Mathematical
Inequalities
& Applications
Volume 7, Number 4 (2004), 543–549

DECAY BOUNDS FOR SOLUTIONS OF SECOND ORDER PARABOLIC PROBLEMS AND THEIR DERIVATIVES II

L. E. PAYNE AND G. A. PHILIPPIN

(*communicated by V. Lakshmikantham*)

Abstract. Extending the investigations initiated in an earlier paper, the authors deal in this paper with the solution to an initial-boundary value problem for a more general quasilinear heat equation in which the nonlinearity is such that the solution, without appropriate data restrictions, may blow up at some finite time. For such an equation they determine conditions on the data and geometry sufficient to insure that the solution remains bounded and then derive exponential decay bounds for the solution and its spatial gradient.

1. Introduction

In [3] the authors studied a quasilinear heat equation, deriving criteria which guaranteed that solutions to a class of initial-boundary value problems remained bounded for all time. They then obtained exponential decay (in time) bounds for these solutions and their spatial derivatives. In subsequent papers [4,5] they examined the question of spatial decay in a long cylindrical region, again deriving exponential decay (in space) bounds for the solution and its cross sectional derivatives. In the present paper the authors derive results similar to those obtained in [3] but for a more general equation. In [3] it was shown that for convex regions, restrictions of the initial data alone were sufficient to guarantee boundedness of solution, but for the more general equation treated in this paper a further restriction on the curvature of the boundary of the region is required. The specific equation to be considered is

$$\Delta u - u_{,t} + f(u) + g(|\nabla u|^2) = 0, \tag{1.1}$$

valid in some region $\Omega \times \mathbb{R}^+$, where Δ is the Laplace operator and the functions f and g are such that without suitable restrictions on the data and geometry the solution may blow up at some point in space-time. As in [3] the goal is to determine specific criteria which will guarantee global boundedness and to demonstrate that with these restrictions the solution and its spatial derivatives decay exponentially.

Mathematics subject classification (2000): 35K55, 35K60, 35K20.
Key words and phrases: quasilinear parabolic problems, maximum principle.

Paper MIA-07-55

2. Main result

In this section we consider the following initial-boundary value problem

$$\Delta u - u_{,t} + f(u) + g(|\nabla u|^2) = 0, \qquad \mathbf{x} \in \Omega, \qquad t > 0, \tag{2.1}$$

$$u(\mathbf{x},t) = 0, \qquad \mathbf{x} \in \partial\Omega, \qquad t > 0, \tag{2.2}$$

$$u(\mathbf{x},0) = h(\mathbf{x}) \geqslant 0, \qquad \mathbf{x} \in \Omega, \tag{2.3}$$

where Ω is a bounded convex domain in $\mathbb{R}^N$ with $C^{2+\varepsilon}$ boundary $\partial\Omega$ and where the given functions f, g, h are assumed to be differentiable and nonnegative. We want to establish the following result:

THEOREM 1. *Let $u(\mathbf{x},t)$ be the classical solution of (2.1), (2.2), (2.3). Assume that the function f and g satisfy the following conditions*

$$f(0) = 0, \qquad \frac{f(s)}{s} \quad \textit{non decreasing} \quad \textit{w.r.t.} \quad s > 0, \tag{2.4}$$

$$g(0) = 0, \qquad \frac{g(s^2)}{s} \quad \textit{non decreasing} \quad \textit{w.r.t.} \quad s > 0. \tag{2.5}$$

Moreover we assume that the initial data $h(\mathbf{x})$ $(\geqslant 0)$ is small enough in the following sense

$$\frac{f(\Gamma_1)}{\Gamma_1} < \frac{\pi^2}{4d^2} - \alpha, \tag{2.6}$$

and

$$\frac{g(\Gamma_2^2)}{\Gamma_2} \leqslant (N-1)K_{ave} - \varepsilon. \tag{2.7}$$

In (2.6), d is the inradius of Ω, α is some positive constant, and Γ_1 is defined as

$$\Gamma_1 := \max_{\mathbf{x}\in\Omega} \sqrt{h^2(\mathbf{x}) + \frac{4d^2}{\pi^2}|\nabla h|^2}. \tag{2.8}$$

In (2.7), K_{ave} (> 0) is the smallest value of the average curvature of $\partial\Omega$, ε is an arbitrary positive constant and Γ_2 is defined as

$$\Gamma_2 := \max_{\mathbf{x}\in\Omega} \sqrt{|\nabla h|^2 + \alpha h^2 + 2F(h)}, \tag{2.9}$$

with

$$F(s) := \int_0^s f(\sigma)d\sigma. \tag{2.10}$$

We then conclude that the auxiliary function ϕ defined as

$$\phi(\mathbf{x},t) := \{|\nabla u|^2 + \alpha u^2 + 2F(u)\}e^{2\alpha t} \tag{2.11}$$

takes its maximum value at $t = 0$, i.e.

$$|\nabla u|^2 + \alpha u^2 + 2F(u) \leqslant \Gamma_2^2 e^{-2\alpha t}, \qquad \mathbf{x} \in \Omega, \qquad t > 0. \tag{2.12}$$

The proof of Theorem 1 will be established in several steps. We first derive the following maximum principle.

LEMMA 1. *Let $w(\mathbf{x},t)$ be the solution of the initial-boundary value problem*

$$\Delta w - w_{,t} + \mu|\nabla w| = 0, \qquad \mathbf{x} \in \Omega, \qquad t > 0, \tag{2.13}$$

$$w(\mathbf{x},t) = 0, \qquad \mathbf{x} \in \partial\Omega, \qquad t > 0, \tag{2.14}$$

$$w(\mathbf{x},0) = h(\mathbf{x}) \geqslant 0, \qquad \mathbf{x} \in \Omega, \tag{2.15}$$

where μ is a positive constant that satisfies the condition

$$\mu \leqslant (N-1)K_{ave}. \tag{2.16}$$

Let the auxiliary function $\psi(\mathbf{x},t)$ be defined as

$$\psi(\mathbf{x},t) := \{|\nabla w|^2 + aw^2\}e^{2at}, \qquad \mathbf{x} \in \Omega, \qquad t > 0, \tag{2.17}$$

where a is a nonnegative parameter subject to the condition

$$0 \leqslant a < a_0 := \frac{\pi^2}{4d^2}. \tag{2.18}$$

We then conclude that $\psi(\mathbf{x},t)$ takes its maximum value initially, so that we have

$$|\nabla w|^2 + a_0 w^2 \leqslant \Gamma^2 e^{-2a_0 t}, \qquad \mathbf{x} \in \Omega, \qquad t > 0, \tag{2.19}$$

with

$$\Gamma^2 := \max_{\mathbf{x}\in\Omega}\{|\nabla h|^2 + a_0 h^2\}. \tag{2.20}$$

For the proof of Lemma 1 we compute

$$\psi_{,k} = \{2w_{,i}w_{,ik} + 2aww_{,k}\}e^{2at}, \tag{2.21}$$

$$\begin{aligned} \Delta\psi &= \{2w_{,ik}w_{,ik} + 2w_{,i}(\Delta w)_{,i} + 2a|\nabla w|^2 + 2aw\Delta w\}e^{2at} \\ &= \{2w_{,ik}w_{,ik} + 2w_{,i}w_{,it} - 2\mu w_{,ik}w_{,i}w_{,k}|\nabla w|^{-1} \\ &\quad + 2a|\nabla w|^2 + 2aw[w_{,t} - \mu|\nabla w|]\}e^{2at}, \end{aligned} \tag{2.22}$$

$$\psi_{,t} = \{2w_{,i}w_{,it} + 2aww_{,t} + 2a|\nabla w|^2 + 2a^2w^2\}e^{2at}, \tag{2.23}$$

from which we obtain

$$\Delta\psi - \psi_{,t} = \{2w_{,ik}w_{,ik} - 2\mu w_{,ik}w_{,i}w_{,k}|\nabla w|^{-1} - 2a\mu w|\nabla w| - 2a^2w^2\}e^{2at}. \tag{2.24}$$

Moreover we have

$$w_{,ik}w_{,ik} \geqslant w_{,ik}w_{,k}w_{,ij}w_{,j}|\nabla w|^{-2} = a^2w^2 + \dots, \tag{2.25}$$

and

$$w_{,ik}w_{,i}w_{,k} = -aw|\nabla w|^2 + \dots, \tag{2.26}$$

where dots in (2.25), (2.26) stand for terms containing $\psi_{,k}$. From (2.24) (2.25), (2.26) we obtain the differential inequality

$$\Delta\psi - \psi_{,t} + \dots \geqslant 0, \qquad \mathbf{x} \in \Omega\backslash\omega, \qquad t > 0, \tag{2.27}$$

where ω is the set of critical points of $w(\mathbf{x},t)$. It follows from Nirenberg's maximum principle [2,6] that ψ takes its maximum value either

(i) at a point $\hat{\mathbf{p}} = (\hat{\mathbf{x}}, \hat{t})$ with $\hat{\mathbf{x}} \in \partial\Omega$, or
(ii) at a critical point $\bar{\mathbf{p}} = (\bar{\mathbf{x}}, \bar{t})$, $\bar{\mathbf{x}} \in \omega$ such that $\nabla w(\bar{\mathbf{x}}, \bar{t}) = 0$, or
(iii) at a point $\tilde{\mathbf{p}} = (\tilde{\mathbf{x}}, 0)$, $\tilde{\mathbf{x}} \in \Omega$.

Since $\partial\Omega \in C^{2+\varepsilon}$, the PDE (2.13) is satisfied on $\partial\Omega$, so that the outward normal derivative of ψ at $\mathbf{x} \in \partial\Omega$ may be expressed as follows

$$\frac{1}{2}\frac{\partial \psi}{\partial n} e^{-2at} = w_n w_{nn} \leqslant |\nabla w|^2 [\mu - (N-1)K_{ave}(\mathbf{x})] \leqslant 0, \tag{2.28}$$

where w_n and w_{nn} are the first and second normal derivatives of w on $\partial\Omega$. Friedman's maximum principle [1,6] then implies that $\psi(\mathbf{x}, t)$ cannot take its maximum value on $\partial\Omega$, unless the equality sign holds in (2.16). In the latter case the maximum principle implies that ψ must be constant for $0 \leqslant t \leqslant \hat{t}$, and if $\psi(\mathbf{x}, \hat{t}) = \psi(\mathbf{x}, 0)$ inequality (2.12) follows. Thus the first possibility (i) is eliminated unless equality holds in (2.16), and if the equality sign holds then (2.12) is automatically satisfied. Now suppose that the second possibility (ii) holds. Then we would have $\psi(\mathbf{x}, \bar{t}) \leqslant \psi(\bar{\mathbf{x}}, \bar{t})$, i.e.

$$\frac{|\nabla w(\mathbf{x}, \bar{t})|}{\sqrt{w_m^2 - w^2(\mathbf{x}, \bar{t})}} \leqslant \sqrt{a}, \qquad \mathbf{x} \in \Omega, \tag{2.29}$$

with $w_m := \max_{\Omega} w(\mathbf{x}, \bar{t})$. Integrating the inequality (2.29) on a straight line from $\bar{\mathbf{x}}$ to the nearest point $\mathbf{x}_0 \in \partial\Omega$, we obtain

$$a \geqslant \frac{\pi^2}{4d^2} =: a_0. \tag{2.30}$$

The inequality (2.30) is a necessary condition in order that ψ takes its maximum at a critical point, so that the second possibility (ii) is eliminated if $a < a_0$. This achieves the proof of Lemma 1. It is perhaps worth noting that if (2.16) is violated then the maximum value of ψ might well occur on the boundary, in which case the bound for ψ would not be explicit.

Let us remark that the solution $u(\mathbf{x}, t)$ of problem (2.1), (2.2), (2.3) or its gradient ∇u may blow up in finite time t^*. We shall see that this is not the case under the hypotheses of Theorem 1. To this end we assume the possibility that t^* is finite and consider a time interval $[0, \tau]$ with $\tau < t^*$ on which we define the quantities

$$\lambda := \frac{f(u_m)}{u_m} \quad \text{with} \quad u_m := \max_{\Omega \times [0,\tau]} u(\mathbf{x}, t), \tag{2.31}$$

$$\mu := \frac{g(|\nabla u|_m^2)}{|\nabla u|_m} \quad \text{with} \quad |\nabla u|_m := \max_{\Omega \times [0,\tau]} |\nabla u(\mathbf{x}, t)|. \tag{2.32}$$

An upper bound for $u(\mathbf{x}, t)$ valid on $[0, \tau]$ will be obtained in the next Lemma.

LEMMA 2. *Under the assumption*

$$\mu \leqslant (N-1)K_{ave}, \tag{2.33}$$

the solution of problem (2.1), (2.2), (2.3) satisfies the inequality

$$(0 \leqslant) u(\mathbf{x}, t) \leqslant \Gamma_1 \exp(\lambda - a_0)t, \qquad 0 \leqslant t \leqslant \tau, \tag{2.34}$$

where Γ_1 *is defined in (2.8) and* a_0 *in (2.18).*

For the proof of Lemma 2 we write

$$u(\mathbf{x},t) = v(\mathbf{x},t)\exp(\lambda t), \tag{2.35}$$

and make use of (2.4), (2.5) to compute on $[0,\tau]$

$$\begin{aligned} 0 &= \Delta u - u_{,t} + f(u) + g(|\nabla u|^2) \\ &\leqslant \Delta u - u_{,t} + \lambda u + \mu|\nabla u| \\ &= (\Delta v - v_t + \mu|\nabla v|)\exp(\lambda t). \end{aligned} \tag{2.36}$$

The auxiliary function $v(\mathbf{x},t)$ satisfies therefore the following conditions

$$\Delta v - v_{,t} + \mu|\nabla v| \geqslant 0, \qquad \mathbf{x}\in\Omega, \qquad 0\leqslant t\leqslant\tau, \tag{2.37}$$

$$v(\mathbf{x},t) = 0, \qquad \mathbf{x}\in\partial\Omega, \qquad 0\leqslant t\leqslant\tau, \tag{2.38}$$

$$v(\mathbf{x},0) = h(\mathbf{x}), \qquad \mathbf{x}\in\Omega. \tag{2.39}$$

We then conclude that

$$v(\mathbf{x},t) \leqslant w(\mathbf{x},t), \qquad \mathbf{x}\in\Omega, \qquad 0\leqslant t\leqslant\tau, \tag{2.40}$$

where $w(\mathbf{x},t)$ is the solution of problem (2.13), (2.14), (2.15). As a consequence of Lemma 1 we have

$$w(\mathbf{x},t) \leqslant \Gamma_1 e^{-a_0 t}, \qquad \mathbf{x}\in\Omega, \qquad 0\leqslant t\leqslant\tau. \tag{2.41}$$

The conclusion of Lemma 2 follows from (2.35), (2.40), (2.41).

We are now in position to show that $t^* = \infty$.

LEMMA 3. *Suppose that the assumptions (2.4), (2.5), (2.6), (2.7) of Theorem 1 are all satisfied. We then conclude that* $|\nabla u|^2$ *and* u^2 *cannot blow up in finite time, i.e. we have* $t^* = \infty$. *Moreover we have*

$$\frac{f(u(\mathbf{x},t))}{u(\mathbf{x},t)} < \frac{\pi^2}{4d^2}, \qquad \mathbf{x}\in\Omega, \quad t>0, \tag{2.42}$$

and

$$\frac{g(|\nabla u(\mathbf{x},t)|^2)}{|\nabla u(\mathbf{x},t)|} < (N-1)K_{ave}, \qquad \mathbf{x}\in\Omega, \qquad t>0. \tag{2.43}$$

For the proof of Lemma 3 we suppose the contrary, i.e. we suppose that t^* is finite. Then by continuity there exists a first time τ such that equality takes place in at least one of the inequalities (2.42), (2.43). On the time interval $[0,\tau)$ we have $u(\mathbf{x},t)\leqslant\Gamma_1$ by Lemma 2. It then follows from (2.4), (2.6) that

$$\frac{f(u(\mathbf{x},t))}{u(\mathbf{x},t)} \leqslant \frac{f(\Gamma_1)}{\Gamma_1} < \frac{\pi^2}{4d^2} - \alpha, \qquad 0\leqslant t\leqslant\tau. \tag{2.44}$$

This eliminates the possibility of equality being achieved in (2.42). We next show that it is not attained in (2.43). In analogy to the differential inequality (2.27) for $\psi(\mathbf{x},t)$, we show that the auxiliary function $\phi(\mathbf{x},t)$ satisfies a similar differential inequality on (o,τ). We compute

$$\Delta\phi - \phi_{,t} = \{2u_{,ik}u_{,ik} - 2(\alpha u + f)(f+g) - 4g' u_{,ik}u_{,i}u_{,k} - 2\alpha^2u^2 - 4\alpha F\}e^{2\alpha t}. \quad (2.45)$$

Moreover rewriting (2.4) as

$$f(\xi) \leqslant \frac{f(u)}{u}\xi, \qquad 0 \leqslant \xi \leqslant u, \quad (2.46)$$

and integrating both sides of (2.46) between 0 and u, we obtain

$$F(u) = \int_0^u f(\xi)d\xi \leqslant \frac{1}{2}uf(u). \quad (2.47)$$

Finally we have the (in)equalities

$$u_{,ik}u_{,ik} \geqslant -2(\alpha u + f)^2 + \ldots, \quad (2.48)$$

$$u_{,ik}u_{,i}u_{,k} = -(\alpha u + f)|\nabla u|^2 + \ldots, \quad (2.49)$$

where dots stand for terms containing $\phi_{,k}$. Inserting (2.47), (2.48), (2.49) in (2.45) and making use of (2.5), we obtain

$$\Delta\phi - \phi_{,t} + \cdots \geqslant 2(\alpha u + f)[2g'|\nabla u|^2 - g]e^{2\alpha t} \geqslant 0, \qquad \mathbf{x} \in \Omega\backslash\omega, \qquad 0 < t < \tau, \quad (2.50)$$

where ω is the set of critical points of $u(\mathbf{x},t)$. It then follows from Nirenberg's maximum principle that ϕ takes its maximum value either

(i) at a point $\hat{\mathbf{P}} = (\hat{\mathbf{x}},\hat{t})$ with $\hat{\mathbf{x}} \in \partial\Omega$, or
(ii) at a critical point $\mathbf{P} = (\bar{\mathbf{x}},\bar{t})$, $\bar{\mathbf{x}} \in \omega$, such that $\nabla u(\bar{\mathbf{x}},\bar{t}) = 0$, or
(iii) at a point $\tilde{\mathbf{P}} = (\tilde{\mathbf{x}},0)$, $\tilde{\mathbf{x}} \in \Omega$.

However the first possibility (i) cannot occur since we have on $\partial\Omega \times (0,\tau)$

$$\frac{1}{2}\frac{\partial\phi}{\partial n}e^{-2\alpha t} = |\nabla u|^2 \left\{ \frac{g(|\nabla u|^2)}{|\nabla u|} - (N-1)K_{ave}(\mathbf{x}) \right\} \leqslant 0, \quad (2.51)$$

where the above inequality follows from the definition of τ.

We now investigate the second possibility (ii). Let us suppose that ϕ takes its maximum value at a critical point $(\bar{\mathbf{x}},\bar{t})$. Then we have

$$|\nabla u(\mathbf{x},\bar{t})|^2 \leqslant \alpha\left(u_m^2 - u^2(\mathbf{x},\bar{t})\right) + 2[F(u_m) - F(u(\mathbf{x},\bar{t})], \qquad \mathbf{x} \in \Omega, \qquad 0 \leqslant t \leqslant \tau, \quad (2.52)$$

with $u_m = \max_{\mathbf{x}\in\Omega} u(\mathbf{x},\bar{t})$. We now make use of the generalized mean value theorem to write

$$2[F(u_m) - F(u(\mathbf{x},\bar{t}))] \leqslant \frac{f(u_m)}{u_m}[u_m^2 - u^2(\mathbf{x},\bar{t})]. \quad (2.53)$$

Combining (2.52) and (2.53) we obtain the inequality

$$\frac{|\nabla u(\mathbf{x},\bar{t})|}{\sqrt{u_m^2 - u^2(\mathbf{x},\bar{t})}} \leqslant \sqrt{\alpha + \frac{f(u_m)}{u_m}}. \quad (2.54)$$

An integration of (2.54) over a straight line from $\bar{\mathbf{x}}$ to the nearest point on $\partial\Omega$ leads to the inequality

$$\alpha \geqslant \frac{\pi^2}{4d^2} - \frac{f(u_m)}{u_m}. \tag{2.55}$$

Since (2.55) is in contradiction to (2.44), we deduce that the second possibility (ii) cannot occur, so that ϕ takes its maximum value initially, i.e. we have

$$|\nabla u(\mathbf{x},t)|^2 + \alpha u^2(\mathbf{x},t) + 2F(u) \leqslant \Gamma_2^2 e^{-2\alpha t}, \qquad \mathbf{x} \in \Omega, \qquad 0 \leqslant t \leqslant \tau. \tag{2.56}$$

It then follows that

$$|\nabla u(\mathbf{x},t)| \leqslant \Gamma_2, \qquad \mathbf{x} \in \Omega, \qquad 0 \leqslant t \leqslant \tau, \tag{2.57}$$

and that

$$\frac{g(|\nabla u(\mathbf{x},t)|^2)}{|\nabla u(\mathbf{x},t)|} \leqslant \frac{g(\Gamma_2^2)}{\Gamma_2} < (N-1)K_{ave}, \qquad \mathbf{x} \in \Omega, \qquad 0 \leqslant t \leqslant \tau, \tag{2.58}$$

by assumptions (2.5) and (2.7). Inequalities (2.44) and (2.58) are in contradiction to the definition of τ. This achieves the proof of Lemma 3. It follows that the differential inequality (2.50) is justified for all time, and that the function $\phi(\mathbf{x},t)$ takes its maximum initially, i.e. at $t = 0$. This achieves the proof of Theorem 1.

REFERENCES

[1] A. FRIEDMAN, *Remarks on the maximum principle for parabolic equations and its applications*, Pacific J. Math., **8**, (1958), pp. 201–211.

[2] L. NIRENBERG, *A strong maximum principle for parabolic equations*, Comm. Pure Appl. Math., **6**, (1953), pp. 167–177.

[3] L. E. PAYNE AND G. A. PHILIPPIN, *Decay bounds for solutions of second order parabolic problems and their derivatives*, Math. Models Methods Appl. Sci., **5**, (1995), pp. 95–110.

[4] L. E. PAYNE AND G. A. PHILIPPIN, *Pointwise bounds and spatial decay estimates in heat conduction problems*, Math. Models Methods Appl. Sci., **5**, (1995), pp. 755–775.

[5] L. E. PAYNE AND G. A. PHILIPPIN, *On the spatial decay of solutions to a quasilinear parabolic initial-boundary value problem and their derivatives*, SIAM J. Math. Anal., **32**, (2000), pp. 291–303.

[6] M. H. PROTTER AND H. F. WEINBERGER, *Maximum principles in differential equations*, Prentice-Hall, Englewood Cliffs, NJ, 1967.

(Received October 1, 2002)

L. E. Payne
Department of mathematics
Cornell university
Ithaca., N.Y, 14853
USA

G. A. Philippin
Département de mathématiques et de statistique
Université Laval
Québec
Canada, G1K 7P4