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ON (p,k)–QUASIHYPONORMAL OPERATORS

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Abstract. An operator T is called (p,k)-quasihyponormal if ${T^*}^k(|T|^{2p}-|T^*|^{2p})T^k \geqslant 0$, ($0 < p \leqslant 1$; $k \in \mathbb{Z}^+$), which is a common generalization of p-quasihyponormality and k-quasihyponormality. In this paper we consider the Putnam's inequality, the Berger-Shaw's inequality, the Weyl's theorem and the tensor product for (p,k)-quasihyponormal operators.

1. Introduction

Throughout this paper let H be a separable complex Hilbert space with inner product $\langle \cdot,\cdot \rangle$. Let $B(H)$ denote the C^*-algebra of all bounded linear operators on H and let $K(H)$ be the ideal of all compact operators on H. For an operator $T \in B(H)$, let $\sigma(T), \sigma_p(T), \sigma_e(T)$ and $\pi_{00}(T)$ denote the spectrum, the point spectrum, the essential spectrum and the set of all isolated eigenvalues of finite multiplicity of T, respectively. An operator $T \in B(H)$ is called *Fredholm*, denoted by $T \in F$, if $\operatorname{ran}(T)$ is closed and both $\ker(T)$ and $H/\ ran(T)$ are finite dimensional. The *index* of a Fredholm operator $T \in B(H)$, denoted by $\operatorname{ind}(T)$, is given by the integer

$$\operatorname{ind}(T) = \dim\ker(T) - \dim\left(H/\operatorname{ran}(T)\right).$$

An operator $T \in B(H)$ is called *Weyl*, denoted by $T \in F_0$, if it is Fredholm of index zero. The Weyl spectrum $w(T)$ of $T \in B(H)$ is defined by

$$w(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin \mathcal{F}_0\}.$$

It is well known that $w(T)$ is non-empty and $w(T) = \bigcap_{K\in K(H)} \sigma(T+K)$. According to Corburn [3], we say that Weyl's theorem holds for $T \in B(H)$ if

$$\sigma(T) \setminus w(T) = \pi_{00}(T).$$

For p such as $0 < p \leqslant 1$, an operator $T \in B(H)$ is called p-*hyponormal* if $(T^*T)^p - (TT^*)^p \geqslant 0$, and is called (p,k)-*quasihyponormal* if ${T^*}^k(|T|^{2p}-|T^*|^{2p})T^k \geqslant 0$, where $0 < p \leqslant 1$ and k is a positive integer. Especially, when $p = 1, k = 1$ and $p = k = 1$, T

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is called k-*quasihyponormal*, p-*quasihyponormal* and *quasihyponormal*, respectively. It is clear that

$$\begin{aligned}\text{the class of hyponormal operators} &\subseteq \text{the class of } p-\text{hyponormal operators}\\ &\subseteq \text{the class of } p-\text{quasihyponormal operators}\\ &\subseteq \text{the class of } (p,k)-\text{quasihyponormal operators.}\end{aligned}$$

and

$$\begin{aligned}\text{the class of hyponormal operators} &\subseteq \text{the class of } k-\text{quasihyponormal operators}\\ &\subseteq \text{the class of } (p,k)-\text{quasihyponormal operators.}\end{aligned}$$

Corburn [5], Cho-Itoh-Oshio [4], Campbell-Gupta [2] and Uchiyama-Djordjevic [20] showed that Weyl's theorem holds for hyponormal operators, p-hyponormal operators, k-quasihyponormal operators and p-quasihyponormal operators, respectively.

On the other hand, J. Hou [12] and J. Stochel [17] showed that $T \otimes S$ is hyponormal on $H \otimes H$ if and only if each of T and S is hyponormal. More recently, B.P. Duggal [6] demonstrated that the Hou–Stochel theorem remains true when one substitutes the term "p-hyponormal" for "hyponormal". Very recently, in [8], it was shown that Hou–Stochel theorem remains true when one substitutes the term "p-quasihyponormal or w-hyponormal" for "hyponormal".

In this paper we consider the Putnam's inequality, the Berger-Shaw's inequality, the Weyl's theorem and the tensor product for (p,k)-quasihyponormal operators. To do this we adopt Uchiyama's ideas (see [18, 19, 20]) and W.Y. Lee's ideas (see [14]).

2. Main Results

We begin with:

LEMMA 1. *If T is (p,k)-quasihyponormal operator, then T has the following matrix representation:*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix},$$

where T_1 is p-hyponormal on $\overline{\operatorname{ran}(T^k)}$ and ${T_3}^k = 0$. Furthermore, $\sigma(T) = \sigma(T_1) \cup \{0\}$.

Proof. Consider the matrix representation of T with respect to the decomposition $H = \overline{ran(T^k)} \oplus \ker(T^{*k})$: $\quad T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$. Let P be the projection onto $\overline{\operatorname{ran}(T^k)}$. Since $T_1 = TP$, we have ${T_1}^* T_1 = PT^*TP$. By Hansen's inequality [11] we have

$$(T_1^* T_1)^p = (PT^*TP)^p \geqslant P(T^*T)^p P,$$

while $T_1 T_1^* = TPT^* = PTPT^*P$. So we have

$$(T_1 T_1^*)^p = (TPT^*)^p = P(TPT^*)^p P \leqslant P(TT^*)^p P.$$

Therefore if T is (p,k)-quasihyponormal operator, then

$$(T_1^*T_1)^p \geqslant P(T^*T)^pP \geqslant P(TT^*)^pP \geqslant (T_1T_1^*)^p.$$

That is, T_1 is p-hyponormal on the $\overline{\operatorname{ran}(T^k)}$.

On the other hand, for any $x = (x_1, x_2) \in H$,

$$\langle T_3{}^k x_2, x_2\rangle = \langle T^k(I-P)x, (I-P)x\rangle = \langle (I-P)x, T^{*k}(I-P)x\rangle = 0,$$

which implies $T_3{}^k = 0$.

Since $\sigma(T_1)\cup\sigma(T_3) = \sigma(T)\cup\mathfrak{G}$, where $\mathfrak{G}$ is the union of the holes in $\sigma(T)$ which happen to be subset of $\sigma(T_1)\cap\sigma(T_3)$ by [10, Corollary 7], and $\sigma(T_1)\cap\sigma(T_3)$ has no interior points and T_3 is nilpotent, we have $\sigma(T) = \sigma(T_1)\cup\{0\}$. □

COROLLARY 2. *If T is a (p,k)-quasihyponormal and the restriction T_1 of T on $\overline{\operatorname{ran}(T^k)}$ is invertible, then T is similar to a direct sum of a p-hyponormal operator and a nilpotent operator.*

Proof. Let $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $H = \overline{\operatorname{ran}(T^k)} \oplus \ker(T^{*k})$. By Lemma 1, T_1 is p-hyponormal and $T_3{}^k = 0$. By assumption we have $\sigma(T_1)\cap\sigma(T_3) = \phi$. Hence by Rosenblum's Corollary there exists an operator S such that $T_1S - ST_3 = T_2$. Therefore

$$\begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} = \begin{pmatrix} I & S \\ 0 & I \end{pmatrix}^{-1} \begin{pmatrix} T_1 & 0 \\ 0 & T_3 \end{pmatrix} \begin{pmatrix} I & S \\ 0 & I \end{pmatrix},$$

which gives the result. □

COROLLARY 3. *If T is a (p,k)-quasihyponormal and λ_0 is an isolated point of $\sigma(T)$ then λ_0 is an eigenvalue, i.e., T is isoloid.*

Proof. Suppose T is a (p,k)-quasihyponormal operator and let $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $H = \overline{\operatorname{ran}(T^k)} \oplus \ker(T^{*k})$. Then from $(1,1)$, $\sigma(T) = \sigma(T_1)\cup\{0\}$. Assume that $\lambda_0 \in \operatorname{iso}\sigma(T)$. Then $\lambda_0 \in \operatorname{iso}\sigma(T_1)$ or $\lambda_0 = 0$. If $\lambda_0 \in \operatorname{iso}\sigma(T_1)$, then $\lambda_0 \in \sigma_p(T_1)$ because T_1 is p-hyponormal. Thus we may assume $\lambda_0 = 0$ and $\lambda_0 \notin \sigma(T_1)$, so $\dim\ker(T_3) > 0$. Therefore if $x \in \ker(T_3)$, then $-T_1{}^{-1}T_2x \oplus x \in \ker(T)$. Thus λ_0 is an eigenvalue of T. □

For some operators, there is an intimate relationship between the plannar Lebesgue measure of its spectrum and its self-commutator. For example, Putnam [16] obtained the norm estimation for the self-commutator of a hyponormal operator, called Putnam's inequality. This inequality is extended for a p-hyponormal operator by Xia [21], Cho-Itoh [3] and Duggal [7]. Also, this is extended for a p-quasihyponormal operator by Uchiyama [19]. On the other hand, Berger-Shaw [1] showed the trace norm estimation for the self-commutator of n-multicyclic hyponormal operator, called Berger-Shaw's inequality. This is extended for a p-hyponormal and p-quasihyponormal operator by Uchiyama [18, 19].

In the sequel we need:

LEMMA 4. [3, 7, 21] *If T is p-hyponormal operator, then*

$$\|(T^*T)^p - (TT^*)^p\| \leqslant \min\left\{\frac{p}{\pi}\int_{\sigma(T)} r^{2p-1}drd\theta, \quad \left(\frac{1}{\pi}\int_{\sigma(T)} rdrd\theta\right)^p\right\}.$$

The following theorem extends a result of M. Cho and M. Itoh [3].

THEOREM 5. *If T is a (p,k)-quasihyponormal operator, then*

$$\|P\{(T^*T)^p - (TT^*)^p\}P\| \leqslant \min\left\{\frac{p}{\pi}\int_{\sigma(T)} r^{2p-1}drd\theta, \quad \left(\frac{1}{\pi}\int_{\sigma(T)} rdrd\theta\right)^p\right\},$$

where P is the projection onto $\overline{\mathrm{ran}(T^k)}$.

Proof. Let $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $H = \overline{\mathrm{ran}(T^k)} \oplus \ker(T^{*k})$. From Lemma 1 we have

$$0 \leqslant P\{(T^*T)^p - (TT^*)^p\}P \leqslant ({T_1}^*T_1)^p - (T_1{T_1}^*)^p$$

and T_1 is p-hyponormal. Hence by Lemmas 1 and 4,

$$\begin{aligned}\|P\{(T^*T)^p - (TT^*)^p\}P\| &\leqslant \|({T_1}^*T_1)^p - (T_1{T_1}^*)^p\| \\ &\leqslant \min\left\{\frac{p}{\pi}\int_{\sigma(T_1)} r^{2p-1}drd\theta, \quad \left(\frac{1}{\pi}\int_{\sigma(T_1)} rdrd\theta\right)^p\right\} \\ &= \min\left\{\frac{p}{\pi}\int_{\sigma(T)} r^{2p-1}drd\theta, \quad \left(\frac{1}{\pi}\int_{\sigma(T)} rdrd\theta\right)^p\right\}.\end{aligned}$$

□

COROLLARY 6. *If T is a (p,k)-quasihyponormal operator and $\sigma(T)$ is Lebesgue null-set, then T is the direct sum of normal operator and nilpotent operator.*

Proof. Let $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $\overline{\mathrm{ran}(T^k)} \oplus \ker(T^{*k})$ and let P be the orthogonal projection onto $\overline{\mathrm{ran}(T^k)}$. Then T_1 is p-hyponormal and $T_3^k = 0$ by Lemma 1 and $||(T_1^*T_1)^p - (T_1T_1^*)^p|| = 0$ by Theorem 5. Hence T_1 is normal. Since

$$\begin{pmatrix} (T_1^*T_1)^p & 0 \\ 0 & 0 \end{pmatrix} \geqslant P(T^*T)^pP \geqslant P(TT^*)^pP \geqslant \begin{pmatrix} (T_1T_1^*)^p & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} (T_1^*T_1)^p & 0 \\ 0 & 0 \end{pmatrix},$$

$(TT^*)^p$ is of the form $\begin{pmatrix} (T_1^*T_1)^p & A \\ A^* & B \end{pmatrix}$. Put $(TT^*)^{\frac{p}{2}} = \begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix}$. Then

$$\begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} = P(TT^*)^{\frac{p}{2}}P \geqslant P(TPT^*)^{\frac{p}{2}}P = \begin{pmatrix} (T_1^*T_1)^{\frac{p}{2}} & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence, $X \geqslant (T_1^*T_1)^{\frac{p}{2}}$. Since $X^2 + YY^* = (T_1^*T_1)^p$, we have $X = (T_1^*T_1)^{\frac{p}{2}}$ and $Y = 0$. Therefore,

$$\begin{pmatrix} T_1T_1^* + T_2T_2^* & T_2T_3^* \\ T_3T_2^* & T_3T_3^* \end{pmatrix} = TT^* = \begin{pmatrix} T_1T_1^* & 0 \\ 0 & Z^{\frac{2}{p}} \end{pmatrix},$$

and $T_2 = 0$. This completes the proof. $\square$

For $T \in B(H)$, $\mathcal{R}(\sigma(T))$ denotes the set of all rational functions being analytic on $\sigma(T)$. The operator T is said to be *n-multicyclic* if there are n vectors $x_1, \cdots, x_n \in H$, called generating vectors, such that $\bigvee\{g(T)x_i \mid i = 1, \cdots, n \quad \text{and} \quad g \in \mathcal{R}(\sigma(T))\} = H$.

LEMMA 7. [18, Theorem] *If T is an n-multicyclic p-hyponormal operator, then $(T^*T)^p - (TT^*)^p$ belongs to the Schatten $\frac{1}{p}$-class and*

$$tr\left(\{(T^*T)^p - (TT^*)^p\}^{\frac{1}{p}}\right) \leqslant \frac{n}{\pi} Area(\sigma(T)).$$

The following theorem is an extension of Berger-Shaw's inequality to the case of (p,k)-quasihyponormal operators.

THEOREM 8. *If T is an n-multicyclic (p,k)-quasihyponormal operator, then we have:*

(i) *The restriction T_1 of T on $\overline{\operatorname{ran}(T^k)}$ is also an n-multcyclic operator;*

(ii) *$\{P(T^*T)^pP - P(TT^*)^pP\}^{\frac{1}{p}}$ belongs to the Schatten $\frac{1}{p}$-class and*

$$tr\left(\{P(T^*T)^pP - P(TT^*)^pP\}^{\frac{1}{p}}\right) \leqslant \frac{n}{\pi} Area(\sigma(T)),$$

where P is the projection onto $\overline{\operatorname{ran}(T^k)}$.

Proof. Let $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $H = \overline{\operatorname{ran}(T^k)} \oplus \ker(T^{*k})$. Since $\sigma(T_1) \subset \sigma(T)$ by Lemma 1, $\mathcal{R}(\sigma(T)) \subset \mathcal{R}(\sigma(T_1))$. By hypothesis there exist n vectors, $x_1, \cdots, x_n \in H$, such that

$$H = \bigvee\{g(T)x_i \mid i = 1, \cdots, n \quad \text{and} \quad g \in \mathcal{R}(\sigma(T))\}.$$

Now let $y_i = T^k x_i, \quad i = 1 \cdots, n$. Then we have the following

$$\begin{aligned} \bigvee\{g(T_1)y_i \mid i = 1, \cdots, n,\ g \in \mathcal{R}(\sigma(T_1))\} &\supset \bigvee\{g(T_1)y_i \mid i = 1, \cdots, n,\ g \in \mathcal{R}(\sigma(T))\} \\ &= \bigvee\{g(T_1)T^k x_i \mid i = 1, \cdots, n,\ g \in \mathcal{R}(\sigma(T))\} \\ &= \bigvee\{g(T)T^k x_i \mid i = 1, \cdots, n,\ g \in \mathcal{R}(\sigma(T))\} \\ &= \bigvee\{T^k g(T)x_i \mid i = 1, \cdots, n,\ g \in \mathcal{R}(\sigma(T))\} \\ &= \overline{\operatorname{ran}(T^k)} \end{aligned}$$

and $\{y_1, \cdots, y_n\}$ are n-multicyclic vectors of T_1. This proves the (i).

On the other hand, since T_1 is an n-multicyclic p-hyponormal operator by (i), it follows from Lemma 7

$$tr\left(\{({T_1}^*T_1)^p - (T_1{T_1}^*)^p\}^{\frac{1}{p}}\right) \leqslant \frac{n}{\pi}\text{Area}(\sigma(T_1)) = \frac{n}{\pi}\text{Area}(\sigma(T)).$$

On the other hand, since

$$0 \leqslant P(T^*T)^pP - P(TT^*)^pP \leqslant (T_1{}^*T_1)^p - (T_1T_1{}^*)^p,$$

we have that $P(T^*T)^pP - P(TT^*)^pP$ and $(T_1{}^*T_1)^p - (T_1T_1{}^*)^p$ are both positive compact operators and

$$s_j\left(P(T^*T)^pP - P(TT^*)^pP\right) \leqslant s_j\left((T_1{}^*T_1)^p - (T_1T_1{}^*)^p\right) \quad \text{for} \quad j = 1, 2, \cdots,$$

where $s_j(T)$ is the n-th singular number of T. Therefore

$$s_j\left(\{P(T^*T)^pP - P(TT^*)^pP\}^{\frac{1}{p}}\right) \leqslant s_j\left(\{(T_1{}^*T_1)^p - (T_1T_1{}^*)^p\}^{\frac{1}{p}}\right) \quad \text{for} \quad j = 1, 2, \cdots.$$

Hence we have

$$\begin{aligned} 0 &\leqslant tr\left(\{P(T^*T)^pP - P(TT^*)^pP\}^{\frac{1}{p}}\right) \\ &\leqslant tr\left(\{(T_1{}^*T_1)^p - (T_1T_1{}^*)^p\}^{\frac{1}{p}}\right) \leqslant \frac{n}{\pi}\mathrm{Area}(\sigma(T)). \qquad \square \end{aligned}$$

The following lemma shows that the passage from $w(A) \cup w(B)$ to $w\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$.

LEMMA 9. [13, Theorem 6] *For a given operators $A, B, C \in B(H)$ there is equality*

$$w(A) \cup w(B) = w(M_C) \cup \mathfrak{G},$$

where $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ and $\mathfrak{G}$ is the union of certain of the holes in $w(M_C)$ which happen to be subsets of $w(A) \cap w(B)$.

The following theorem shows that the spectral mapping theorem for Weyl spectrum holds for (p,k)-quasihyponormal operators.

THEOREM 10. *If T is (p,k)-quasihyponormal, then $f(w(T)) = w(f(T))$ for any analytic function f on a neighborhood of $\sigma(T)$.*

Proof. We need only to prove that $w(p(T)) = p(w(T))$ for any polynomial p. Since T has the matrix representation $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$, where T_1 is p-hyponormal and $T_3{}^k = 0$, and the spectral mapping theorem for Weyl spectrum holds for p-hyponormal operator, it follows that

$$\begin{aligned} w(p(T)) &= w(p(T_1)) \cup w(p(T_3)) \\ &= p(w(T_1)) \cup p(w(T_3)) \\ &= p\left(w(T_1) \cup w(T_3)\right) \\ &= p(w(T)). \qquad \square \end{aligned}$$

The following corollary is immediate result from above theorem.

COROLLARY 11. *The spectral mapping theorem for Weyl spectrum holds for quasi-hyponormal operators, p-quasihyponormal operators and k-quasihyponormal operators.*

It was known [13] if A and B are isoloid and if Weyl's theorem holds for A and B then

$$\text{Weyl's theorem holds for } \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \Leftrightarrow w\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = w(A) \cup w(B).$$

The "spectral picture" [15] of the operator $T \in B(H)$, denoted by $SP(T)$, which consists of the set $\sigma_e(T)$, the collection of holes and pseudoholes in $\sigma_e(T)$, and the indices associated with these holes and pseudoholes.

In general, Weyl's theorem does not hold for operator matrix $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ even though Weyl's theorem holds for $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ (see [14]). But W.Y. Lee [14] showed that following Lemma:

LEMMA 12. *If either $SP(A)$ or $SP(B)$ has no pseudoholes and if A is an isoloid operator for which Weyl's theorem holds then for every $C \in B(H)$,*

$$\textit{Weyl's theorem holds for } \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \Rightarrow \textit{Weyl's theorem holds for } \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}.$$

We have the following result from above Lemma.

COROLLARY 13. *Weyl's theorem holds for every (p,k)-quasihyponormal operator.*

Proof. Let $T \in B(H)$ is a (p,k)-quasihyponormal operator. Then by Lemma 1 T has the following matrix representation:

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad \text{on } H = \overline{\operatorname{ran}(T^k)} \oplus \ker(T^{*k}),$$

where T_1 is p-hyponormal operator and T_3 is nilpotent operator. Therefore Weyl's theorem holds for $\begin{pmatrix} T_1 & 0 \\ 0 & T_3 \end{pmatrix}$ because Weyl's theorem holds for p-hyponormal operator and nilpotent operator and both p-hyponormal operator and nilpotent operator are isoloid. Hence by Lemma 11 Weyl's theorem holds for $\begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ because $SP(T_3)$ has no pseudoholes. □

The next theorem extends J. Hou [12, Theorem 1.4] and Farenick and Kim [7, Theorem 9].

THEOREM 14. *Let $A \in B(H)$ and $B \in B(K)$ are nonzero operators. Then $A \otimes B$ is (p,k)-quasihyponormal if and only if one of the following holds:*

1. *$A^k = 0$ or $B^k = 0$,*
2. *A and B are (p,k)-quasihyponormal.*

Proof. It is clear that $A \otimes B$ is (p,k)-quasihyponormal if and only if

$$\begin{aligned} &A^{*k}\left(|A|^{2p}-|A^*|^{2p}\right)A^k \otimes B^{*k}|B|^{2p}B^k \\ &+A^{*k}|A^*|^{2p}A^k \otimes B^{*k}\left(|B|^{2p}-|B^*|^{2p}\right)B^k \geqslant 0. \end{aligned} \tag{14.1}$$

Therefore the sufficiency is clear.

To prove the necessity, let $\xi \in H$ and $\eta \in K$ be arbitrary. Then we have

$$\begin{aligned} &\left\langle A^{*k}\left(|A|^{2p}-|A^*|^{2p}\right)A^k\xi,\xi\right\rangle \left\langle B^{*k}|B|^{2p}B^k\eta,\eta\right\rangle \\ &+\left\langle A^{*k}|A^*|^{2p}A^k\xi,\xi\right\rangle \left\langle B^{*k}\left(|B|^{2p}-|B^*|^{2p}\right)B^k\eta,\eta\right\rangle \geqslant 0 \end{aligned} \tag{14.2}$$

and

$$\begin{aligned} &\left\langle A^{*k}\left(|A|^{2p}-|A^*|^{2p}\right)A^k\xi,\xi\right\rangle \left\langle B^{*k}|B^*|^{2p}B^k\eta,\eta\right\rangle \\ &+\left\langle A^{*k}|A|^{2p}A^k\xi,\xi\right\rangle \left\langle B^{*k}\left(|B|^{2p}-|B^*|^{2p}\right)B^k\eta,\eta\right\rangle \geqslant 0. \end{aligned} \tag{14.3}$$

Suppose that $A^k \neq 0$ and $B^k \neq 0$. To the contrary, assume that A is not (p,k)-quasihyponormal, then there exists a vector $\xi_0 \in H$ such that

$$\left\langle A^{*k}\left(|A|^{2p}-|A^*|^{2p}\right)A^k\xi_0,\xi_0\right\rangle = \alpha < 0 \quad \text{and} \quad \left\langle A^{*k}|A^*|^{2p}A^k\xi_0,\xi_0\right\rangle = \beta > 0.$$

From (14.2) we have

$$(\alpha+\beta)\left\langle B^{*k}|B|^{2p}B^k\eta,\eta\right\rangle \geqslant \beta\left\langle B^{*k}|B^*|^{2p}B^k\eta,\eta\right\rangle \text{ for all } \eta. \tag{14.4}$$

By using Hölder-McCarcy inequality: For $A \geqslant 0$ and $x \in H$,

(1) $\langle Ax,x\rangle \leqslant ||x||^{2(1-\frac{1}{p})}\langle A^px,x\rangle^{\frac{1}{p}}$ if $p \geqslant 1$

(2) $\langle Ax,x\rangle \geqslant ||x||^{2(1-\frac{1}{p})}\langle A^px,x\rangle^{\frac{1}{p}}$ if $0 < p \leqslant 1$,

we have

$$\begin{aligned} \left\langle B^{*k}|B|^{2p}B^k\eta,\eta\right\rangle &= \left\langle (B^*B)^pB^k\eta,B^k\eta\right\rangle \\ &\leqslant \left\langle (B^*B)B^k\eta,B^k\eta\right\rangle^p ||B^k\eta||^{2(1-p)} \ \text{(by(2))} \\ &= ||B^{k+1}\eta||^{2p}||B^k\eta||^{2(1-p)} \ \text{for all } \eta, \end{aligned}$$

$$\begin{aligned} \left\langle B^{*k}|B^*|^{2p}B^k\eta,\eta\right\rangle &= \left\langle B^{*k-1}(B^*B)^{p+1}B^{k-1}\eta,\eta\right\rangle \\ &= \left\langle (B^*B)^{p+1}B^{k-1}\eta,B^{k-1}\eta\right\rangle \\ &\geqslant \left\langle (B^*B)B^{k-1}\eta,B^{k-1}\eta\right\rangle^{p+1} ||B^{k-1}\eta||^{-2p} \ \text{(by (1))} \\ &= ||B^k\eta||^{2(p+1)}||B^{k-1}\eta||^{-2p} \ \text{for all } \eta, \end{aligned}$$

and

$$(\alpha+\beta)||B^{k+1}\eta||^{2p}||B^k\eta||^{2(1-p)} \geqslant \beta||B^k\eta||^{2(p+1)}||B^{k-1}\eta||^{-2p} \text{ for all } \eta \text{ by (14.4).}$$

Hence, we have

$$\beta||B^k\eta||^2 \leqslant (\alpha+\beta)||B^{k+1}\eta||\,||B^{k-1}\eta|| \text{ for all } \eta, \tag{14.5}$$

and (replacing η by $B\eta$) we have

$$\beta||B^{k+1}\eta||^2 \leqslant (\alpha+\beta)||B^{k+2}\eta||\,||B^k\eta|| \text{ for all } \eta. \tag{14.6}$$

Now let $B = \begin{pmatrix} B_1 & B_2 \\ 0 & B_3 \end{pmatrix}$ on $\overline{\operatorname{ran}(B^k)} \oplus \ker(B^{*k})$. Then B is (p,k)-quasihyponormal by (14.4) and B_1 is p-hyponormal (hence it is normaloid) by Lemma 1. By (14.6) we have

$$\beta||B_1\zeta||^2 \leqslant (\alpha+\beta)||B_1^2\zeta||\,||\zeta|| \text{ for all } \zeta \in \overline{\operatorname{ran}(B^k)},$$

so we have

$$\beta||B_1||^2 \leqslant (\alpha+\beta)||B_1^2|| = (\alpha+\beta)||B_1||^2 \text{ (since } B_1 \text{ is normaloid).}$$

This implies that $B_1 = 0$. Since $B^{k+1}\eta = B_1B^k\eta = 0$ for all η, $B^{k+1} = 0$ and $B^k = 0$ by (14.5). This contradicts the assumption $B^k \neq 0$. Hence A must be (p,k)-quasihyponormal. A similar argument shows that B is also (p,k)-quasihyponormal. □

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