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HARDY'S INEQUALITY FOR JACOBI EXPANSIONS

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Abstract. If an analytic function $F(z) = \sum_{n=0}^{\infty} a_n z^n$ belongs to the Hardy space on the unit disc, then the sequence of coefficients satisfies $\sum_{n=0}^{\infty} |a_n|/(n+1) < \infty$, which is well-known as Hardy's inequality. This type of inequality is obtained with respect to the Jacobi expansions.

Hardy's inequality [2] says that there exists a constant C such that $\sum_{n=0}^{\infty} |a_n|/(n+1) \leqslant C\|F\|_{H^1}$ for $F(z) = \sum_{n=0}^{\infty} a_n z^n$ in $H^1(\mathbb{D})$, where $H^1(\mathbb{D})$ is the Hardy space on the unit disc $\mathbb{D}$ which consists of the analytic functions $F(z)$ on $\mathbb{D}$ satisfying $\|F\|_{H^1} = \sup_{0<r<1} \int_0^{2\pi} |F(re^{i\theta})|\, d\theta < \infty$. For our purpose, we restate this inequality in terms of the real Hardy space. Let $\Re H^1$ be the real Hardy space, that is, the space consisting of the boundary functions $f(\theta) = \lim_{r\to 1} \Re F(re^{i\theta})$ of $F \in H^1(\mathbb{D})$ and $\|f\|_{\Re H^1} = \|F\|_{H^1}$ with real $F(0)$. Then, Hardy's inequality turns to

$$\sum_{n=-\infty}^{\infty} \frac{|c_n|}{|n|+1} \leqslant C\|f\|_{\Re H^1} \tag{1}$$

for $f(\theta) \sim \sum_{n=-\infty}^{\infty} c_n e^{in\theta}$ in $\Re H^1$, where C is a constant independent of f. The purpose of this note is to obtain this type of inequality with respect to the Jacobi expansions.

Let $R_n^{(\alpha,\beta)}(\theta)$ be the Jacobi functions defined by

$$R_n^{(\alpha,\beta)}(\theta) = t_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(\cos\theta) \left(\sin\frac{\theta}{2}\right)^{\alpha+1/2} \left(\cos\frac{\theta}{2}\right)^{\beta+1/2},$$

where $P_n^{(\alpha,\beta)}(x)$ is the Jacobi polynomial of degree n and of order $\alpha, \beta > -1$, and

$$t_n^{(\alpha,\beta)} = \left(\frac{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)\Gamma(n+1)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}\right)^{1/2}.$$

Then, the system $\{R_n^{(\alpha,\beta)}\}_{n=0}^{\infty}$ is complete and orthonormal in $L^2(0,\pi)$ with respect to the ordinary Lebesgue measure $d\theta$. For a function f on $(0,\pi)$, we have the Jacobi

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$$f(\theta) \sim \sum_{n=0}^{\infty} c_n^{(\alpha,\beta)}(f) R_n^{(\alpha,\beta)}(\theta), \quad c_n^{(\alpha,\beta)}(f) = \int_0^{\pi} f(\theta) R_n^{(\alpha,\beta)}(\theta)\, d\theta.$$

We remark that

$$R_n^{(-1/2,-1/2)}(\theta) = \begin{cases} \sqrt{2/\pi} & (n=0), \\ \sqrt{1/\pi}\ \cos n\theta & (n>0), \end{cases} \tag{2}$$
$$R_n^{(1/2,1/2)}(\theta) = \sqrt{1/\pi}\ \sin(n+1)\theta \quad (n \geqslant 0),$$

that is, the Jacobi expansions are the cosine and sine expansions when $(\alpha,\beta) = (-1/2,-1/2)$ and $(\alpha,\beta) = (1/2,1/2)$. We refer to the work of Szegö [4] for the Jacobi polynomials.

Let $H^1(0,\pi)$ be the space defined by

$$H^1(0,\pi) = \{h|_{(0,\pi)}; h \in \Re H^1,\ \text{even}\}.$$

We endow the space $H^1(0,\pi)$ with the norm $\|f\|_{H^1(0,\pi)} = \|h\|_{\Re H^1}$, where $f = h|_{(0,\pi)}$.

Our theorem is as follows:

THEOREM . *Let* $\alpha,\beta \geqslant -1/2$. *Then, the Jacobi coefficients* $c_n^{(\alpha,\beta)}(f)$ *of a function* $f \in H^1(0,\pi)$ *satisfy*

$$\sum_{n=0}^{\infty} \frac{|c_n^{(\alpha,\beta)}(f)|}{n+1} \leqslant C\|f\|_{H^1(0,\pi)}, \tag{3}$$

where C *is a constant independent of* f.

Another well-known inequality for the Hardy space is Paley's inequality:

$$\left\{\sum_{k=1}^{\infty}(|c_{n_k}|^2 + |c_{-n_k}|^2)\right\}^{1/2} \leqslant C\|f\|_{\Re H^1} \quad \text{for} \quad f(\theta) \sim \sum_{n=-\infty}^{\infty} c_n e^{in\theta} \quad \text{in} \quad \Re H^1,$$

where $\{n_k\}_{k=1}^{\infty}$ is an Hadamard sequence, that is, a sequence of positive integers such that $n_{k+1}/n_k \geqslant \rho$ with a constant $\rho > 1$. In [3], we have established an analogue of this inequality with respect to the Jacobi expansions by proving the following Lipschitz property of the Jacobi functions:

LEMMA . ([3]) *Let* $\alpha,\beta \geqslant -1/2$. *Then, there exists a constant* C *such that*

$$|R_n^{(\alpha,\beta)}(\theta_1) - R_n^{(\alpha,\beta)}(\theta_2)| \leqslant Cn^{\delta}|\theta_1 - \theta_2|^{\delta} \tag{4}$$

for $0 \leqslant \theta_1 < \theta_2 \leqslant \pi$, *where* $\delta = \min\{\alpha+1/2, \beta+1/2\}$ *if* $0 < \alpha+1/2 < 1$ *or* $0 < \beta+1/2 < 1$, *and* $\delta = 1$ *otherwise, and* C *is independent of* θ_1, θ_2 *and* n.

The atomic decomposition of functions in the real Hardy space allows us to apply the lemma to proving the theorem. An atom is a real valued function a supported in an interval I satisfying $|a(\theta)| \leqslant |I|^{-1}$ almost everywhere and $\int a(\theta)d\theta = 0$, where $|I|$ is the length of I. We may refer to [1] for the atomic decomposition.

Proof of the theorem. Let $f \in H^1(0,\pi)$. Then, there exist a sequence $\{a_j\}_{j=0}^{\infty}$ of atoms and a sequence $\{\lambda_j\}_{j=0}^{\infty}$ of real numbers such that

$$f(\theta) = \sum_{j=0}^{\infty} \lambda_j a_j(\theta) \text{ a.e.}\theta, \tag{5}$$

$$\sum_{j=0}^{\infty} |\lambda_j| \leqslant C\|f\|_{H^1(0,\pi)} \tag{6}$$

with a constant C independent of f. Here and below, C denotes a positive constant which may differ at each different occurrence. Further, we may assume that $I_j \subset [0,\pi]$, where I_j is the support interval of a_j. For this, see [1, p. 608,the last line to p. 609, line 9].

It follows from (5) and (6) that

$$c_n^{(\alpha,\beta)}(f) = \sum_{j=0}^{\infty} \lambda_j c_n^{(\alpha,\beta)}(a_j).$$

Here, we used the fact $|R_n^{(\alpha,\beta)}(\theta)| \leqslant C$ with a constant C depending only on α and β for $0 \leqslant \theta \leqslant \pi$ and $\alpha, \beta \geqslant -1/2$ (see [4, (7.32.5)]). Thus, we have

$$\sum_{n=0}^{\infty} \frac{|c_n^{(\alpha,\beta)}(f)|}{n+1} \leqslant \sum_{j=0}^{\infty} |\lambda_j| \sum_{n=0}^{\infty} \frac{|c_n^{(\alpha,\beta)}(a_j)|}{n+1}.$$

By (6), we see that it is enough to show

$$\sum_{n=0}^{\infty} \frac{|c_n^{(\alpha,\beta)}(a)|}{n+1} \leqslant C \tag{7}$$

for every atom a with C not depending on atoms. In order to prove this inequality, let us evaluate $c_n^{(\alpha,\beta)}(a)$. Let $I = [b, b+h]$ be the support interval of a. We have

$$|c_n^{(\alpha,\beta)}(a)| = \left| \int_b^{b+h} a(\theta) \left(R_n^{(\alpha,\beta)}(\theta) - R_n^{(\alpha,\beta)}(b) \right) d\theta \right|$$

by the fact $\int a(\theta)\, d\theta = 0$. Our lemma leads to

$$|c_n^{(\alpha,\beta)}(a)| \leqslant C \int_b^{b+h} |a(\theta)| n^{\delta} (\theta - b)^{\delta}\, d\theta,$$

where δ means the one in the lemma. By Schwarz's inequality we see that the right-hand side of the inequality is bounded by $Cn^{\delta}\|a\|_2 h^{\delta+1/2}$, where $\|a\|_2 = (\int |a(\theta)|^2\, d\theta)^{1/2}$. Since atoms satisfy the fact $h \leqslant \|a\|_2^{-2}$, it follows that

$$|c_n^{(\alpha,\beta)}(a)| \leqslant Cn^{\delta}\|a\|_2^{-2\delta}. \tag{8}$$

To estimate the sum on the left-hand side of (7), we choose γ as $\gamma = \|a\|_2^2$ and write

$$\sum_{n=0}^{\infty} \frac{|c_n^{(\alpha,\beta)}(a)|}{n+1} = \left(\sum_{n\leqslant\gamma} + \sum_{n>\gamma}\right) \frac{|c_n^{(\alpha,\beta)}(a)|}{n+1}. \tag{9}$$

For the sum $\sum_{n>\gamma}$, we use Parseval's identity and Schwarz's inequality and get

$$\sum_{n>\gamma} \frac{|c_n^{(\alpha,\beta)}(a)|}{n+1} \leqslant \|a\|_2 \left(\sum_{n>\gamma} \frac{1}{(n+1)^2}\right)^{1/2} \leqslant C\|a\|_2\gamma^{-1/2} \leqslant C. \tag{10}$$

We apply (8) to estimating the sum $\sum_{n\leqslant\gamma}$. It follows that

$$\sum_{n\leqslant\gamma} \frac{|c_n^{(\alpha,\beta)}(a)|}{n+1} \leqslant C\|a\|_2^{-2\delta} \sum_{n\leqslant\gamma} \frac{n^{\delta}}{n+1} \leqslant C\|a\|_2^{-2\delta}\gamma^{\delta} \leqslant C. \tag{11}$$

Combining (10) and (11), we get (7), which completes the proof. □

REMARK 1. The theorem does not hold if we substitute the Lebesgue space $L^1(0,\pi)$ for the Hardy space $H^1(0,\pi)$, that is, there exists a function $f \in L^1(0,\pi)$ such that the series $\sum_{n=0}^{\infty} |c_n^{(\alpha,\beta)}(f)|/(n+1)$ diverges. We show this. Assume that

$$\sum_{n=0}^{\infty} \frac{|c_n^{(\alpha,\beta)}(f)|}{n+1} < \infty$$

for all $f \in L^1(0,\pi)$. Then, by the closed graph theorem, we have

$$\sum_{n=0}^{\infty} \frac{|c_n^{(\alpha,\beta)}(f)|}{n+1} \leqslant C\|f\|_{L^1(0,\pi)},$$

with C independent of f. For $\theta_0 \in (0,\pi)$, we put

$$f_k(\theta) = \begin{cases} k & (|\theta-\theta_0| < 1/(2k) \text{ and } \theta \in (0,\pi)), \\ 0 & (\text{otherwise}) \end{cases}$$

for $k = 1, 2, \ldots$. Since $\|f_k\|_{L^1(0,\pi)} \leqslant 1$, the assumption implies that there exists a constant C such that

$$\sum_{n=0}^{\infty} \frac{|c_n^{(\alpha,\beta)}(f_k)|}{n+1} \leqslant C$$

for every k. By the fact $\lim_{k\to\infty} c_n^{(\alpha,\beta)}(f_k) = R_n^{(\alpha,\beta)}(\theta_0)$, we have

$$\sum_{n=0}^{\infty} \frac{|R_n^{(\alpha,\beta)}(\theta_0)|}{n+1} \leqslant \liminf_{k\to\infty} \sum_{n=0}^{\infty} \frac{|c_n^{(\alpha,\beta)}(f_k)|}{n+1} \leqslant C. \tag{12}$$

Here, we use the following inequality

$$|R_n^{(\alpha,\beta)}(\theta_0)| \geqslant C\left|\cos\left(n\theta_0 + \frac{2(\alpha+\beta+1)\theta_0 - \pi(2\alpha+1)}{4}\right)\right| - \frac{C'}{n\sin\theta_0},$$

where C and C' are independent of n and may depend on α,β and θ_0 (see [4, (8.21.18)]). Taking $\theta_0 = 2\pi/3$, for example, we get

$$\sum_{n=0}^{\infty} \frac{|R_n^{(\alpha,\beta)}(\theta_0)|}{n+1} = \infty,$$

which contradicts (12).

REMARK 2. Our inequality (3) with $(\alpha,\beta) = (-1/2,-1/2)$ implies the classical Hardy inequality (1). For, if $f \in \Re H^1$, then $\tilde{f} \in \Re H^1$ by the definition of $\Re H^1$, where $\tilde{f}(\theta) \sim -i\sum_{n=-\infty}^{\infty} \operatorname{sgn}(n)c_n e^{in\theta}$ is the conjugate function of $f(\theta) \sim \sum_{n=-\infty}^{\infty} c_n e^{in\theta}$. Let f_e and $\tilde{f}_e$ be the even parts of f and $\tilde{f}$, respectively. Then, f_e and $\tilde{f}_e$ belong to $H^1(0,\pi)$. By (2), we have the following relation: $c_0 = \pi^{-1/2}c_0^{(-1/2,-1/2)}(f_e)$ and

$$c_n = \begin{cases} (2\pi)^{-1/2}\left(c_n^{(-1/2,-1/2)}(f_e) + ic_n^{(-1/2,-1/2)}(\tilde{f}_e)\right) & (n>0), \\ (2\pi)^{-1/2}\left(c_{-n}^{(-1/2,-1/2)}(f_e) - ic_{-n}^{(-1/2,-1/2)}(\tilde{f}_e)\right) & (n<0). \end{cases}$$

This allows to deduce (1) from (3) with $(\alpha,\beta) = (-1/2,-1/2)$.

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