modified network, as done before, then let us set:

$$\widetilde{\widetilde{C}}^j(H) = \min_{R_r \in \mathcal{R}_j} \widetilde{C}_r(H); \quad \widetilde{C}^j(H^*) = \min_{R_r \in \mathcal{R}_j} \widetilde{C}_r^*(H^*).$$

Our purpose is to find the conditions which imply the following paradoxical result: the increase in the travel demand leads to a decrease in the total travel cost. To this end we prove the theorem:

THEOREM 1. *A sufficient condition which makes the travel cost decrease is given by:*

$$\sum_{j=1}^{l} \tilde{\tilde{C}}^j(H)\rho_j > \sum_{j=1}^{l} \tilde{C}^j(H^*)\rho_j + \sum_{h=1}^{k} \tilde{C}^{\nu_h}(H^*)d_h^*.$$

Proof. By applying Wardrop's principle, we have that the difference between the total cost in the initial network and in the modified one results in:

$$C^*(H^*)H^* - C(H)H = \sum_{j=1}^{l} \sum_{R_r \in \mathcal{R}_j} \tilde{C}_r^*(H^*)H_r^* - \sum_{j=1}^{l} \sum_{R_r \in \mathcal{R}_j} \tilde{C}_r(H)H_r =$$

$$= \sum_{j=1}^{l} \Bigg(\sum_{\substack{R_r \in \mathcal{R}_j \\ \tilde{C}_r^*(H^*)=\tilde{C}^j(H^*)}} \tilde{C}_r^*(H^*)H_r^* + \sum_{\substack{R_r \in \mathcal{R}_j \\ \tilde{C}_r^*(H^*)>\tilde{C}^j(H^*)}} \tilde{C}_r^*(H^*)H_r^* +$$

$$- \sum_{\substack{R_r \in \mathcal{R}_j \\ \tilde{C}_r(H)=\tilde{\tilde{C}}^j(H)}} \tilde{C}_r(H)H_r - \sum_{\substack{R_r \in \mathcal{R}_j \\ \tilde{C}_r(H)>\tilde{\tilde{C}}^j(H)}} \tilde{C}_r(H)H_r \Bigg) =$$

$$= \sum_{j=1}^{l} \left(\tilde{C}^j(H^*) \sum_{R_r \in \mathcal{R}_j} H_r^* - \tilde{\tilde{C}}^j(H) \sum_{R_r \in \mathcal{R}_j} H_r \right) =$$

$$= \sum_{j \neq \nu_1,\dots,\nu_k} \tilde{C}^j(H^*)\rho_j - \sum_{j=1}^{l} \tilde{\tilde{C}}^j(H)\rho_j + \sum_{h=1}^{k} \tilde{C}^{\nu_h}(H^*)(\rho_{\nu_h} + d_h^*) =$$

$$= \sum_{j=1}^{l} \tilde{C}^j(H^*)\rho_j - \sum_{j=1}^{l} \tilde{\tilde{C}}^j(H)\rho_j + \sum_{h=1}^{k} \tilde{C}^{\nu_h}(H^*)d_h^*.$$

Thus we are entitled to deduce the following sufficient condition which allows us to have the paradox:

$$\sum_{j=1}^{l} \tilde{\tilde{C}}^j(H)\rho_j > \sum_{j=1}^{l} \tilde{C}^j(H^*)\rho_j + \sum_{h=1}^{k} \tilde{C}^{\nu_h}(H^*)d_h^*. \tag{3}$$

We observe that if cost functions fulfill the strong monotonicity condition, then it is possible to derive a more general relationship which provides the decrease in the cost. We find indeed the following result, whose proof we omit:

THEOREM 2. *Let C be a strongly monotone function, namely*

$$\exists \alpha > 0 \text{s.t.} \Big[C(F_1) - C(F_2)\Big](F_1 - F_2) \geqslant \alpha \|F_1 - F_2\|^2 \quad \forall F_1,\ F_2 \in \mathbb{K},$$

then the paradoxical behavior is obtained if:

1. $C(F)(F - H) + C^*(H^*)H^* \leqslant \alpha \|F - H\|^2 \quad \forall F \in \mathbb{K}$, *or equivalently if*

2. $C^*(H^*)H^* \leqslant \alpha\|F - H\|^2 \quad \forall F \in \mathbb{K}.$

In particular if $H^* \in \mathbb{K}$*, the paradox occurs when*

$$C^*(H^*)H^* \leqslant \alpha\|H^* - H\|^2.$$

3. The time-dependent model

In this section we extend the previous results to the case of time-dependent traffic networks. Let us consider, at each $t \in [0,T]$, a traffic network which has the same geometry as the static model before presented, and let us assign a route-flow vector $F(t) \in \mathbb{R}^m_+, \forall t \in [0,T]$. Due to technical reasons, our functional setting is the reflexive Banach space $L^2(0,T;\mathbb{R}^m_+)$. We assume that route flows have to satisfy some time-dependent capacity constraints and demand requirements. Therefore, the set of feasible flows is given by:

$$\begin{aligned}\mathbb{K} = \quad & \Big\{F \in L^2(0,T;\mathbb{R}^m_+) : \quad \lambda_r(t) \leqslant F_r(t) \leqslant \mu_r(t) \quad \text{a.e. in } [0,T], \\ & r = 1,\dots,m; \sum_{r=1}^{m} \varphi_{jr}F_r(t) = \rho_j(t) \quad \text{a.e. in } [0,T], \quad j = 1,\dots,l\Big\},\end{aligned}$$

where $\lambda(t),\ \mu(t) \in L^2(0,T;\mathbb{R}^m_+),\ \lambda(t) \leqslant \mu(t)$ a.e. in $[0,T]$ and $\Phi\lambda(t) \leqslant \Phi F(t) \leqslant \Phi\mu(t)$ a.e. in $[0,T]$. Moreover, let $C : \mathbb{K} \to L^2(0,T;\mathbb{R}^m_+)$ the cost function, then we have that (see [1]):

DEFINITION. $H \in L^2(0,T;\mathbb{R}^m_+)$ is an equilibrium flow if and only if it satisfies the following variational inequality:

$$H \in \mathbb{K} \quad \int_0^T C(H(t))(F(t) - H(t))dt \geqslant 0, \quad \forall F \in \mathbb{K}.$$

Now, let us increase the travel demands for the O/D pairs $w_{\nu_h},\ h = 1,\dots,k$, with $k < l$. Hence, the set of feasible flows of the modified network is:

$$\begin{aligned}\mathbb{K}^* = \quad & \Big\{F^* \in L^2(0,T;\mathbb{R}^m_+) : \quad \lambda_r(t) \leqslant F^*_r(t) \leqslant \mu_r(t) \quad \text{a.e. in } [0,T], \\ & r = 1,\dots,m; \quad \sum_{r=1}^{m} \varphi_{jr}F^*_r(t) = \rho_j(t) \quad \text{a.e. in } [0,T], \\ & j = 1,\dots,l, \quad j \neq \nu_1,\dots,\nu_k, \quad \sum_{r=1}^{m} \varphi_{\nu_h r}F^*_r(t) = \rho^*_{\nu_h}(t) \\ & \text{a.e. in } [0,T], \quad h = 1,\dots,k\Big\}.\end{aligned}$$

where $\rho^*_{\nu_h}(t) = \rho_{\nu_h}(t) + d^*_h,\quad d^*_h > 0,\quad h = 1,\dots,k$ and $\Phi\lambda(t) \leqslant \Phi F^*(t) \leqslant \Phi\mu(t)$ a.e. in $[0,T]$.

The variational inequality associated with the new equilibrium problem assumes the form:

$$H^* \in \mathbb{K}^*, \quad \int_0^T C^*(H^*(t))(F^*(t) - H^*(t))dt \geqslant 0, \quad \forall F^* \in \mathbb{K}^*.$$

Let us denote the total cost in the first network by $\int_0^T C(H(t))H(t)dt$ and the total cost in the modified network by $\int_0^T C^*(H^*(t))H^*(t)dt$.

The following theorem gives a sufficient condition which guarantees the paradox. We inform the reader that the syymbols have the same meaning as the ones introduced in section 2 for the static case.

THEOREM 3. *The total cost decreases after the increase in the travel demands if, it results that:*

$$\int_0^T \Big(\sum_{j=1}^{l} \widetilde{C}^j(H^*(t))\rho_j(t) - \sum_{j=1}^{l} \widetilde{\widetilde{C}}^j(H(t))\rho_j(t) + \sum_{h=1}^{k} \widetilde{C}^{\nu_h}(H^*(t))d_h^* \Big) dt < 0.$$

The proof of the above theorem is on the lines of Theorem 1 and therefore will be omitted.

4. An example

Let us consider a time-dependent traffic network as in Figure 1, where the set of nodes is $N = \{P_1, P_2, P_3\}$ and the set of links, which coincides with the set of origin-destination pairs, is $L = \{(P_3, P_2), (P_3, P_1), (P_2, P_1)\}$. The set of routes connecting the O/D pairs is $R = \{R_1, R_2, R_3, R_4\}$, where $R_1 = P_3P_2$, $R_2 = P_3P_1$, $R_3 = P_2P_1$, $R_4 = P_3P_2 \cup P_2P_1$.

Let us assume that link costs are:

$$\begin{aligned} c_1(f(t)) &= f_1(t), \\ c_2(f(t)) &= f_2(t) + \alpha, \\ c_3(f(t)) &= f_3(t), \end{aligned}$$

where $f(t) = (f_1(t), f_2(t), f_3(t))$ is the link flow vector and α is a non negative constant.
Since link flows can be expressed in terms of the route flows F_r, $r = 1, \dots, 4$, we find the following relationships:

$$\begin{aligned} f_1(t) &= F_1(t) + F_4(t), \\ f_2(t) &= F_2(t), \\ f_3(t) &= F_3(t) + F_4(t). \end{aligned}$$

Thus, the route costs $C_1(F(t)), C_2(F(t)), C_3(F(t)), C_4(F(t))$ are given by:

$$\begin{aligned} C_1(F(t)) &= F_1(t) + F_4(t), \\ C_2(F(t)) &= F_2(t) + \alpha, \\ C_3(F(t)) &= F_3(t) + F_4(t), \\ C_4(F(t)) &= F_1(t) + F_3(t) + 2F_4(t). \end{aligned}$$

We assume that the route flows have to satisfy some capacity constraints so that the set of all feasible flows is given by:

$$\begin{aligned}\mathbb{K} = \quad & \{F \in L^2(0,T;\mathbb{R}^4_+) : \lambda_r(t) \leqslant F_r(t) \leqslant \mu_r(t) \text{a.e. in } [0,T],\\ & r = 1,2,3; \quad F_1(t) = \rho_1(t), F_2(t) + F_4(t) = \rho_2(t),\\ & F_3(t) = \rho_3(t) \text{a.e. in } [0,T]\},\end{aligned}$$

where $\lambda(t) \leqslant \mu(t) \in L^2(0,T;\mathbb{R}^3_+)$, $\lambda_1(t) \leqslant \rho_1(t) \leqslant \mu_1(t)$, $\rho_2(t) > \lambda_2(t)$, $\lambda_3(t) \leqslant \rho_3(t) \leqslant \mu_3(t)$ a. e. in $[0,T]$. For simplicity, we confine our study to the case $\rho_2(t) > \mu_2(t)$ and apply the computational procedure shown in [8].

The variational inequality which expresses the equilibrium problem is:

$$H \in \mathbb{K} \quad \int_0^T C(H(t))(F(t) - H(t))dt \geqslant 0, \qquad \forall F \in \mathbb{K}. \tag{4}$$

Deducing $F_4(t)$ from $F_2(t)+F_4(t) = \rho_2(t)$ the variational inequality (4) becomes:

$$\widetilde{H} \in \widetilde{\mathbb{K}} \quad \int_0^T \Gamma(\widetilde{H}(t))(\widetilde{F}(t) - \widetilde{H}(t))dt \geqslant 0, \quad \forall \widetilde{F} \in \widetilde{\mathbb{K}}, \tag{5}$$

with

$$\widetilde{\mathbb{K}} = \{\widetilde{F} \in L^2(0,T) : \lambda_2(t) \leqslant F_2(t) \leqslant \mu_2(t) \text{a.e. in } [0,T]\}$$

and

$$\Gamma(\widetilde{H}(t)) = C_2(\widetilde{H}(t)) - C_4(\widetilde{H}(t)).$$

It is immediate to show that if $\widetilde{H}$ satisfies the system:

$$\begin{cases} \Gamma(\widetilde{H}) = 0 \\ \widetilde{H} \in \widetilde{\mathbb{K}} \end{cases} \tag{6}$$

then it solves (5). We find that if the following condition holds:

$$3\lambda_2(t) + \alpha - 2\rho_2(t) \leqslant \rho_1(t) + \rho_3(t) \leqslant 3\mu_2(t) + \alpha - 2\rho_2(t) \quad \text{a.e. in } [0,T], \tag{7}$$

the solution of the system:

$$\begin{cases} C_2(\widetilde{H}(t)) = C_4(\widetilde{H}(t)) \\ H_1(t) = \rho_1(t) \\ H_3(t) = \rho_3(t) \\ H_2(t) + H_4(t) = \rho_2(t) \end{cases}$$

is given by:

$$\begin{cases} H_1(t) = \rho_1(t) \\ H_2(t) = \frac{\rho_1(t)+2\rho_2(t)+\rho_3(t)-\alpha}{3} \\ H_3(t) = \rho_3(t) \\ H_4(t) = \frac{-\rho_1(t)+\rho_2(t)-\rho_3(t)+\alpha}{3}. \end{cases}$$

Now, let us increase the travel demand for the pair (P_3, P_2) so as to have the new travel demand vector: $(\rho_1(t)+d, \rho_2(t), \rho_3(t))$, $d>0$. If

$$3\lambda_2(t)+\alpha-2\rho_2(t) \leqslant \rho_1(t)+d+\rho_3(t) \leqslant 3\mu_2(t)+\alpha-2\rho_2(t) \quad \text{a.e. in } [0,T],$$

then the solution $H^*(t)=(H_1^*(t), H_2^*(t), H_3^*(t), H_4^*(t))$ is given by:

$$\begin{cases} H_1^*(t)=H_1(t)+d \\ H_2^*(t)=H_2(t)+\frac{d}{3} \\ H_3^*(t)=H_3(t) \\ H_4^*(t)=H_4(t)-\frac{d}{3}. \end{cases}$$

The new cost functions are:

$$\begin{cases} C_1^*(H^*(t))=C_1(H(t))+\frac{2d}{3} \\ C_2^*(H^*(t))=C_2(H(t))+\frac{d}{3} \\ C_3^*(H^*(t))=C_3(H(t))-\frac{d}{3} \\ C_4^*(H^*(t))=C_4(H(t))+\frac{d}{3}. \end{cases}$$

It is easy to show that under convenient conditions the increase in the travel demand can make the total cost decrease. If we denote the total cost in the initial network by $\int_0^T C(H(t))H(t)dt$ and the travel cost in the modified one by $\int_0^T C^*(H^*(t))H^*(t)dt$, we have that:

$$\int_0^T C^*(H^*(t))H^*(t)dt = \int_0^T \Big[C(H(t))H(t)+\frac{d}{3}(2(2\rho_1(t)+ \rho_2(t)-\rho_3(t))+\alpha)+\frac{2d^2}{3}\Big]dt.$$

The paradox occurs if:

$$\int_0^T C^*(H^*(t))H^*(t)dt-\int_0^T C(H(t))H(t)dt<0,$$

namely

$$\bar{\rho}_3(t)>2\bar{\rho}_1(t)+\bar{\rho}_2(t)+\frac{\alpha}{2}+d,$$

where $\bar{\rho}_j(t)=\frac{1}{T}\int_0^T \rho_j(t)dt, \quad j=1,2,3$ represents the travel demand on average with respect to time. A numerical example can be obtained by choosing: $T=1$, $\rho_1(t)=t$, $\rho_2(t)=10t+5$, $\rho_3(t)=4t+30$, $\lambda_2(t)=7t$, $\mu_2(t)=9t+4$, $\alpha=30$, $d=2$.

5. The elastic demand model

We consider a time-dependent model of traffic networks and suppose that the travel demand depends not only on time but also on the equilibrium pattern. Our functional setting is again the space $L^2(0,T;\mathbb{R}^m_+)$.

Let us assume that:

a) $C : [0,T] \times L^2(0,T;\mathbb{R}^m_+) \to \mathbb{R}^m_+$ are the route cost function;
b) $\rho : [0,T] \times L^2(0,T;\mathbb{R}^m_+) \to \mathbb{R}^m_+$ are the elastic demand, depending on the equilibrium pattern;
c) $C(t,v)$ is measurable in $t\ \forall v \in L^2(0,T;\mathbb{R}^m_+)$, continuous in v for t a.e. in $[0,T]$,

$$\exists \gamma \in L^2(0,T) : |C(t,v)| \leqslant \gamma(t) + |v|;$$

d) $\rho(t,v)$ is measurable in $t\ \forall v \in L^2(0,T;\mathbb{R}^m_+)$, continuous in v for t a.e. in $[0,T]$,

$$\exists \psi \in L^1(0,T) : |\rho(t,v)| \leqslant \psi(t) + |v|^2;$$

e) $\exists h(t) \geqslant 0$ a.e. in $[0,T]$, $h \in L^2(0,T)$:

$$\forall v_1, v_2 \in L^2(0,T;\mathbb{R}^m_+), \quad |\rho(t,v_1) - \rho(t,v_1)| \leqslant h(t)|v_1 - v_2|;$$

f) $\lambda(t),\ \mu(t) \in L^2(0,T;\mathbb{R}^m_+),\ \lambda(t) \leqslant \mu(t)$ a.e. in $[0,T]$ are the capacity restrictions.

Then if D is a non-empty, compact, convex subset of $L^2(0,T;\mathbb{R}^m_+)$, the set of feasible flows is the set-valued function defined as follows:

$$\mathbb{K} : D \to 2^{L^2(0,T;\mathbb{R}^m_+)};$$

$$\begin{aligned}\mathbb{K}(H) = \quad & \Big\{F \in L^2(0,T;\mathbb{R}^m_+) : \lambda_r(t) \leqslant F_r(t) \leqslant \mu_r(t) \text{a.e. in } [0,T], \\ & r = 1,2,\ldots,m; \sum_{r=1}^{m} \varphi_{jr} F_r(t) = \frac{1}{T}\int_0^T \rho_j(t,H(\tau))d\tau \\ & \text{a.e. in } [0,T], j = 1,2,\ldots,l\Big\};\end{aligned}$$

where $\Phi\lambda(t) \leqslant \Phi F(t) \leqslant \Phi\mu(t)$ a.e. in $[0,T]$. We refer to a relaxed formulation of the equilibrium problems (see [2]), where the demand requirements, $\sum_{r=1}^m \varphi_{jr} F_r(t) = \frac{1}{T}\int_0^T \rho_j(t,H(\tau))d\tau$, are considered on average with respect to time.

DEFINITION. $H \in L^2(0,T;\mathbb{R}^m_+)$ is an equilibrium flow if and only if it solves the following quasi-variational inequality:

$$H \in \mathbb{K}(H) \quad \int_0^T C(t,H(t))(F(t) - H(t))dt \geqslant 0, \quad \forall F \in \mathbb{K}(H).$$

Under the above assumptions on C and ρ, and if $K(H) \subset E$, $\forall H \in E$, then the quasi-variational inequality associated with the equilibrium problem admits a solution (see [2]).

If we increase the travel demands for the O/D pairs w_{v_h}, $h = 1,\ldots,k$, with $k < l$, we obtain the new set of feasible flows:

$$\begin{aligned}\mathbb{K}^*(H^*) = \quad & \Big\{F^* \in L^2(0,T;\mathbb{R}^m_+) : \lambda_r(t) \leqslant F^*_r(t) \leqslant \mu_r(t) \quad \text{a.e. in } [0,T], \\ & r = 1,2,\dots,m; \sum_{r=1}^{m} \varphi_{jr} F^*_r(t) = \frac{1}{T}\int_0^T \rho_j(t,H(\tau))d\tau \\ & \text{a.e. in } [0,T], \quad j = 1,2,\dots,l, \quad j \neq \nu_1,\dots,\nu_k, \quad \sum_{r=1}^{m} \varphi_{\nu_h r} F^*_r(t) = \\ & = \frac{1}{T}\int_0^T \rho^*_h(t,H(\tau))d\tau \quad \text{a.e. in } [0,T], \quad h = 1,2,\dots,k\Big\};\end{aligned}$$

where $\Phi\lambda(t) \leqslant \Phi F^*(t) \leqslant \Phi\mu(t)$ a.e. in $[0,T]$. Let $H^* \in \mathbb{K}^*(H^*)$ be an equilibrium solution, then it results:

$$\int_0^T C^*(t,H^*(t))(F^*(t) - H^*(t))dt \geqslant 0, \quad \forall F^* \in \mathbb{K}^*(H^*).$$

Therefore, the following theorem can be deduced:

THEOREM 4. *A sufficient condition which allows us to have a decrease in the travel cost is given by:*

$$\int_0^T \Big(\sum_{j=1}^{l} \widetilde{C}^j(t,H^*(t))\bar{\rho}_j(t) - \sum_{j=1}^{l} \widetilde{\widetilde{C}}^j(t,H(t))\bar{\rho}_j(t) + \sum_{h=1}^{k} \widetilde{C}^{\nu_h}(t,H^*(t))d^*_h\Big)dt < 0,$$

where $\bar{\rho}_j(t) = \dfrac{1}{T}\displaystyle\int_0^T \rho_j(t,H(\tau))d\tau$.

6. An example

In this section we present the example of a time-dependent and elastic model of traffic networks. Let us consider the same network considered in section 4, and let us assume we have the same cost functions. Let the set of feasible flows be:

$$\begin{aligned}\mathbb{K}(H) = \quad & \Big\{F \in L^2(0,T;\mathbb{R}^4_+) : \lambda_r(t) \leqslant F_r(t) \leqslant \mu_r(t) \quad \text{a.e. in } [0,T], \\ & r = 1,2,3; \quad F_1(t) = \frac{1}{T}\int_0^T t\, d\tau; \quad F_2(t) + F_4(t) = \\ & = \frac{1}{T}\int_0^T (t + \varepsilon H_2(\tau))d\tau, \quad F_3(t) = \frac{2}{T}\int_0^T t\, d\tau \text{a.e. in } [0,T]\Big\};\end{aligned}$$

where $\Phi\lambda(t) \leqslant \Phi F(t) \leqslant \Phi\mu(t)$ a.e. in $[0,T]$ and $\varepsilon > 0$. The equilibrium flow is a solution of the quasi-variational inequality if:

$$H \in \mathbb{K}(H), \quad \int_0^T \sum_{r=1}^{4} C_r(H(t))(F_r(t) - H_r(t))dt \geqslant 0, \qquad \forall F \in \mathbb{K}(H). \qquad (8)$$

Following the procedure shown in [3, 8], we set:

$$\begin{aligned}
F_4(t) = \quad & t + \frac{\varepsilon}{T}\int_0^T H_2(\tau)d\tau - F_2(t);\\
\widetilde{E} = \quad & \Big\{\widetilde{H} \in L^2(0,T) : \lambda_2(t) \leqslant H_2(t) \leqslant \mu_2(t) \text{a.ein}[0,T];\\
& H_2(t) \leqslant t + \frac{\varepsilon}{T}\int_0^T H_2(\tau)d\tau \text{a.ein}[0,T]\Big\};\\
\widetilde{\mathbb{K}}(H) = \quad & \Big\{\widetilde{F} \in L^2(0,T) : \lambda_2(t) \leqslant F_2(t) \leqslant \mu_2(t) \quad \text{a.e} \in [0,T];\\
& F_2(t) \leqslant t + \frac{\varepsilon}{T}\int_0^T H_2(\tau)d\tau \quad \text{a.e} \in [0,T]\Big\};
\end{aligned}$$

$$\begin{aligned}
\Gamma(\widetilde{F}(t),\widetilde{H}(t)) = \quad & C_2(\widetilde{F}(t),\widetilde{H}(t)) - C_4(\widetilde{F}(t),\widetilde{H}(t))\\
& 3F_2(t) + \alpha - 5t - \frac{2\varepsilon}{T}\int_0^T H_2(\tau)d\tau.
\end{aligned}$$

Thus, the problem (8) can be written as:

$$\widetilde{H} \in \widetilde{\mathbb{K}}(\widetilde{H}), \quad \int_0^T \Gamma(\widetilde{H}(t),\widetilde{H}(t))(\widetilde{F}(t) - \widetilde{H}(t)dt \geqslant 0 \qquad \forall \widetilde{F} \in \widetilde{\mathbb{K}}(\widetilde{H}). \tag{9}$$

It is immediate to show that if $\widetilde{H}$ satisfies the system:

$$\begin{cases} \Gamma(\widetilde{H},\widetilde{H}) = 0\\ \widetilde{H} \in \widetilde{\mathbb{K}}, \end{cases}$$

then it solves (9). Thus we find the equilibrium solution:

$$\left(t, \frac{5}{3}t + \frac{5\varepsilon T}{3(3-2\varepsilon)} - \frac{\alpha}{3-2\varepsilon}, 2t, -\frac{2}{3}t + \frac{5\varepsilon T}{6(3-2\varepsilon)} + \frac{\alpha(1-\varepsilon)}{3-2\varepsilon}\right)$$

under these conditions:

$$\begin{aligned}
&\lambda_2(t) \leqslant \frac{5}{3}t + \frac{5\varepsilon T}{3(3-2\varepsilon)} - \frac{\alpha}{3-2\varepsilon} \leqslant \mu_2(t),\\
&\frac{2}{3}t - \frac{5\varepsilon T}{6(3-2\varepsilon)} - \frac{\alpha(1-\varepsilon)}{3-2\varepsilon} \leqslant 0 \quad \text{a.e. in } [0,T].
\end{aligned}$$

Now, let us increase the travel demand for the pair (P_3,P_2), adding the positive quantity d.

The equilibrium solution $H^*(t) = (H_1^*(t), H_2^*(t), H_3^*(t), H_4^*(t))$ is given by:

$$\begin{cases}
H_1^*(t) = H_1(t) + d\\
H_2^*(t) = H_2(t) + \frac{d}{3-2\varepsilon}\\
H_3^*(t) = H_3(t)\\
H_4^*(t) = H_4(t) - \frac{(1-\varepsilon)d}{3-2\varepsilon},
\end{cases}$$

if the following relationships hold:

$$\lambda_2(t) \leqslant \frac{5}{3}t + \frac{5\varepsilon T}{3(3-2\varepsilon)} - \frac{\alpha}{3-2\varepsilon} + \frac{d}{3-2\varepsilon} \leqslant \mu_2(t),$$
$$\frac{2}{3}t - \frac{5\varepsilon T}{6(3-2\varepsilon)} - \frac{\alpha(1-\varepsilon)}{3-2\varepsilon} + \frac{(1-\varepsilon)d}{3-2\varepsilon} \leqslant 0 \quad \text{a.e. in } [0,T].$$

Moreover the new cost functions are:

$$\begin{cases} C_1^*(H^*(t)) = C_1(H(t)) + \frac{(2-\varepsilon)d}{3-2\varepsilon} \\ C_2^*(H^*(t)) = C_2(H(t)) + \frac{d}{3-2\varepsilon} \\ C_3^*(H^*(t)) = C_3(H(t)) - \frac{(1-\varepsilon)d}{3-2\varepsilon} \\ C_4^*(H^*(t)) = C_4(H(t)) + \frac{d}{3-2\varepsilon}. \end{cases}$$

We have that the paradoxical decrease in the total cost occurs if:

$$\int_0^T C^*(H^*(t))H^*(t)dt - \int_0^T C(H(t))H(t)dt < 0,$$

namely if

$$(2\varepsilon^2 - 6\varepsilon - 3)T - d(2\varepsilon^2 - 6\varepsilon + 6) + \alpha(4\varepsilon - 3) > 0$$

A numerical example can be obtained by choosing: $T = 1$, $\varepsilon = 3$, $\lambda_2(t) = t + 12$, $\mu_2(t) = 2t + 18$, $\alpha = 50$, $d = 4$.

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Laura Scrimali
Department of Mathematics and Computer Science
University of Catania
Viale Andrea Doria 6 - 95125 Catania
Italy
e-mail: scrimali@dmi.unict.it