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INEQUALITIES BETWEEN $f(\|A\|)$ AND $\|f(|A|)\|$

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Abstract. Let f be a nonnegative concave function on $[0,\infty)$, and let $\|\cdot\|$ be a unitarily invariant norm on the space of $n\times n$ complex matrices. We prove that, for any $n\times n$ complex matrix A, $f(\|A\|)\leqslant\|f(|A|)\|$ provided the norm $\|\cdot\|$ is normalized. On the other hand, if the norm of the identity matrix is 1, then $f(\|A\|)\geqslant\|f(|A|)\|$ for any matrix A. These results extend the theorems of F. Hiai and X. Zhan that were proved in the case when f is an operator monotone function.

1. Introduction

Let M_n be the space of $n\times n$ complex matrices. The singular values of $A\in M_n$ are denoted by $s_1(A)\geqslant s_2(A)\geqslant\ldots\geqslant s_n(A)$. A norm $\|\cdot\|$ on M_n is called *unitarily invariant* if $\|UAV\|=\|A\|$ for all $A,U,V\in M_n$ with U,V unitary. Throughout the paper, let E and I denote the matrix $\mathrm{diag}(1,0,\ldots,0)$ and the identity matrix, respectively. A norm $\|\cdot\|$ on M_n is called *normalized* whenever $\|E\|=1$. A real-valued function f on $[0,\infty)$ is said to be *operator monotone* if $0\leqslant A\leqslant B$ implies that $f(A)\leqslant f(B)$ for any Hermitian matrices A, $B\in M_n$ of all orders n. Here $\leqslant$ denotes the Löwner partial order, i.e., $A\leqslant B$ iff $B-A$ is a positive-semidefinite matrix.

The following results have been recently proved by F. Hiai and X. Zhan [5] (see also [6]).

THEOREM 1. *Let f be a nonnegative operator monotone function on $[0,\infty)$ and $\|\cdot\|$ be a normalized unitarily invariant norm on M_n. Then for every $A\in M_n$,*

$$f(\|A\|)\leqslant\|f(|A|)\|.$$

THEOREM 2. *Let f be a nonnegative operator monotone function on $[0,\infty)$ and $\|\cdot\|$ be a unitarily invariant norm on M_n with $\|I\|=1$. Then for every $A\in M_n$,*

$$\|f(|A|)\|\leqslant f(\|A\|).$$

COROLLARY 3. *Let f be a nonnegative operator monotone function on $[0,\infty)$ and $\|\cdot\|$ be a unitarily invariant norm on M_n. Then for every $A\in M_n$,*

$$\|E\|\cdot f\left(\frac{\|A\|}{\|E\|}\right)\leqslant\|f(|A|)\|\leqslant\|I\|\cdot f\left(\frac{\|A\|}{\|I\|}\right).$$

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COROLLARY 4. *Let g be a strictly increasing function on $[0,\infty)$ such that $g(0)=0$, $g(\infty)=\infty$ and the inverse function g^{-1} is operator monotone. Let $\|\cdot\|$ be a unitarily invariant norm on M_n. Then for every $A \in M_n$,*

$$\|I\| \cdot g\left(\frac{\|A\|}{\|I\|}\right) \leqslant \|g(|A|)\| \leqslant \|E\| \cdot g\left(\frac{\|A\|}{\|E\|}\right).$$

For the sake of completeness we now give a short proof of the well-known fact that every nonnegative operator monotone function on $[0,\infty)$ is concave. In fact, it is even operator concave, i.e., $f((1-\lambda)A+\lambda B) \geqslant (1-\lambda)f(A)+\lambda f(B)$ for any $\lambda \in [0,1]$ and for any positive-semidefinite matrices A, $B \in M_n$ of all orders n. For the proof of this last assertion see [1, Theorem V.2.5], where the proof is taken from the papers [2] and [4]. As the editor pointed out, the following proof was essentially given in [3, p. 3].

PROPOSITION 5. *If a continuous function $f:[0,\infty)\to[0,\infty)$ is operator monotone, then f is concave.*

Proof. Since f is a continuous function, it is not difficult to verify that it suffices to show that

$$\frac{f(x)+f(y)}{2} \leqslant f\left(\frac{x+y}{2}+\varepsilon\right)$$

for all $x,y \geqslant 0$ and $\varepsilon > 0$. To prove this, we start with the following equality

$$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} \frac{x+y}{2} & \frac{y-x}{2} \\ \frac{y-x}{2} & \frac{x+y}{2} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}.$$

Then, with $\lambda = \frac{(x-y)^2}{4\varepsilon} + \frac{x+y}{2}$, we have

$$\begin{pmatrix} \frac{x+y}{2} & \frac{y-x}{2} \\ \frac{y-x}{2} & \frac{x+y}{2} \end{pmatrix} \leqslant \begin{pmatrix} \frac{x+y}{2}+\varepsilon & 0 \\ 0 & \lambda \end{pmatrix},$$

and so

$$\begin{pmatrix} f(x) & 0 \\ 0 & f(y) \end{pmatrix} \leqslant \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} f(\frac{x+y}{2}+\varepsilon) & 0 \\ 0 & f(\lambda) \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}.$$

It follows that

$$\begin{aligned}
\frac{f(x)+f(y)}{2} &= \left\langle \begin{pmatrix} f(x) & 0 \\ 0 & f(y) \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}, \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix} \right\rangle \\
&\leqslant \left\langle \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} f(\frac{x+y}{2}+\varepsilon) & 0 \\ 0 & f(\lambda) \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}, \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix} \right\rangle \\
&= \left\langle \begin{pmatrix} f(\frac{x+y}{2}+\varepsilon) & 0 \\ 0 & f(\lambda) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle = f(\tfrac{x+y}{2}+\varepsilon). \qquad \square
\end{aligned}$$

In view of Proposition 5 one may conjecture that Theorems 1 and 2 hold if f is a concave nonnegative function on $[0,\infty)$. In this paper we give very short proofs of both conjectures.

2. Results

We first extend Theorem 1 to a class of nonnegative functions on $[0,\infty)$ containing all concave functions.

THEOREM 6. *Let f be a nonnegative function on $[0,\infty)$, and let $\|\cdot\|$ be a normalized unitarily invariant norm.*

(i) Suppose that f satisfies the condition $f(\lambda x) \leqslant \lambda f(x)$ for every $x \geqslant 0$ and $\lambda \geqslant 1$. Then for every $A \in M_n$,

$$f(\|A\|) \leqslant \|f(|A|)\|.$$

(ii) Suppose that f satisfies the condition $f(\lambda x) \geqslant \lambda f(x)$ for every $x \geqslant 0$ and $\lambda \geqslant 1$, and that $f(0)=0$. Then for every $A \in M_n$,

$$f(\|A\|) \geqslant \|f(|A|)\|.$$

Proof. We may assume with no loss of generality that $A \geqslant 0, A \neq 0$.

(i) Since $\|\cdot\|$ is a normalized unitarily invariant norm, the spectrum of A is contained in $[0,\|A\|]$. The assumption on f yields $f(\mu x) \geqslant \mu f(x)$ for every $x \geqslant 0$ and $\mu \in [0,1]$. For $t \in [0,\|A\|]$ we therefore have

$$f(t) \geqslant f(\|A\|)\frac{t}{\|A\|}.$$

Using the functional calculus we obtain

$$f(A) \geqslant \frac{f(\|A\|)}{\|A\|}A, \quad \text{i.e.,} \quad s_j(f(A)) \geqslant s_j\left(\frac{f(\|A\|)}{\|A\|}A\right)$$

for all $j=1,2,\ldots,n$. Since every unitarily invariant norm is a monotone function of singular values, we conclude that

$$\|f(A)\| \geqslant \frac{f(\|A\|)}{\|A\|}\|A\| = f(\|A\|).$$

(ii) Similarly as above, the assumption on f implies that $f(\mu x) \leqslant \mu f(x)$ for every $x \geqslant 0$ and $\mu \in [0,1]$. For $t \in [0,\|A\|]$ we then have

$$f(t) \leqslant f(\|A\|)\frac{t}{\|A\|},$$

and so

$$\|f(A)\| \leqslant \frac{f(\|A\|)}{\|A\|}\|A\| = f(\|A\|). \qquad \square$$

Imitating the well-known proof of Jensen's inequality we now prove the following extension of Theorem 2.

THEOREM 7. *Let f be a nonnegative function on $[0,\infty)$, and let $\|\cdot\|$ be a unitarily invariant norm with $\|I\| = 1$.*

(i) If f is a concave function, then for every $A \in M_n$,

$$f(\|A\|) \geqslant \|f(|A|)\|.$$

(ii) If f is a convex function with $f(0) = 0$, then for every $A \in M_n$,

$$f(\|A\|) \leqslant \|f(|A|)\|.$$

Proof. We may assume that $A \geqslant 0, A \neq 0$.

(i) We define

$$\beta = \inf_{t<\|A\|} \frac{f(\|A\|) - f(t)}{\|A\| - t}.$$

Since the function f is nonnegative, it is increasing, and therefore $\beta \geqslant 0$. The concavity of f implies that

$$\beta \geqslant \frac{f(t) - f(\|A\|)}{t - \|A\|}$$

for $t > \|A\|$. From this inequality and the definition of β we conclude that the inequality

$$f(t) \leqslant f(\|A\|) + \beta(t - \|A\|)$$

holds for all $t \geqslant 0$, and therefore

$$s_j(f(A)) \leqslant s_j((f(\|A\|) - \beta\|A\|)I + \beta A)$$

for all $j = 1, 2, \ldots, n$. Since every unitarily invariant norm is a monotone function of singular values and since $\beta\|A\| \leqslant f(\|A\|) - f(0) \leqslant f(\|A\|)$, we obtain

$$\begin{aligned}\|f(A)\| &\leqslant \|(f(\|A\|) - \beta\|A\|)I + \beta A\| \\ &\leqslant (f(\|A\|) - \beta\|A\|)\|I\| + \beta\|A\| = f(\|A\|),\end{aligned}$$

applying the triangle inequality and the assumption that $\|I\| = 1$.

(ii) Similarly as above, we define

$$\beta = \sup_{t<\|A\|} \frac{f(\|A\|) - f(t)}{\|A\| - t},$$

and note that $\beta\|A\| \geqslant f(\|A\|) - f(0) = f(\|A\|)$. The convexity of f gives

$$f(t) \geqslant f(\|A\|) + \beta(t - \|A\|)$$

for all $t \geqslant 0$, which implies that

$$f(A) + (\beta\|A\| - f(\|A\|))I \geqslant \beta A.$$

Using the triangle inequality we conclude from this that

$$\|f(A)\| + (\beta\|A\| - f(\|A\|)\|I\| \geqslant \beta\|A\|.$$

Since $\|I\| = 1$ by the assumption, we finally obtain the desired inequality

$$\|f(A)\| \geqslant f(\|A\|). \qquad \square$$

Combining Theorems 6 and 7 we obtain the following generalizations of Corollaries 3 and 4.

COROLLARY 8. *Let f be a nonnegative function on $[0,\infty)$, and let $\|\cdot\|$ be a unitarily invariant norm.*

(i) Let f be a concave function. Then for every $A \in M_n$,

$$\|E\| \cdot f\left(\frac{\|A\|}{\|E\|}\right) \leqslant \|f(|A|)\| \leqslant \|I\| \cdot f\left(\frac{\|A\|}{\|I\|}\right).$$

(ii) Let f be a convex function with $f(0) = 0$. Then for every $A \in M_n$,

$$\|I\| \cdot f\left(\frac{\|A\|}{\|I\|}\right) \leqslant \|f(|A|)\| \leqslant \|E\| \cdot f\left(\frac{\|A\|}{\|E\|}\right).$$

Proof. We introduce the unitarily invariant norms $\|\cdot\|^E := \frac{\|\cdot\|}{\|E\|}$ and $\|\cdot\|^I := \frac{\|\cdot\|}{\|I\|}$, which obviously satisfy the conditions $\|E\|^E = 1$ and $\|I\|^I = 1$.

(i) Every nonnegative concave function on $[0,\infty)$ satisfies the condition $f(\lambda x) \leqslant \lambda f(x)$ for every $x \geqslant 0$ and $\lambda \geqslant 1$. By means of $\|\cdot\|^E$ and Theorem 6 (i) we obtain the first inequality and similarly with $\|\cdot\|^I$ and Theorem 7 (i) the second one.

(ii) Since the proof is similar, we omit it. $\square$

Theorem 6 remains valid under a weaker assumption that $\|E\| \geqslant 1$. However, Corollary 8 gives a better estimate, which makes the case $\|E\| > 1$ irrelevant. Similarly, Theorem 7 is true in the case $\|I\| \leqslant 1$, but only the case $\|I\| = 1$ is interesting.

As in [5] we now consider the case of the equality in Theorem 6. Recall that a matrix norm $\|\cdot\|$ is *strictly increasing* if $0 \leqslant A \leqslant B$ and $\|A\| = \|B\|$ imply $A = B$.

THEOREM 9. *Let f be a nonnegative concave function on $[0,\infty)$ with the property that there is no $\varepsilon > 0$ such that the restriction of f on $[0,\varepsilon]$ is linear. Let $\|\cdot\|$ be a strictly increasing normalized unitarily invariant norm on M_n with $n \geqslant 2$. Then $f(\|A\|) = \|f(|A|)\|$ if and only if $f(0) = 0$ and* $\operatorname{rank} A \leqslant 1$.

Proof. We may assume without any loss of generality that $A = \operatorname{diag}(s_1, s_2, \ldots, s_n)$, where $s_1 \geqslant s_2 \geqslant \cdots s_n \geqslant 0$. First assume that $f(0) = 0$ and $A = \operatorname{diag}(\lambda, 0, \ldots, 0)$ with $\lambda \geqslant 0$. Then $\|A\| = \lambda$ by the normalization assumption, and so

$$\|f(|A|)\| = \|f(\lambda)\operatorname{diag}(1,0,\ldots,0)\| = f(\lambda) = f(\|A\|).$$

Conversely, assume that $f(\|A\|) = \|f(|A|)\|$. If $A = 0$, then $f(0) = 0$. Indeed, otherwise we would have $f(\|A\|) = f(0) < \|f(0)I\| = \|f(|A|)\|$, since the norm $\|\cdot\|$ is normalized and strictly increasing.

Suppose now that $A = \mathrm{diag}(s_1, s_2, \ldots, s_n) \neq 0$, so that $s_1 > 0$. Putting $t_i := s_i/s_1$, we have

$$\begin{aligned}\|f(A)\| &= \|\mathrm{diag}(f(s_1), f(t_2 s_1), \ldots, f(t_n s_1))\| \\ &\geqslant \|\mathrm{diag}(f(s_1), t_2 f(s_1), \ldots, t_n f(s_1))\| \\ &= f(s_1)\|\mathrm{diag}(1, t_2, \ldots, t_n)\|,\end{aligned}$$

since the assumption on f imply that $f(tx) \geqslant tf(x)$ for every $x > 0$ and every $t \in [0,1]$. Assume that $t_2 > 0$. Since the norm $\|\cdot\|$ is strictly increasing and normalized, we have $\|\mathrm{diag}(1, t_2, \ldots, t_n)\| > \|\mathrm{diag}(1, 0, \ldots, 0)\| = 1$. The assumptions on f yield $f(tx) < tf(x)$ for every $x > 0$ and $t > 1$, so that

$$\begin{aligned}f(\|A\|) &= f(s_1\|\mathrm{diag}(1, t_2, \ldots, t_n)\|) \\ &< f(s_1)\|\mathrm{diag}(1, t_2, \ldots, t_n)\| \\ &\leqslant \|\mathrm{diag}(f(s_1), f(s_2), \ldots, f(s_n))\| = \|f(A)\|,\end{aligned}$$

which is a contradiction. It follows that $s_2 = \cdots = s_n = 0$, or equivalently $\mathrm{rank}\, A = 1$. Since $f(\|A\|) = \|f(|A|)\|$ means

$$f(s_1) = \|f(s_1)\mathrm{diag}(1, 0, \ldots, 0)\| = \|f(s_1)\mathrm{diag}(1, 0, \ldots, 0) + f(0)\mathrm{diag}(0, 1, \ldots, 1)\|,$$

we have $f(0) = 0$ because of the strict increasingness of $\|\cdot\|$ again. $\square$

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REFERENCES

[1] R. BHATIA, *Matrix Analysis*, Springer-Verlag, New York, 1997.
[2] F. HANSEN, *An operator inequality*, Math. Ann., **246** 3 (1979/80), 249–250.
[3] F. HANSEN, *Operator monotone functions of several variables*, Math. Inequal. Appl., **6** (2003), no. 1, 1–17.
[4] F. HANSEN AND G. K. PEDERSEN, *Jensen's inequality for operators and Löwner's theorem*, Math. Ann. **258** 3 (1981/82), 229–241.
[5] F. HIAI AND X. ZHAN, *Inequalities involving unitarily invariant norms and operator monotone functions*, Lin. Alg. Appl., **341** (2002), 151–169.
[6] X. ZHAN, *Matrix inequalities*, Springer-Verlag, Berlin, 2002.

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