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POSITIVE SOLUTIONS FOR CONTINUOUS AND DISCRETE BOUNDARY VALUE PROBLEMS TO THE ONE–DIMENSION p–LAPLACIAN

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Abstract. New existence results (for positive solutions) for continuous and discrete boundary value problems to the one-dimension p-Laplacian are presented in this paper. Here we use a well-known fixed point theorem in cones. Our results improve several recent results established in the literature.

1. Introduction

In this paper, we discuss the existence of positive solutions to the boundary value problem

$$\begin{cases} (\phi(u'))' + g(t)f(u) = 0, & a.e. \ t \in [0,1]; \\ u(0) = u(1) = 0 \end{cases} \tag{1.1}$$

and the discrete boundary value problem

$$\begin{cases} \Delta(\phi(\Delta u(i-1))) + q(i)f(u(i)) = 0, & i \in N; \\ u(0) = u(T+1) = 0, \end{cases} \tag{1.2}$$

where $\phi(s) = |s|^{p-2}s$, $p > 1$, $N = \{1, 2, \ldots, T\}$ and $T \geqslant 1$ is a fixed positive integer.

It is of interest to note here that the existence of positive solutions to problem (1.1) and (1.2) has been studied in great detail in the literature. For results the continuous case, we refer the reader to [3, 7, 8, 11, 18, 24] $(p = 2)$ and [5, 6, 9, 12, 16, 19, 20, 21]. For results in the discrete case, we refer the reader to [1, 2, 3, 4] $(p = 2)$ and [22, 23].

In this paper, we present a new existence theory for the continuous case in section 2 and the discrete case in section 3. In both sections we employ a well-known fixed point theorem in cones (see Theorem 1.1). One of the key steps is to find a function ψ such that the appropriate operator Φ satisfies the condition $u \neq \Phi u + \lambda\psi$ in the cited fixed point theorem. It seems to be difficult to utilize the norm-type expansion and compression theorem to prove our main results (see [24] for a discussion when $p = 2$).

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As an application of our new results, we respectively consider the existence of eigenvalues of the problem

$$\begin{cases} (\phi(u'))' + \lambda g(t)f(u) = 0, \quad a.e. \ t \in [0,1]; \\ u(0) = u(1) = 0 \end{cases} \tag{1.3}$$

in section 2 and the problem

$$\begin{cases} \Delta(\phi(\Delta u(i-1))) + \lambda q(i)f(u(i)) = 0, \quad i \in N; \\ u(0) = u(T+1) = 0 \end{cases} \tag{1.4}$$

in section 3.

To conclude this section, we state a fixed point theorem in cones which will be needed in this paper.

THEOREM 1.1. ([10], [24]) *Let X be a Banach space and K is a cone in X. Assume Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1, \bar{\Omega}_1 \subset \Omega_2$. Let*

$$\Phi : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K$$

be a continuous and completely continuous operator such that

(i) $\|\Phi x\| \leqslant \|x\|$ *for* $x \in K \cap \partial\Omega_1$,

(ii) there exists $\psi \in K\backslash\{0\}$ *such that* $x \neq \Phi x + \lambda\psi$ *for* $x \in K \cap \partial\Omega_2$ *and* $\lambda > 0$.

Then Φ *has a fixed point in* $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

REMARK 1.1. In Theorem 1.1, if (i) and (ii) are replaced by

(i)* $\|\Phi x\| \leqslant \|x\|$ for $x \in K \cap \partial\Omega_2$,

and

(ii)* there exists $\psi \in K\backslash\{0\}$ such that $x \neq \Phi x + \lambda\psi$ for $x \in K \cap \partial\Omega_1$ and $\lambda > 0$,

then Φ has a fixed point in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

2. Continuous Case

In this section we establish the existence of positive solutions to problem (1.1). Throughout this section we will assume the following two conditions hold:

$(H1)$ $f : [0,\infty) \to [0,\infty)$ is continuous.

$(H2)$ $g \in L^1[0,1]$ and $g(t) \geqslant 0$ a.e. for $t \in [0,1]$; in addition, it is assumed that there exists $a \in (0, \frac{1}{4}]$ such that $g(t) > 0$ a.e. for $t \in [a, 1-a]$.

Let $X = C[0,1]$, with the norm $\|u\| = \max_{t\in[0,1]} |u(t)|$, so X is a Banach space. By a solution u to (1.1) we mean a function $u \in C^1[0,1]$, $\phi(u') \in AC[0,1]$ such that u satisfies (1.1) and the boundary condition; here $AC[0,1]$ denotes the space of absolutely continuous functions defined on [0,1]. Also, we define

$$K = \{u \in X : \ u \text{ is concave on } [0,1] \text{ and } u(0) = u(1) = 0\}. \tag{2.1}$$

One may readily verify that K is a cone in X.

First we present some useful results.

LEMMA 2.1. ([16]) *Let* $u \in K$, *then*

$$u(t) \geqslant \min\{t, 1-t\}\|u\| \text{ for all } t \in [0,1].$$

LEMMA 2.2. *If* $u \in C^1[0,1]$ *satisfies*

$$\begin{cases} (\phi(u'))' \leqslant 0, & a.e.\ t \in [0,\ 1], \\ u(0) = u(1) = 0, \end{cases}$$

then $u(t) \geqslant \min\{t, 1-t\}\|u\|$ *for all* $t \in [0,1]$.

Proof. Notice that $(\phi(u'))' \leqslant 0$ for a.e. $t \in [0,1]$ implies u is concave on $[0,1]$, so the result follows from Lemma 2.1. □

LEMMA 2.3. ([17]) *If* $u,\ v \in C^1[0,1]$ *satisfies*

$$\begin{cases} (\phi(u'))' \leqslant (\phi(v'))', & a.e.\ t \in [0,1] \\ u(0) \geqslant v(0), & u(1) \geqslant v(1), \end{cases}$$

then $u(t) \geqslant v(t)$ *for all* $t \in [0,1]$.

In order to prove the existence of positive solutions to problem (1.1), we consider the following boundary value problem

$$\begin{cases} (\phi(w'))' + g(t)f(u) = 0, & a.e.\ t \in [0,1]; \\ w(0) = w(1) = 0, \end{cases} \tag{2.2}$$

for any $u \in K$, where K is a cone in X given in (2.1).

It follows from [17] that, for each fixed $u \in K$, problem (2.2) has a solution w and (2.2) is equivalent to

$$w(t) = \int_0^t \phi^{-1}(\tau + \int_s^1 g(r)f(u(r))dr)ds =: (\Phi u)(t),\ 0 \leqslant t \leqslant 1 \tag{2.3}$$

where $\tau = \phi(w'(1))$ is a solution of the equation

$$\int_0^1 \phi^{-1}(\tau + \int_s^1 g(r)f(u(r))dr)ds = 0. \tag{2.4}$$

Moreover, the operator $\Phi : K \to X$ is continuous and completely continuous. Now since

$$\begin{cases} (\phi((\Phi u)'))' + g(t)f(u) = 0,\ t \in (0,1); \\ (\Phi u)(0) = (\Phi u)(1) = 0, \end{cases}$$

and $g(t)f(u) \geqslant 0$, it follows that $w = \Phi u \in K,$ so $\Phi : K \to K.$

From Lemmas 2.1–2.3 we have the following results:

LEMMA 2.4. *Let* $u \in K$, *then* $u(t) \geqslant a\|u\|$ *for* $\forall\, t \in [a, 1-a]$.

LEMMA 2.5. *Let $P(t)$ be a solution to problem (2.2) with $f(u) \equiv 1$, then*

(I) if $w(t)$ is a solution to problem (2.2) with $f(u) \leqslant \phi(M)$, then $w(t) \leqslant MP(t)$, i.e., $(\Phi u)(t) \leqslant MP(t)$ for $t \in [0,1]$;

(II) if $w(t)$ is a solution to problem (2.2) with $f(u) \geqslant \phi(M)$, then $w(t) \geqslant MP(t)$, i.e., $(\Phi u)(t) \geqslant MP(t)$ for $t \in [0,1]$.

Proof. Notice that for a.e. $t \in [0,1]$ we have

$$(\phi(w'(t)))' - g(t)f(u(t)) \geqslant -M^{p-1}g(t) = (\phi(MP'(t)))',$$

so the result in (I) follows from Lemma 2.3. Similarly we can prove (II). □

Let $V \in C^1[0,1]$ be a solution to the following problem

$$\begin{cases} (\phi(V'))' + g(t)\chi_{[a,1-a]}(t) = 0, \quad a.e. \ t \in [0,1]; \\ V(0) = V(1) = 0, \end{cases} \tag{2.5}$$

where

$$\chi_{[a,1-a]}(t) = \begin{cases} 1, & t \in [a, 1-a]; \\ 0, & t \in [0,a) \cup (1-a, 1] \end{cases} \tag{2.6}$$

is the characteristic function on $[a, 1-a]$.

LEMMA 2.6. *Let $V(t)$ be a solution to problem (2.5), then*

(I) if $w(t)$ is a solution to problem (2.2) with $f(u(t)) \leqslant \phi(M)\chi_{[a,1-a]}(t)$, then $w(t) \leqslant MV(t)$, i.e., $(\Phi u)(t) \leqslant MV(t)$ for $t \in [0,1]$;

(II) if $w(t)$ is a solution to problem (2.2) with $f(u(t)) \geqslant \phi(M)\chi_{[a,1-a]}(t)$, then $w(t) \geqslant MV(t)$, i.e., $(\Phi u)(t) \geqslant MV(t)$ for $t \in [0,1]$.

The proof will be omitted only because it is similar to that of Lemma 2.5.

In order to state our main results, we let

$$f_0 = \lim_{u \to 0^+} \frac{f(u)}{u^{p-1}} \quad \text{and} \quad f_\infty = \lim_{u \to \infty} \frac{f(u)}{u^{p-1}}.$$

THEOREM 2.1. *Suppose that $(H1)$ and $(H2)$ hold. In addition, we assume that one of the following two conditions holds:*

(h_1) $0 \leqslant f_0 < A^{p-1}$ *and* $B^{p-1} < f_\infty \leqslant \infty$;

(h_2) $0 \leqslant f_\infty < A^{p-1}$ *and* $B^{p-1} < f_0 \leqslant \infty$;

where $A = (\max_{0 \leqslant t \leqslant 1} P(t))^{-1}$ and $B = (\min_{a \leqslant t \leqslant 1-a} V(t))^{-1}$.

Then problem (1.1) has at least one solution $u \in K$ with $u(t) \not\equiv 0$ for $t \in (0,1)$.

Proof. (I) Assume that (h_1) holds.

By the first part of (h_1), there exists $r > 0$ such that

$$f(u) \leqslant (Au)^{p-1} \quad \text{for} \quad 0 \leqslant u \leqslant r.$$

For any $u \in K$ with $\|u\| = r$, we have $f(u(t)) \leqslant (Ar)^{p-1}$ for $t \in [0,1]$.

It follows from Lemma 2.5 that

$$w(t) = (\Phi u)(t) \leqslant ArP(t) \leqslant Ar \max_{0 \leqslant t \leqslant 1} P(t) = r,$$

so,

$$\|\Phi u\| \leqslant \|u\|, \quad \forall\, u \in K \cap \partial\Omega_1, \tag{2.7}$$

where $\Omega_1 = \{u \in X : \|u\| < r\}$.

By the second part of (h_1), there exists $\eta > 0$ such that

$$f(u) \geqslant (Bu)^{p-1} \quad \text{for} \quad u \geqslant \eta.$$

Let $R = \max\{2r,\ a^{-1}\eta\}$. For any $u \in K$ with $\|u\| = R$, then it follows from Lemma 2.4 that

$$u(t) \geqslant aR \geqslant \eta \quad \text{for} \quad t \in [a, 1-a].$$

As a result

$$\begin{aligned} f(u(t)) &\geqslant \begin{cases} 0, & t \in [0,a) \cup (1-a, 1] \\ (B \min\limits_{a \leqslant t \leqslant 1-a} u(t))^{p-1}, & t \in [a, 1-a] \end{cases} \\ &= (B \min_{a \leqslant t \leqslant 1-a} u(t))^{p-1} \chi_{[a,1-a]}(t). \end{aligned}$$

Then it follows from Lemma 2.6 that we have

$$w(t) = (\Phi u)(t) \geqslant BV(t) \min_{a \leqslant t \leqslant 1-a} u(t),\ t \in [0,1].$$

Let $\psi \equiv 1$ for $t \in [0,1]$, so $\psi \in K \backslash \{0\}$. We shall prove that

$$u \neq \Phi u + \lambda \psi \ \text{ for } \ u \in K \cap \partial\Omega_2 \text{ and } \lambda > 0, \tag{2.8}$$

where $\Omega_2 = \{u \in X : \|u\| < R\}$.

If not, there exists $u_0 \in K \cap \partial\Omega_2$ and $\lambda_0 > 0$ such that

$$u_0 = \Phi u_0 + \lambda_0 \psi.$$

Let $\mu = \min_{t \in [a,1-a]} u_0(t)$. Then for $t \in [a, 1-a]$ we have

$$u_0(t) = (\Phi u_0)(t) + \lambda_0 \geqslant B\mu V(t) + \lambda_0 \geqslant \mu + \lambda_0,$$

and this implies $\mu \geqslant \mu + \lambda_0$, a contradiction.

It follows from Theorem 1.1, (2.7) and (2.8) that Φ has a fixed point $u \in K \cap (\bar{\Omega}_2 \setminus \Omega_1)$; notice $u(t) \geqslant \min\{t, 1-t\}\, r$. Clearly, this fixed point is a positive solution of (1.1).

(II) Assume that (h_2) holds.

By the first part of (h_2), there exists $\beta > 0$ such that

$$f(u) \leqslant (Au)^{p-1} \quad \text{for} \quad u \geqslant \beta.$$

Let $b = \max_{0 \leqslant u \leqslant \beta} f(u)$, $R \geqslant \phi^{-1}(b)/A$, and it is easy to see that

$$\max_{0 \leqslant u \leqslant R} f(u) \leqslant \phi(AR).$$

For any $u \in K$ with $\|u\| = R$, we have $f(u(t)) \leqslant (AR)^{p-1}$.

It follows from Lemma 2.5 that

$$w(t) = (\Phi u)(t) \leqslant ARP(t) \leqslant AR \max_{0\leqslant t\leqslant 1} P(t) = R, \ t \in [0,1],$$

so,

$$\|\Phi u\| \leqslant \|u\|, \quad \forall\, u \in K \cap \partial\Omega_2, \tag{2.9}$$

where $\Omega_2 = \{u \in X : \ \|u\| < R\}$.

By the second part of (h_2), there exists $r > 0 (r < R)$ such that

$$f(u) \geqslant (Bu)^{p-1} \quad \text{for} \quad 0 \leqslant u \leqslant r.$$

For any $u \in K$ with $\|u\| = r$, then

$$\begin{aligned} f(u(t)) &\geqslant \begin{cases} 0, & t \in [0,a) \cup (1-a,1] \\ (B \min\limits_{a\leqslant t\leqslant 1-a} u(t))^{p-1}, & t \in [a, 1-a] \end{cases} \\ &= (B \min_{a\leqslant t\leqslant 1-a} u(t))^{p-1} \chi_{[a,1-a]}(t). \end{aligned}$$

It follows from Lemma 2.6 that $(\Phi u)(t) \geqslant BV(t) \min_{a\leqslant t\leqslant 1-a} u(t)$. Let $\psi \equiv 1$ for $t \in [0,1]$, and we prove that

$$u \neq \Phi u + \lambda \psi \ \text{ for } \ u \in K \cap \partial\Omega_1 \text{ and } \lambda > 0, \tag{2.10}$$

where $\Omega_1 = \{u \in X : \|u\| < r\}$.

If not, there exists $u_0 \in K \cap \partial\Omega_2$ and $\lambda_0 > 0$ such that

$$u_0 = \Phi u_0 + \lambda_0 \psi.$$

Let $\mu = \min_{t\in[a,1-a]} u_0(t)$. Then for $t \in [a, 1-a]$ we have

$$u_0(t) = (\Phi u_0)(t) + \lambda_0 \geqslant B\mu V(t) + \lambda_0 \geqslant \mu + \lambda_0,$$

and this implies $\mu \geqslant \mu + \lambda_0$, a contradiction.

It follows from Theorem 1.1, (2.9) and (2.10) that Φ has a fixed point $u \in K \cap (\bar{\Omega}_2 \setminus \Omega_1)$. Clearly, this fixed point is a positive solution of (1.1). □

COROLLARY 2.1. *Assume $(H1)$ and $(H2)$ hold. Also suppose $f_0 = 0$ and $f_\infty = \infty$; or $f_0 = \infty$ and $f_\infty = 0$. Then problem (1.1) has at least one solution $u \in K$ with $u(t) \not\equiv 0$ for $t \in (0,1)$.*

As an application of Theorem 2.1 we consider the following eigenvalue problem

$$\begin{cases} (\phi(u'))' + \lambda g(t) f(u) = 0, \ \ a.e. \ t \in [0,1]; \\ u(0) = u(1) = 0. \end{cases} \tag{1.3}$$

Consider the following conditions:

$(L1)$ $\quad f_\infty > 0, f_0 \neq \infty$ and $\frac{B^{p-1}}{f_\infty} < \frac{A^{p-1}}{f_0}$;

$(L2)$ $\quad f_0 > 0, f_\infty \neq \infty$ and $\frac{B^{p-1}}{f_0} < \frac{A^{p-1}}{f_\infty}$;

where A and B are given in Theorem 2.1.

THEOREM 2.2. *Assume that* $(H1)$ *and* $(H2)$ *hold. If* $(L1)$ *holds, then for* $\forall\,\lambda \in (\frac{B^{p-1}}{f_\infty}, \frac{A^{p-1}}{f_0})$, *problem (1.3) has at least one solution* u *with* $u(t) \geqslant 0$ *for* $t \in [0,1]$ *and* $u(t) \not\equiv 0$ *on [0,1]. The same result remains valid for* $\forall\,\lambda \in (\frac{B^{p-1}}{f_0}, \frac{A^{p-1}}{f_\infty})$, *if* $(L2)$ *holds.*

Proof. If $(L1)$ holds, then $\lambda f_\infty > B^{p-1}$ and $\lambda f_0 < A^{p-1}$; if $(L2)$ holds, then $\lambda f_0 > B^{p-1}$ and $\lambda f_\infty < A^{p-1}$. The result follows from Theorem 2.1. □

COROLLARY 2.2. *Assume that* $(H1)$ *and* $(H2)$ *hold. Also suppose* $f_0 = 0$ *and* $f_\infty = \infty$; *or* $f_0 = \infty$ *and* $f_\infty = 0$. *Then problem (1.3) has at least one solution* $u \in K$ *with* $u(t) \not\equiv 0$ *for* $t \in (0,1)$ *for all* $\lambda \in (0,\infty)$.

3. Discrete Case

In this section, we establish the existence of positive solutions to the discrete boundary value problem (1.2). Throughout this section it is assumed that:

(A_1) $q : N \to (0,\infty)$ is continuous and $f : [0,\infty) \to [0,\infty)$ is continuous.

REMARK 3.1. We call a map $q : N \to (0,\infty)$ continuous if it is continuous as a map of the topological space N into the topological space $(0,\infty)$. Throughout this paper the topology on N will be the discrete topology.

Let $C(N^+,\mathbf{R})$ denote the class of map u continuous on N^+, with norm $\|u\| = \max_{i\in N^+}|u(i)|$, here $N^+ = \{0,1,\ldots,T+1\}$, so $C(N^+,\mathbf{R})$ is a Banach space. By a solution u to (1.2) we mean a function $u \in C(N^+,\mathbf{R})$ such that u satisfies (1.2) and the boundary condition. A solution $u(i)$ of (1.2) is called a positive solution if $u(i) > 0$ for $i \in N$.

LEMMA 3.1. ([1]) *Let* $y \in C(N^+,\mathbf{R})$ *satisfy* $y(i) \geqslant 0$ *for* $i \in N^+$. *If* $u \in C(N^+,\mathbf{R})$ *satisfies*

$$\begin{cases} \Delta^2 u(i-1) + y(i) = 0, & i \in N \\ u(0) = u(T+1) = 0, \end{cases}$$

then

$$u(i) \geqslant \mu(i)\|u\| \ \ \textit{for} \ \ i \in N^+;$$

here

$$\mu(i) = \min\left\{\frac{T+1-i}{T+1},\ \frac{i}{T}\right\}. \tag{3.1}$$

LEMMA 3.2. *If* $u \in C(N^+,\mathbf{R})$ *satisfies*

$$\begin{cases} \Delta(\phi(\Delta u(i-1))) \leqslant 0, & i \in N \\ u(0) = u(T+1) = 0, \end{cases}$$

then $u(i) \geqslant \mu(i)\|u\|$ *for* $i \in N^+$*; here* $\mu(i)$ *is defined in Lemma 3.1.*

Proof. Notice that $\Delta(\phi(\Delta u(i-1))) \leqslant 0$ implies $\Delta^2 u(i-1) \leqslant 0$ for $i \in N$, so the result follows from Lemma 3.1. □

LEMMA 3.3. *If* $u,\ v \in C(N^+, \mathbf{R})$ *satisfies*

$$\begin{cases} \Delta(\phi(\Delta u(i-1))) \leqslant \Delta(\phi(\Delta v(i-1))), & i \in N \\ u(0) \geqslant v(0), \quad u(T+1) \geqslant v(T+1), \end{cases}$$

then

$$u(i) \geqslant v(i) \ \ \textit{for } i \in N^+.$$

Proof. Let $z(i) = u(i) - v(i)$. If the lemma were not true, there would exist $i_0 \in N$ such that $z(i_0) = \min_{i \in N} z(i) < 0,\ \Delta z(i_0 - 1) \leqslant 0$. Notice that

$$\Delta(\phi(\Delta u(i-1))) \leqslant \Delta(\phi(\Delta v(i-1))), \quad i \in N.$$

Sum both sides of the above inequality from i_0 to $i\,(i_0 \leqslant i \leqslant T)$ to obtain

$$\phi(\Delta u(i)) - \phi(\Delta u(i_0 - 1)) \leqslant \phi(\Delta v(i)) - \phi(\Delta v(i_0 - 1))$$

i.e.,

$$\phi(\Delta u(i)) - \phi(\Delta v(i)) \leqslant \phi(\Delta u(i_0 - 1)) - \phi(\Delta v(i_0 - 1)) \leqslant 0,$$

so,

$$\Delta z(i) = \Delta u(i) - \Delta v(i) \leqslant 0, \qquad i_0 \leqslant i \leqslant T.$$

This implies $z(i_0) \geqslant z(T+1) = 0$, a contradiction. □

In order to prove the existence of positive solutions to problem (1.2), we consider the following boundary value problem

$$\begin{cases} \Delta(\phi(\Delta w(i-1))) + q(i)f(u(i)) = 0, & i \in N \\ w(0) = w(T+1) = 0, \end{cases} \tag{3.2}$$

for any $u \in K$, where K is a cone in $X = C(N^+, \mathbf{R})$ defined by

$$K = \{u \in X : u(i) \geqslant \mu(i)\|u\| \text{ for } i \in N^+\}, \tag{3.3}$$

where $\mu(i)$ is given in (3.1).

It follows from [23] that, for each fixed $u \in K$, problem (3.2) has a solution w and (3.2) is equivalent to

$$\omega(i) = (\Phi u)(i) = \begin{cases} 0, & i = 0 \text{ or } i = T+1 \\ \sum_{s=i}^{T} \Phi^{-1}(\tau + \sum_{r=1}^{s} q(r)f(u(r))), & i \in N, \end{cases} \tag{3.4}$$

where τ is a solution of the equation

$$Z(\tau) = \phi^{-1}(\tau) + \sum_{s=1}^{T} \phi^{-1}(\tau + \sum_{r=1}^{s} q(r)f(u(r))) = 0. \tag{3.5}$$

Moreover, the operator $\Phi : K \to X$ is continuous and completely continuous. Now since

$$\begin{cases} \Delta(\phi(\Delta(\Phi u)(i-1))) + q(i)f(u(i)) = 0, \ i \in N; \\ (\Phi u)(0) = (\Phi u)(T+1) = 0, \end{cases}$$

and $q(i)f(u(i)) \geqslant 0$, then it follows from Lemma 3.2 that $w = \Phi u \in K$, so $\Phi : K \to K$.

From Lemma 3.3, we have the following result.

LEMMA 3.4. *Let $P(i)$ be a solution to problem (3.2) with $f(u) \equiv 1$,*

(I) if $w(i)$ is a solution to problem (3.2) with $f(u) \leqslant \phi(M)$, then $w(i) \leqslant MP(i)$, i.e., $(\Phi u)(i) \leqslant MP(i)$ for $i \in N^+$.

(II) if $w(i)$ is a solution to problem (3.2) with $f(u) \geqslant \phi(M)$, then $w(i) \geqslant MP(i)$, i.e., $(\Phi u)(i) \geqslant MP(i)$ for $i \in N^+$.

Proof. Notice that

$$\Delta(\phi(\Delta w(i-1))) = -q(i)f(u(i)) \geqslant -M^{p-1}q(i) = \Delta(\phi(\Delta MP(i-1)))$$

for $i \in N$, so the result in (I) follows from Lemma 3.3. Similarly we can prove (II). □

THEOREM 3.1. *Suppose that $(A1)$ holds. In addition, we assume that one of the following conditions holds:*

$(A2)$ $\quad 0 \leqslant f_0 < A^{p-1}$ *and* $B^{p-1} < f_\infty \leqslant \infty$;

$(A3)$ $\quad 0 \leqslant f_\infty < A^{p-1}$ *and* $B^{p-1} < f_0 \leqslant \infty$;

where $A = (\max_{i\in N} P(i))^{-1}$ and $B = (\min_{i\in N} P(i))^{-1}$.

Then problem (1.2) has a solution $u \in K$ with $u(i) \not\equiv 0$ for $i \in N^+$.

Proof. (I) Assume that $(A1)$ and $(A2)$ hold.

Since $0 \leqslant f_0 < A^{p-1}$, we can choose $r > 0$ such that

$$f(u) \leqslant (Au)^{p-1} \quad \text{whenever} \quad 0 \leqslant u \leqslant r.$$

Thus, if $u \in K$ with $\|u\| = r$, then

$$f(u(i)) \leqslant (Au(i))^{p-1} \leqslant (Ar)^{p-1}, \ i \in N.$$

It follows from Lemma 3.4 that

$$\omega(i) \leqslant ArP(i) \leqslant Ar \max_{i\in N} P(i) \leqslant r, \ i \in N^+,$$

i.e.,

$$\|\Phi u\| \leqslant \|u\|, \quad \forall\, u \in K \cap \partial\Omega_1, \tag{3.6}$$

where $\Omega_1 = \{u \in X : \|u\| < r\}$.

Also, since $B^{p-1} < f_\infty \leqslant \infty$, there exists $\eta > \frac{r}{T+1}$ such that

$$f(u) \geqslant (Bu)^{p-1} \ \text{whenever} \ u \geqslant \eta.$$

Let $R = (T+1)\eta > r$. If $u \in K$ with $\|u\| = R$, we have

$$u(i) \geqslant \frac{R}{T+1} = \eta \ \text{for} \ i \in N,$$

so $f(u(i)) \geqslant (Bu(i))^{p-1}$, for $i \in N$.

Let $\psi(i) \equiv 1$ for $i \in N^+$, so $\psi \in \partial K \backslash \{0\}$. We shall prove that

$$u \neq \Phi u + \lambda \psi \ \text{for} \ u \in K \cap \partial\Omega_2 \text{ and } \lambda > 0, \tag{3.7}$$

where $\Omega_2 = \{u \in X : \|u\| < R\}$.

If not, there exists $u_0 \in K \cap \partial\Omega_2$ and $\lambda_0 > 0$ such that

$$u_0 = \Phi u_0 + \lambda_0 \psi.$$

Let $\alpha = \min_{i \in N} u_0(i)$. Then $\alpha \geqslant \eta$, so we have

$$f(u(i)) \geqslant (B\alpha)^{p-1}, \quad \text{for } \forall\, i \in N.$$

Then, for $i \in N$, it follows from Lemma 3.4 that we have

$$u_0(i) = (\Phi u_0)(i) + \lambda_0 \geqslant B\alpha P(i) + \lambda_0 \geqslant \alpha + \lambda_0.$$

This implies $\alpha \geqslant \alpha + \lambda_0$, a contradiction.

Now (3.6), (3.7) and Theorem 1.1 guarantee that Φ has a fixed point $u \in K \cap (\bar{\Omega}_2 \setminus \Omega_1)$ with $r \leqslant \|u\| \leqslant R$.

(II) Assume that $(A1)$ and $(A3)$ hold.

By the first part of $(A3)$, there exists $r_1 > 0$ such that

$$f(u) \leqslant (Au)^{p-1} \quad \text{for } u \geqslant r_1.$$

Let $R = (T+1)r_1$, so we have, for $\forall\, i \in N$,

$$u(i) \geqslant \frac{1}{T+1}\|u\| = \frac{R}{T+1} = r_1 \quad \text{for} \quad u \in K \cap \partial\Omega_2, \tag{3.8}$$

where $\Omega_2 = \{u \in X : \|u\| < R\}$.

Thus

$$f(u(i)) \leqslant (Au(i))^{p-1} \leqslant (AR)^{p-1},\ i \in N.$$

It follows from Lemma 3.4 that

$$\omega(i) \leqslant ARP(i) \leqslant R \quad \text{for all } i \in N,$$

i.e.,

$$\|\Phi u\| \leqslant \|u\| \quad \forall\, u \in K \cap \partial\Omega_2.$$

On the other hand, since $B^{p-1} < f_0 \leqslant \infty$, there exists $r \in (0, r_1)$ such that

$$f(u) \geqslant (Bu)^{p-1} \quad \text{for } 0 \leqslant u \leqslant r.$$

Then for any $u \in K$ with $\|u\| = r$, we have $u(i) \geqslant \frac{r}{T+1}$ for all $i \in N$.

Let $\psi(i) \equiv 1$ for $i \in N^+$. Essentially the same reasoning used in (I), establishes

$$u \neq \Phi u + \lambda \psi \quad \text{for } u \in K \cap \partial\Omega_1 \text{ and } \lambda > 0, \tag{3.9}$$

where $\Omega_1 = \{u \in X : \|u\| < r\}$.

Now from (3.8), (3.9) and Theorem 1.1 we have that Φ has a fixed point $u \in K \cap (\bar{\Omega}_2 \setminus \Omega_1)$ with $r \leqslant \|u\| \leqslant R$.

This completes the proof of the theorem. □

COROLLARY 3.1. *Assume that* $(A1)$ *holds. Also suppose* $f_0 = 0$ *and* $f_\infty = \infty$; *or* $f_0 = \infty$ *and* $f_\infty = 0$. *Then problem (1.2) has at least one solution* $u \in K$ *with* $u(i) \not\equiv 0$ *for* $i \in N^+$.

As an application of Theorem 3.1 we consider the following eigenvalue problem

$$\begin{cases} \Delta(\phi(\Delta u(i-1))) + \lambda q(i) f(u(i)) = 0, & i \in N, \\ u(0) = u(T+1) = 0. \end{cases} \tag{1.4}$$

Consider the following conditions:

$(S1)$ $f_\infty > 0, f_0 \neq \infty$ and $\frac{B^{p-1}}{f_\infty} < \frac{A^{p-1}}{f_0}$,

$(S2)$ $f_0 > 0, f_\infty \neq \infty$ and $\frac{B^{p-1}}{f_0} < \frac{A^{p-1}}{f_\infty}$;

where A and B are given in Theorem 3.1.

THEOREM 3.2. *Assume that* $(A1)$ *holds. If* $(S1)$ *holds, then for all* $\lambda \in (\frac{B^{p-1}}{f_\infty}, \frac{A^{p-1}}{f_0})$, *problem (1.4) has at least one solution* u *with* $u(i) \geqslant 0$ *for* $i \in N^+$ *and* $u(i) \not\equiv 0$ *on* N^+. *The same result remains valid for all* $\lambda \in (\frac{B^{p-1}}{f_0}, \frac{A^{p-1}}{f_\infty})$, *if* $(S2)$ *holds.*

Proof. If $(S1)$ holds, then $\lambda f_\infty > B^{p-1}$ and $\lambda f_0 < A^{p-1}$. If $(S2)$ holds, then $\lambda f_0 > B^{p-1}$ and $\lambda f_\infty < A^{p-1}$. The result follows from Theorem 3.1. □

COROLLARY 3.2. *Assume* $(A1)$ *holds. Also suppose* $f_0 = 0$ *and* $f_\infty = \infty$; *or* $f_0 = \infty$ *and* $f_\infty = 0$. *Then problem (1.4) has at least one solution* $u \in K$ *with* $u(i) \not\equiv 0$ *for* $i \in N$ *for all* $\lambda \in (0, \infty)$.

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