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A SIMPLE PROOF FOR SOME IMPORTANT PROPERTIES OF THE PROJECTION MAPPING

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Abstract. In this paper, we give a simple proof for an important property of projection mapping under general G-norm.

1. Introduction

Let Ω be a nonempty closed convex subset of R^n and F be a continuous monotone mapping from R^n into itself. A variational inequality problem, denoted by VI (Ω, F), is to determine a vector $u^* \in \Omega$ such that

$$(u-u^*)^T F(u^*) \geqslant 0, \qquad \forall u \in \Omega. \tag{1.1}$$

VI (Ω, F) problem includes nonlinear complementarity problems (when $\Omega = R^n_+$) and system of nonlinear equations (when $\Omega = R^n$), and thus it has many important applications [4]. A typical situation where problem (1.1) can be reformulated as an optimization problem is that $F(u)$ is the gradient of a differentiable function $f : R^n \rightarrow R^n$, in which case Problem (1.1) is equivalent to the problem

$$\min_x \{f(x) | x \in \Omega\}.$$

Among the existing methods for nonlinear variational inequality problems, the simplest one is the Goldstein–Levitin–Polyak projection method [5, 11], which starts with any $u^0 \in \Omega$, and iteratively updates u^{k+1} according to the formula

$$u^{k+1} = P_\Omega[u^k - \beta_k F(u^k)], \tag{1.2}$$

where β_k is a chosen positive step size and $P_\Omega(v)$ denotes the projection of v onto Ω. For any $\beta > 0$, denote

$$e(u, \beta) := u - P_\Omega[u - \beta F(u)].$$

Since solving VI (Ω, F) is equivalent to finding a zero point of $e(u, \beta)$, $\|e(u, 1)\|$ is usually viewed as an error bound which measures how much u fails to be a solution point. In projected gradient methods for constrained minimization problems [1, 3], merit function methods [12, 13] and projection-type methods for variational inequalities

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[6, 7, 8, 9, 10], $\|e(u,\beta)\|$ plays an important role in convergence analysis. Especially, the properties

$$\|e(u,\tilde{\beta})\| \geqslant \|e(u,\beta)\|, \ \forall\tilde{\beta} \geqslant \beta > 0 \tag{1.3}$$

and

$$\frac{\|e(u,\tilde{\beta})\|}{\tilde{\beta}} \leqslant \frac{\|e(u,\beta)\|}{\beta}, \ \forall\tilde{\beta} \geqslant \beta > 0 \tag{1.4}$$

are very important and have been studied by Gafni and Bertsekas (Lemma 1 in [3]), Calamai and Moré (Lemma 2.2 in [1]) and Peng and Fukushima (Lemma 3.3 in [13]). In this paper, we give a simple proof for such properties in the sense of projection under G-norm.

2. Preliminaries

Let G be a symmetric positive definite matrix. $P_{\Omega,G}(v)$ denotes the unique solution of the minimization problem

$$\min\{\|v-u\|_G \ \mid u \in \Omega\}.$$

In other words,

$$P_{\Omega,G}(v) = \operatorname{argmin}\{\|v-u\|_G \mid u \in \Omega\}.$$

LEMMA 1. *For all* $u \in \Omega$, *we have*

$$\{v - P_{\Omega,G}(v)\}^T G\{u - P_{\Omega,G}(v)\} \leqslant 0, \quad \forall u \in \Omega. \tag{2.1}$$

Proof. First, according to the definition of $P_{\Omega,G}(v)$, we have

$$\|v - P_{\Omega,G}(v)\|_G \leqslant \|v-w\|_G \quad \forall w \in \Omega.$$

For $u \in \Omega$ and $\theta \in (0,1)$, it holds

$$\theta u + (1-\theta)P_{\Omega,G}(v) = P_{\Omega,G}(v) + \theta(u - P_{\Omega,G}(v)) \in \Omega$$

and

$$\|v - P_{\Omega}(v)\|_G^2 \leqslant \|v - P_{\Omega,G}(v) - \theta(u - P_{\Omega,G}(v))\|_G^2.$$

Hence, we have

$$[v - P_{\Omega,G}(v)]^T G[u - P_{\Omega,G}(v)] \leqslant \frac{\theta}{2}\|u - P_{\Omega,G}(v))\|_G^2 \quad \forall u \in \Omega \text{ and } \theta \in (0,1).$$

Let $\theta \to 0_+$, the assertion is proved. □

We define

$$e_G(u,\beta) = u - P_{\Omega,G}[u - \beta G^{-1}F(u)]. \tag{2.2}$$

LEMMA 2. *Solving VI* (Ω,F) *is equivalent to finding a zero point of* $e_G(u,\beta)$.

Proof. i). Since u is a solution of VI (Ω,F), $u \in \Omega$. Setting $v := u - \beta G^{-1}F(u)$ in (2.1), we get

$$\left(e_G(u,\beta) - \beta G^{-1}F(u)\right)^T G e_G(u,\beta) \leqslant 0$$

and consequently

$$\|e_G(u,\beta)\|_G^2 \leqslant e_G(u,\beta)^T(\beta F(u)). \tag{2.3}$$

Because $P_\Omega[u - G^{-1}F(u)] \in \Omega$ and u is a solution of VI (Ω, F), it follows from (1.1) that

$$\{P_\Omega[u - G^{-1}F(u)] - u\}^T F(u) \geqslant 0,$$

and hence

$$e_G(u,\beta)^T F(u) \leqslant 0. \tag{2.4}$$

Combining (2.3) and (2.4) we get $e_G(u,\beta) = 0$.
ii). Conversely, setting $v = u - \beta G^{-1}F(u)$ in(2.1) we get

$$\{e_G(u,\beta) - \beta G^{-1}F(u)\}^T G\{u' - P_{\Omega,G}[u - \beta G^{-1}F(u)]\} \leqslant 0, \qquad \forall u' \in \Omega. \tag{2.5}$$

Because $e_G(u,\beta) = 0$, we have $u = P_{\Omega,G}(\cdot) \in \Omega$ and $P_{\Omega,G}[u - \beta G^{-1}F(u)] = u$. Substituting it in (2.5) we get

$$u \in \Omega, \qquad (u' - u)^T F(u) \geqslant 0, \qquad \forall u' \in \Omega$$

This means that u is a solution of VI (Ω, F).

3. The main Theorem

THEOREM 1. *For any $u \in R^n$ and $\tilde{\beta} \geqslant \beta > 0$, we have*

$$\|e_G(u,\tilde{\beta})\|_G \geqslant \|e_G(u,\beta)\|_G \tag{3.1}$$

and

$$\frac{\|e_G(u,\tilde{\beta})\|_G}{\tilde{\beta}} \leqslant \frac{\|e_G(u,\beta)\|_G}{\beta}. \tag{3.2}$$

Proof. Let $t = \|e_G(u,\tilde{\beta})\| / \|e_G(u,\beta)\|$, we need only to prove that $1 \leqslant t \leqslant \tilde{\beta}/\beta$. Notice that it's equivalent expression is

$$(t-1)(t - \tilde{\beta}/\beta) \leqslant 0. \tag{3.3}$$

According to the basic property of the projection mapping, we have

$$(v - P_{\Omega,G}(v))^T G(P_{\Omega,G}(v) - w) \geqslant 0 \quad \forall w \in \Omega. \tag{3.4}$$

Setting $w := P_{\Omega,G}[u - \tilde{\beta}G^{-1}F(u)]$ and $v := u - \beta F(u)$ in (3.4), and using

$$P_{\Omega,G}[u - \beta G^{-1}F(u)] - P_{\Omega,G}[u - \tilde{\beta}G^{-1}F(u)] = e_G(u,\tilde{\beta}) - e_G(u,\beta),$$

we obtain

$$\{e_G(u,\beta) - \beta G^{-1}F(u)\}^T G\{e_G(u,\tilde{\beta}) - e_G(u,\beta)\} \geqslant 0. \tag{3.5}$$

Similarly, we have

$$\{\tilde{\beta}G^{-1}F(u) - e_G(u,\tilde{\beta})\}^T G\{e_G(u,\tilde{\beta}) - e_G(u,\beta)\} \geqslant 0. \tag{3.6}$$

Multiplying (3.5) and (3.6) by $\tilde{\beta}$ and β, respectively, and then adding them, we get

$$\{\tilde{\beta} e_G(u,\beta) - \beta e_G(u,\tilde{\beta})\}^T G \{e_G(u,\tilde{\beta}) - e_G(u,\beta)\} \geqslant 0 \tag{3.7}$$

and consequently

$$\tilde{\beta}\|e_G(u,\beta)\|_G^2 + \beta\|e_G(u,\tilde{\beta})\|_G^2 \leqslant (\beta+\tilde{\beta}) e_G(u,\beta)^T G e_G(u,\tilde{\beta}). \tag{3.8}$$

From *Cauchy-Schwarz* inequality, we have

$$e_G(u,\beta)^T G e_G(u,\tilde{\beta}) \leqslant \|e_G(u,\beta)\|_G \|e_G(u,\tilde{\beta})\|_G$$

Then

$$\tilde{\beta}\|e_G(u,\beta)\|_G^2 + \beta\|e_G(u,\tilde{\beta})\|_G^2 \leqslant (\beta+\tilde{\beta}) \|e_G(u,\beta)\|_G \|e_G(u,\tilde{\beta})\|_G \tag{3.9}$$

Dividing (3.9) by $\|e_G(u,\beta)\|_G^2$ we obtain

$$\tilde{\beta} + \beta t^2 \leqslant (\beta+\tilde{\beta}) t$$

and thus (3.3) holds and the theorem is proved. □

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