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GENERAL HILBERT'S AND HARDY'S INEQUALITIES

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Abstract. In this paper we make some further generalizations of well known Hilbert's inequality and its equivalent form in two dimensional case. We also derive some results on Hardy's inequality. Then we apply our general results to homogeneous functions. A reverses of Hilbert's inequality are also given in integral case. Many other results of this type in recent years, follows as a special case of our results.

1. Introduction

Let us, firstly, repeat the well known Hilbert's inequality and its equivalent form in both integral and discrete case

THEOREM A. *If f and $g \in L^2[0,\infty)$, then the following inequalities hold and are equivalent*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy \leqslant \pi\left(\int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx\right)^{\frac{1}{2}},$$

and

$$\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y}dx\right)^2 dy \leqslant \pi^2 \int_0^\infty f^2(x)dx,$$

where π and π^2 are the best constants.

THEOREM B. *The following inequalities hold and are equivalent*

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{m+n} \leqslant \pi\left(\sum_{n=1}^\infty {a_n}^2 \sum_{m=1}^\infty {b_m}^2\right)^{\frac{1}{2}},$$

$$\sum_{m=1}^\infty \left(\sum_{n=1}^\infty \frac{a_m}{m+n}\right)^2 \leqslant \pi^2\left(\sum_{m=1}^\infty {a_n}^2\right),$$

where π and π^2 are the best constants.

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In recent years there were lots of generalizations of these theorems. Let's mention some of the authors who gave many results: Jichang, Yang , Hong Yong, Gavrea, Peachey, Rassias.

Brnetić and Pečarić ([2],[3]) considered the case when the kernel is $K(x,y) = (x+y)^{-s}$, and they obtained the following result in both equivalent forms:

THEOREM C. *If $s > 0$, $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, then the following inequalities hold and are equivalent*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^s} dxdy$$

$$< P\left(\int_0^\infty x^{1-s+p(A_1-A_2)} f^p(x)dx\right)^{\frac{1}{p}} \left(\int_0^\infty x^{1-s+q(A_2-A_1)} g^q(x)dx\right)^{\frac{1}{q}}, \tag{1}$$

and

$$\int_0^\infty y^{(s-1)(p-1)+p(A_1-A_2)} \left(\int_0^\infty \frac{f(x)}{(x+y)^s} dx\right)^p dy$$

$$< P^p \left(\int_0^\infty x^{1-s+p(A_1-A_2)} f^p(x)dx\right) \tag{2}$$

where $P = B(1-A_2p, s-1+A_2p)^{\frac{1}{p}} B(1-A_1q, s-1+A_1q)^{\frac{1}{q}}$, $A_1 \in (\frac{1-s}{q}, \frac{1}{q})$, $A_2 \in (\frac{1-s}{p}, \frac{1}{p})$ and B is a beta function.

Further, Jichang and Rassias gave more general results ([7]), concerning symmetrical homogeneous functions. More precisely, they obtained following

THEOREM D. *Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $K(x,y)$ be nonnegative, symmetrical and homogeneous function of degree $-s$, $\max\{\frac{1}{p}, \frac{1}{q}\} < s$, $K(1,y)$ be strictly decreasing function of y, $f(x)$, $g(y)$ nonnegative functions and $I(r) = \int_0^\infty K(1,u)u^{-\frac{1}{r}}$. Then the following inequality is valid*

$$\int_a^b \int_a^b K(x,y)f(x)g(y)dxdy$$

$$\leqslant \left(\int_a^b \Big(I(q) - \varphi(q,x)\Big) x^{1-s} f(x)^p dx\right)^{\frac{1}{p}} \left(\int_a^b \Big(I(p) - \varphi(p,y)\Big) y^{1-s} g(y)^q dy\right)^{\frac{1}{q}}, \tag{3}$$

where

$$\varphi(r,x) = \left(\frac{a}{x}\right)^{1-\frac{1}{r}} \int_0^1 K(1,u)u^{-\frac{1}{r}} du + \left(\frac{x}{b}\right)^{s+\frac{1}{r}-1} \int_0^1 K(1,u)u^{s+\frac{1}{r}-2} du.$$

In this paper we obtain some general results for estimating the integral

$$\int_{\Omega}\int_{\Omega} K(x,y)f(x)g(y)d\mu_1(x)d\mu_2(y),$$

and we apply those results on kernel $K(x,y)$. We obtain many inequalities which are generalizations of the previously mentioned results. We also apply reverse Hölder's inequality ([14]) to obtain reverse inequalities. So let's start with the general case.

2. General case

In this section we shall state our general results. We suppose that all integrals converges and shall omit these types of conditions. So we have following

THEOREM 1. *If* $\frac{1}{p}+\frac{1}{q}=1$ *with* $p>1$ *and* $K(x,y), f(x), g(y), \varphi(x), \psi(y)$ *be nonnegative functions, then the following inequalities hold and are equivalent*

$$\int_{\Omega}\int_{\Omega} K(x,y)f(x)g(y)d\mu_1(x)d\mu_2(y)$$
$$\leqslant \left(\int_{\Omega}\varphi(x)^p F(x)f(x)^p d\mu_1(x)\right)^{\frac{1}{p}}\left(\int_{\Omega}\psi(y)^q G(y)g(y)^q d\mu_2(y)\right)^{\frac{1}{q}} \tag{4}$$

and

$$\int_{\Omega} G(y)^{1-p}\psi(y)^{-p}\left(\int_{\Omega}K(x,y)f(x)d\mu_1(x)\right)^p d\mu_2(y) \tag{5}$$
$$\leqslant \int_{\Omega}\varphi(x)^p F(x)f(x)^p d\mu_1(x),$$

where $F(x)=\int_{\Omega}\frac{K(x,y)}{\psi(y)^p}d\mu_2(y)$ *and* $G(y)=\int_{\Omega}\frac{K(x,y)}{\varphi(x)^q}d\mu_1(x)$.
If $0<p<1$, *then the reverse inequalities in (4) and (5) are valid as well as the inequality*

$$\int_{\Omega} F(x)^{1-q}\varphi(x)^{-q}\left(\int_{\Omega}K(x,y)g(y)d\mu_2(y)\right)^q d\mu_1(x)$$
$$\leqslant \int_{\Omega}\psi(y)^q G(y)g(y)^q d\mu_2(y). \tag{6}$$

Proof. We start with the following identity

$$\int_{\Omega}\int_{\Omega} K(x,y)f(x)g(y)d\mu_1(x)d\mu_2(y)=\int_{\Omega}\int_{\Omega} K(x,y)f(x)\frac{\varphi(x)}{\psi(y)}g(y)\frac{\psi(y)}{\varphi(x)}d\mu_1(x)d\mu_2(y).$$

Now, if we apply Hölder's inequality, we obtain

$$\int_{\Omega}\int_{\Omega} K(x,y)f(x)g(y)d\mu_1(x)d\mu_2(y)$$

$$\leqslant \left(\int_\Omega \varphi(x)^p F(x) f(x)^p d\mu_1(x) \right)^{\frac{1}{p}} \left(\int_\Omega \psi(y)^q G(y) g(y)^q d\mu_2(y) \right)^{\frac{1}{q}}.$$

Let us show that the inequalities (4) and (5) are equivalent. Suppose that the inequality (4) is valid. If we put

$$g(y) = G(y)^{1-p}\psi(y)^{-p}\left(\int_\Omega K(x,y) f(x) d\mu_1(x) \right)^{p-1},$$

taking into account $\frac{1}{p} + \frac{1}{q} = 1$ and using (4), we have

$$\int_\Omega G(y)^{1-p}\psi(y)^{-p}\left(\int_\Omega K(x,y) f(x) d\mu_1(x) \right)^{p} d\mu_2(y)$$

$$= \int_\Omega \int_\Omega K(x,y) f(x) g(y) d\mu_1(x) d\mu_2(y)$$

$$\leqslant \left(\int_\Omega \varphi(x)^p F(x) f(x)^p d\mu_1(x) \right)^{\frac{1}{p}} \left(\int_\Omega \psi(y)^q G(y) g(y)^q d\mu_2(y) \right)^{\frac{1}{q}}$$

$$= \left(\int_\Omega \varphi(x)^p F(x) f(x)^p d\mu_1(x) \right)^{\frac{1}{p}}$$

$$\cdot \left(\int_\Omega G(y)^{1-p}\psi(y)^{-p}\left(\int_\Omega K(x,y) f(x) d\mu_1(x) \right)^{p} d\mu_2(y) \right)^{\frac{1}{q}}$$

from where we have (5).

Now let's suppose that the inequality (5) is valid. By applying Hölder's inequality and (5), we obtain

$$\int_\Omega \int_\Omega K(x,y) f(x) g(y) d\mu_1(x) d\mu_2(y)$$

$$= \int_\Omega \left(\psi(y)^{-1} G(y)^{-\frac{1}{q}} \int_\Omega K(x,y) f(x) d\mu_1(x) \right) \psi(y) G(y)^{\frac{1}{q}} g(y) d\mu_2(y)$$

$$\leqslant \left(\int_\Omega G(y)^{1-p}\psi(y)^{-p}\left(\int_\Omega K(x,y) f(x) d\mu_1(x) \right)^{p} d\mu_2(y) \right)^{\frac{1}{p}}$$

$$\cdot \left(\int_\Omega \psi(y)^q G(y) g(y)^q d\mu_2(y) \right)^{\frac{1}{q}}$$

$$\leqslant \left(\int_\Omega \varphi(x)^p F(x) f(x)^p d\mu_1(x) \right)^{\frac{1}{p}} \left(\int_\Omega \psi(y)^q G(y) g(y)^q d\mu_2(y) \right)^{\frac{1}{q}},$$

so we have (4). While inequality (4) is valid, the inequality (5) holds, too.
We obtain the reverse inequalities in a similar way, by using reverse Hölder's inequality ([14]). That completes the proof. □

REMARK 1. Equality in the previous theorem is possible if and only if it holds in Hölder's inequality i.e.

$$\left(f(x)\frac{\varphi(x)}{\psi(y)}\right)^p = K\left(g(y)\frac{\psi(y)}{\varphi(x)}\right)^q,$$

wherefrom we obtain $f(x)^p = K_1\varphi(x)^{-(p+q)}$ and $g(y)^q = K_2\psi(y)^{-(p+q)}$, for arbitrary constants K_1 and K_2. It is possible only if

$$\int_\Omega F(x)\varphi(x)^{-q}d\mu_1(x) < \infty \quad \text{and} \quad \int_\Omega G(y)\psi(y)^{-p}d\mu_2(y) < \infty.$$

Otherwise, inequalities in the Theorem 1 are strict.

It is of great importance to consider the case when the functions $F(x)$ and $G(y)$, from the Theorem 1, are bounded. More precisely, we have the following result

THEOREM 2. *Let* $\frac{1}{p}+\frac{1}{q}=1$ *with* $p>1$, $K(x,y), f(x), g(y), \varphi(x), \psi(y)$ *be non-negative functions and* $F(x)=\int_\Omega \frac{K(x,y)}{\psi(y)^p}d\mu_2(y) \leqslant F_1(x)$, $G(y)=\int_\Omega \frac{K(x,y)}{\varphi(x)^q}d\mu_1(x) \leqslant G_1(x)$. *Then the following inequalities hold and are equivalent*

$$\int_\Omega\int_\Omega K(x,y)f(x)g(y)d\mu_1(x)d\mu_2(y)$$

$$\leqslant \left(\int_\Omega \varphi(x)^pF_1(x)f(x)^pd\mu_1(x)\right)^{\frac{1}{p}}\left(\int_\Omega \psi(y)^qG_1(y)g(y)^qd\mu_2(y)\right)^{\frac{1}{q}} \tag{7}$$

and

$$\int_\Omega G_1(y)^{1-p}\psi(y)^{-p}\left(\int_\Omega K(x,y)f(x)d\mu_1(x)\right)^p d\mu_2(y)$$

$$\leqslant \int_\Omega \varphi(x)^pF_1(x)f(x)^pd\mu_1(x), \tag{8}$$

If $0<p<1$, $F(x)\geqslant F_1(x)$ *and* $G(y)\leqslant G_1(y)$, *then the reverse inequalities in (7) and (8) are valid as well as the inequality*

$$\int_\Omega F_1(x)^{1-q}\varphi(x)^{-q}\left(\int_\Omega K(x,y)g(y)d\mu_2(y)\right)^q d\mu_1(x)$$

$$\leqslant \int_\Omega \psi(y)^qG_1(y)g(y)^qd\mu_2(y). \tag{9}$$

3. Hardy type inequalities

Inequalities (5) and (8) are so called Hardy type inequalities. Now we shall consider some special cases of Hardy's inequalities. If we put

$$K(x,y)=\begin{cases} h(y), & x\leqslant y\\ 0, & x>y\end{cases}$$

in Theorem 1, where $\Omega=[a,b]$, $a<b$, we obtain following result

THEOREM 3. *Let* $\frac{1}{p}+\frac{1}{q}=1$ *with* $p>1$, *and let* $h(y)$, $f(x)$, $g(y)$, $\varphi(x)$, $\psi(y)$ *be nonnegative functions. Then the following inequalities hold and are equivalent*

$$\begin{aligned}&\int_a^b\int_a^y h(y)f(x)g(y)d\mu_1(x)d\mu_2(y)\\ &\leqslant\left(\int_a^b\varphi(x)^pf(x)^p\Big(\int_x^b H(y)d\mu_2(y)\Big)d\mu_1(x)\right)^{\frac{1}{p}}\\ &\cdot\left(\int_a^b\psi(y)^qg(y)^qh(y)\Big(\int_a^y\varphi(x)^{-q}d\mu_1(x)\Big)d\mu_2(y)\right)^{\frac{1}{q}}\end{aligned}\tag{10}$$

and

$$\begin{aligned}&\int_a^b H(y)\Big(\int_a^y\varphi(x)^{-q}d\mu_1(x)\Big)^{1-p}\Big(\int_a^y f(x)d\mu_1(x)\Big)^pd\mu_2(y)\\ &\leqslant\int_a^b\varphi(x)^pf(x)^p\Big(\int_x^b H(y)d\mu_2(y)\Big)d\mu_1(x),\end{aligned}\tag{11}$$

where $H(y)=h(y)\psi(y)^{-p}$. *If* $0<p<1$, *then the reverse inequalities in (10) and (11) are valid. Further, if* $p<0$, *then the inequality (11) is valid, as well as the reverse in (10).*

Also, if we put

$$K(x,y)=\begin{cases} 0, & x\leqslant y\\ h(y), & x>y\end{cases}$$

in Theorem 1 we have

THEOREM 4. *Let* $\frac{1}{p}+\frac{1}{q}=1$ *with* $p>1$, *and let* $h(y)$, $f(x)$, $g(y)$, $\varphi(x)$, $\psi(y)$ *be nonnegative functions. Then the following inequalities hold and are equivalent*

$$\begin{aligned}&\int_a^b\int_y^b h(y)f(x)g(y)d\mu_1(x)d\mu_2(y)\leqslant\left(\int_a^b\varphi(x)^pf(x)^p\Big(\int_a^x H(y)d\mu_2(y)\Big)d\mu_1(x)\right)^{\frac{1}{p}}\\ &\cdot\left(\int_a^b\psi(y)^qg(y)^qh(y)\Big(\int_y^b\varphi(x)^{-q}d\mu_1(x)\Big)d\mu_2(y)\right)^{\frac{1}{q}}\end{aligned}\tag{12}$$

and

$$\int_a^b H(y)\Big(\int_y^b \varphi(x)^{-q} d\mu_1(x)\Big)^{1-p}\Big(\int_y^b f(x)d\mu_1(x)\Big)^p d\mu_2(y)$$

$$\leqslant \int_a^b \varphi(x)^p f(x)^p \Big(\int_a^x H(y)d\mu_2(y)\Big)d\mu_1(x). \tag{13}$$

If $0 < p < 1$*, then the reverse inequalities in (12) and (13) are valid. Further, if* $p < 0$*, then the inequality (13) is valid, as well as the reverse in (12).*

Note that some authors obtained similar inequalities of Hardy type. Such inequalities can be found in [1] and [16]. See also [10] (page 163).

Further,we shall consider some special cases of Theorems 3 and 4. Namely, if we put $h(y) = \frac{1}{y}$, $\varphi(x) = x^{A_1}$, $\psi(y) = y^{A_2}$ in these theorems, we obtain following results

COROLLARY 1. *Let* $\frac{1}{p} + \frac{1}{q} = 1$ *with* $p > 1$*, and let* $f(x)$*,* $g(y)$ *be nonnegative functions and* $0 \leqslant a < b \leqslant \infty$*. Then the following inequalities hold and are equivalent*

$$\int_a^b \int_a^y \frac{f(x)g(y)}{y} dxdy \leqslant \frac{|1-qA_1|^{-\frac{1}{q}}}{|pA_2|^{\frac{1}{p}}} \left(\int_a^b x^{p(A_1-A_2)}\left|1-\left(\frac{x}{b}\right)^{pA_2}\right| f(x)^p dx\right)^{\frac{1}{p}}$$

$$\cdot \left(\int_a^b y^{q(A_2-A_1)}\left|1-\left(\frac{a}{y}\right)^{1-qA_1}\right| g(y)^q dy\right)^{\frac{1}{q}} \tag{14}$$

and

$$\int_a^b y^{p(A_1-A_2)-p}\left|1-\left(\frac{a}{y}\right)^{1-qA_1}\right|^{1-p}\Big(\int_a^y f(x)dx\Big)^p dy$$

$$\leqslant \frac{|1-qA_1|^{1-p}}{|pA_2|} \int_a^b x^{p(A_1-A_2)}\left|1-\left(\frac{x}{b}\right)^{pA_2}\right| f(x)^p dx \tag{15}$$

for any constants $A_1 \neq \frac{1}{q}$ *and* $A_2 \neq 0$ *such that all integrals converges. If* $a = 0$*, inequalities (14) and (15) hold under the condition* $1 - qA_1 > 0$*, and the case* $b = \infty$ *holds if* $pA_2 > 0$*. The reverse inequalities, when* $p < 1$*, are fulfilled as in Theorem 3.*

COROLLARY 2. *Let* $\frac{1}{p} + \frac{1}{q} = 1$ *with* $p > 1$*, and let* $f(x)$*,* $g(y)$ *be nonnegative functions and* $0 \leqslant a < b \leqslant \infty$*. Then the following inequalities hold and are equivalent*

$$\int_a^b \int_y^b \frac{f(x)g(y)}{y} dxdy \leqslant \frac{|1-qA_1|^{-\frac{1}{q}}}{|pA_2|^{\frac{1}{p}}} \left(\int_a^b x^{p(A_1-A_2)}\left|1-\left(\frac{x}{a}\right)^{pA_2}\right| f(x)^p dx\right)^{\frac{1}{p}}$$

$$\cdot \left(\int_a^b y^{q(A_2-A_1)}\left|1-\left(\frac{b}{y}\right)^{1-qA_1}\right| g(y)^q dy\right)^{\frac{1}{q}} \tag{16}$$

and

$$\int_a^b y^{p(A_1-A_2)-p}\left|1-\left(\frac{b}{y}\right)^{1-qA_1}\right|^{1-p}\left(\int_y^b f(x)dx\right)^p dy$$
$$\leqslant \frac{|1-qA_1|^{1-p}}{|pA_2|}\int_a^b x^{p(A_1-A_2)}\left|1-\left(\frac{x}{a}\right)^{pA_2}\right| f(x)^p dx \tag{17}$$

for any constants $A_1 \neq \frac{1}{q}$ *and* $A_2 \neq 0$ *such that all integrals converges. If* $a = 0$*, inequalities (16) and (17) hold under the condition* $pA_2 < 0$*, and the case* $b = \infty$ *holds if* $1 - qA_1 < 0$*. The reverse inequalities, when* $p < 1$*, are fulfilled as in Theorem 4.*

We see that the cases $a = 0$ and $b = \infty$ have additional conditions on the constants A_1 and A_2. For example, if $a = 0$ and $b = \infty$, we have

COROLLARY 3. *Let* $\frac{1}{p} + \frac{1}{q} = 1$. *Then the following inequalities hold*

$$\int_0^\infty \int_0^y \frac{f(x)g(y)}{y} dxdy$$
$$< \frac{|1-qA_1|^{-\frac{1}{q}}}{|pA_2|^{\frac{1}{p}}}\left(\int_0^\infty x^{p(A_1-A_2)} f(x)^p dx\right)^{\frac{1}{p}}\left(\int_0^\infty y^{q(A_2-A_1)} g(y)^q dy\right)^{\frac{1}{q}} \tag{18}$$

if $p > 1,\ A_1 < \frac{1}{q},\ A_2 > 0$,

$$\int_0^\infty y^{p(A_1-A_2)}\left(\frac{1}{y}\int_0^y f(x)dx\right)^p dy < \frac{|1-qA_1|^{1-p}}{|pA_2|}\int_0^\infty x^{p(A_1-A_2)} f(x)^p dx \tag{19}$$

if $p > 1,\ A_1 < \frac{1}{q},\ A_2 > 0$ *or* $p < 0,\ A_1 < \frac{1}{q},\ A_2 < 0$,

and

$$\int_0^\infty \int_y^\infty \frac{f(x)g(y)}{y} dxdy$$
$$< \frac{|1-qA_1|^{-\frac{1}{q}}}{|pA_2|^{\frac{1}{p}}}\left(\int_0^\infty x^{p(A_1-A_2)} f(x)^p dx\right)^{\frac{1}{p}}\left(\int_0^\infty y^{q(A_2-A_1)} g(y)^q dy\right)^{\frac{1}{q}} \tag{20}$$

if $p > 1,\ A_1 > \frac{1}{q},\ A_2 < 0$,

$$\int_0^\infty y^{p(A_1-A_2)}\left(\frac{1}{y}\int_y^\infty f(x)dx\right)^p dy < \frac{|1-qA_1|^{1-p}}{|pA_2|}\int_0^\infty x^{p(A_1-A_2)} f(x)^p dx \tag{21}$$

if $p > 1,\ A_1 > \frac{1}{q},\ A_2 < 0$ *or* $p < 0,\ A_1 > \frac{1}{q},\ A_2 > 0$.
The reverse in (18) holds if $p < 0,\ A_1 < \frac{1}{q},\ A_2 < 0$ *or* $0 < p < 1,\ A_1 > \frac{1}{q},\ A_2 > 0$*, and the reverse in (19) holds if* $0 < p < 1,\ A_1 > \frac{1}{q},\ A_2 > 0$ *. Further, the reverse in (20) is valid if* $p < 0,\ A_1 > \frac{1}{q},\ A_2 > 0$ *or* $0 < p < 1,\ A_1 < \frac{1}{q},\ A_2 < 0$*, as well as the reverse in (21) if* $0 < p < 1,\ A_1 < \frac{1}{q},\ A_2 < 0$.

Note that the inequalities in Corollary 3 are strict (see Remark 1).

REMARK 2. Let's put $p(A_1 - A_2) = \varepsilon$ and observe the constant $\dfrac{|1-qA_1|^{1-p}}{|pA_2|}$ as the function of A_1. We obtain that this function reaches the minimum value in $A_1 = \dfrac{1+\varepsilon}{pq}$, if $p > 1$ or $p < 0$, and the maximum if $0 < p < 1$. Therefrom we obtain inequalities

$$\int_0^\infty \int_0^y \frac{f(x)g(y)}{y} dxdy < \left|\frac{p}{\varepsilon - p + 1}\right| \left(\int_0^\infty x^\varepsilon f(x)^p dx\right)^{\frac{1}{p}} \left(\int_0^\infty y^{\varepsilon(1-q)} g(y)^q dy\right)^{\frac{1}{q}} \tag{22}$$

if $p > 1$, $\varepsilon < p - 1$ (the reverse if $p < 0$, $\varepsilon > p - 1$ or $0 < p < 1$, $\varepsilon > p - 1$),

$$\int_0^\infty y^{\varepsilon - p} \left(\int_0^y f(x)dx\right)^p dy < \left|\frac{p}{\varepsilon - p + 1}\right|^p \int_0^\infty x^\varepsilon f(x)^p dx \tag{23}$$

if $p > 1$, $\varepsilon < p - 1$ or $p < 0$, $\varepsilon > p - 1$ (the reverse if $0 < p < 1$, $\varepsilon > p - 1$),

$$\int_0^\infty \int_0^y \frac{f(x)g(y)}{y} dxdy < \left|\frac{p}{\varepsilon - p + 1}\right| \left(\int_0^\infty x^\varepsilon f(x)^p dx\right)^{\frac{1}{p}} \left(\int_0^\infty y^{\varepsilon(1-q)} g(y)^q dy\right)^{\frac{1}{q}} \tag{24}$$

if $p > 1$, $\varepsilon > p - 1$ (the reverse if $p < 0$, $\varepsilon < p - 1$ or $0 < p < 1$, $\varepsilon < p - 1$),

$$\int_0^\infty y^{\varepsilon - p} \left(\int_y^\infty f(x)dx\right)^p dy < \left|\frac{p}{\varepsilon - p + 1}\right|^p \int_0^\infty x^\varepsilon f(x)^p dx, \tag{25}$$

if $p > 1$, $\varepsilon > p - 1$ or $p < 0$, $\varepsilon < p - 1$ (the reverse if $0 < p < 1$, $\varepsilon < p - 1$). We see that $\left|\dfrac{p}{\varepsilon - p + 1}\right|$ is the best possible constant. Note also that for $\varepsilon = 0$ we obtain inequality

$$\int_0^\infty \left(\frac{1}{y}\int_0^y f(x)dx\right)^p dy < q^p \int_0^\infty f(x)^p dx.$$

These results are the generalizations of Kufner's paper [9]. Moreover, if $\varepsilon = p - k$ we obtain results from [14].

REMARK 3. If we put $a = 0$, $A_1 = \dfrac{p+1-k}{pq}$, $A_2 = \dfrac{k-1}{p^2}$ in Corollary 1, the inequality (15) becomes

$$\int_0^b y^{-k} \left(\int_0^y f(x)dx\right)^p dy \leqslant \left(\frac{p}{k-1}\right)^p \int_0^b x^{p-k}\left(1 - \left(\frac{x}{b}\right)^{\frac{k-1}{p}}\right) f(x)^p dx,$$

if $1-qA_1>0$. It is easy to see that the inequality is strict, which is the result from [4]. Further, by putting $b=\infty$, $A_1=\dfrac{p+1-k}{pq}$, $A_2=\dfrac{k-1}{p^2}$ in the inequality (17), we again obtain the result from [4].

REMARK 4. It is obvious that $\left|1-\left(\frac{x}{b}\right)^{pA_2}\right|\leqslant 1-\left(\frac{a}{b}\right)^{pA_2}$ and $\left|1-\left(\frac{a}{y}\right)^{1-qA_1}\right|\leqslant 1-\left(\frac{a}{b}\right)^{1-qA_1}$ if $1-qA_1>0$ and $pA_2>0$, so from the inequalities (14) and (15) we obtain

$$\int_a^b\int_a^y \frac{f(x)g(y)}{y}dxdy\leqslant K\left(\int_a^b x^{p(A_1-A_2)}f(x)^p dx\right)^{\frac{1}{p}}\left(\int_a^b y^{q(A_2-A_1)}g(y)^q dy\right)^{\frac{1}{q}},$$

and

$$\int_a^b y^{p(A_1-A_2)-p}\left(\int_a^y f(x)dx\right)^p dy\leqslant K^p\int_a^b x^{p(A_1-A_2)}f(x)^p dx,$$

where

$$K=\frac{|1-qA_1|^{-\frac{1}{q}}}{|pA_2|^{\frac{1}{p}}}\left(1-\left(\frac{a}{b}\right)^{pA_2}\right)^{\frac{1}{p}}\left(1-\left(\frac{a}{b}\right)^{1-qA_1}\right)^{\frac{1}{q}}.$$

Now, if $A_1=\dfrac{p+1-k}{pq}$, $A_2=\dfrac{k-1}{p^2}$, then the second inequality is the result from [5].

4. Homogeneous functions

In this section we apply our main results to homogeneous functions. Recall that for homogeneous function of degree $-s$, $s>0$, equality $K(tx,ty)=t^{-s}K(x,y)$ is satisfied.

THEOREM 5. *Let* $\frac{1}{p}+\frac{1}{q}=1$ *with* $p>1$ *and let* $K(x,y)$ *be homogeneous function of degree* $-s$, $s>0$, *strictly decreasing in both variables* x *and* y. *Then the following inequalities hold and are equivalent*

$$\int_a^b\int_a^b K(x,y)f(x)g(y)dxdy$$
$$\leqslant\left(\int_a^b\big(k(pA_2)-\varphi_1(pA_2,x)\big)x^{1-s+p(A_1-A_2)}f(x)^p dx\right)^{\frac{1}{p}}$$
$$\cdot\left(\int_a^b\big(k(2-s-qA_1)-\varphi_2(2-s-qA_1,y)\big)y^{1-s+q(A_2-A_1)}g(y)^q dy\right)^{\frac{1}{q}} \tag{26}$$

and

$$\int_a^b \big(k(2-s-qA_1)-\varphi_2(2-s-qA_1,y)\big)^{1-p} y^{(p-1)(s-1)+p(A_1-A_2)}$$

$$\cdot\Big(\int_a^b K(x,y)f(x)dx\Big)^p dy$$

$$\leqslant \int_a^b \big(k(pA_2)-\varphi_1(pA_2,x)\big)x^{1-s+p(A_1-A_2)}f(x)^p dx, \tag{27}$$

where $A_1 \in (\frac{1-s}{q},\frac{1}{q})$, $A_2 \in (\frac{1-s}{p},\frac{1}{p})$, $k(\alpha)=\int_0^\infty K(1,u)u^{-\alpha}du$ *and*

$$\varphi_1(\alpha,x)=\Big(\frac{a}{x}\Big)^{1-\alpha}\int_0^1 K(1,u)u^{-\alpha}du+\Big(\frac{x}{b}\Big)^{s+\alpha-1}\int_0^1 K(u,1)u^{s+\alpha-2}du,$$

$$\varphi_2(\alpha,y)=\Big(\frac{a}{y}\Big)^{s+\alpha-1}\int_0^1 K(u,1)u^{s+\alpha-2}du+\Big(\frac{y}{b}\Big)^{1-\alpha}\int_0^1 K(1,u)u^{-\alpha}du.$$

If $0<p<1$, $b=\infty$ *and* $K(x,y)$ *is strictly decreasing in* x *and strictly increasing in* y, *then the reverses in (26) and (27) are valid for any* $A_1 \in (\frac{1}{q},\frac{1-s}{q})$ *and* $A_2 \in (\frac{1-s}{p},\frac{1}{p})$ *as well as the inequality*

$$\int_a^\infty \big(k(pA_2)-\varphi_1(pA_2,x)\big)^{1-q}x^{(q-1)(s-1)+q(A_2-A_1)}\Big(\int_a^\infty K(x,y)g(y)dy\Big)^q dx$$

$$\leqslant \int_a^\infty \big(k(2-s-qA_1)-\varphi_2(2-s-qA_1,y)\big)y^{1-s+q(A_2-A_1)}g(y)^q dy.$$

Further,if $0<p<1$, $a=0$ *and* $K(x,y)$ *is strictly increasing in* x *and strictly decreasing in* y, *then the reverses in (26) and (27) are valid for any* $A_1 \in (\frac{1}{q},\frac{1-s}{q})$ *and* $A_2 \in (\frac{1-s}{p},\frac{1}{p})$ *as well as the inequality*

$$\int_0^b \big(k(pA_2)-\varphi_1(pA_2,x)\big)^{1-q}x^{(q-1)(s-1)+q(A_2-A_1)}\Big(\int_0^b K(x,y)g(y)dy\Big)^q dx$$

$$\leqslant \int_0^b \big(k(2-s-qA_1)-\varphi_2(2-s-qA_1,y)\big)y^{1-s+q(A_2-A_1)}g(y)^q dy.$$

Proof. We prove inequality (26). If we put $\varphi(x)=x^{A_1}$ and $\psi(y)=y^{A_2}$ in Theorem 1, we obtain

$$\int_a^b\int_a^b K(x,y)f(x)g(y)dxdy$$

$$\leqslant \left(\int_a^b f(x)^p x^{1-s+p(A_1-A_2)}\left(\int_{\frac{a}{x}}^{\frac{b}{x}} K(1,u)u^{-pA_2}du\right)dx\right)^{\frac{1}{p}}$$

$$\cdot\left(\int_a^b g(y)^q y^{1-s+q(A_2-A_1)}\left(\int_{\frac{y}{b}}^{\frac{y}{a}} K(1,u)u^{qA_1+s-2}du\right)dy\right)^{\frac{1}{q}}$$

Here, we used substitution $u=\frac{y}{x}$. Further, it can be easily shown (see [7]) that if $l(y)=y^{\alpha-1}\int_0^y K(1,u)u^{-\alpha}du\alpha<1$, then

$$l'(y)=y^{\alpha-2}\int_0^y u^{1-\alpha}\frac{\partial K(1,u)}{\partial u}du. \tag{28}$$

Now, since

$$\begin{aligned}\int_{\frac{a}{x}}^{\frac{b}{x}} K(1,u)u^{-pA_2}du &= \int_0^\infty K(1,u)u^{-pA_2}du-\int_0^{\frac{a}{x}} K(1,u)u^{-pA_2}du\\ &\quad -\int_0^{\frac{x}{b}} K(u,1)u^{pA_2+s-2}du,\end{aligned}$$

we obtain, using (28), $\int_{\frac{a}{x}}^{\frac{b}{x}} K(1,u)u^{-pA_2}du \leqslant k(pA_2)-\varphi_1(pA_2,x)$ and analogously $\int_{\frac{y}{b}}^{\frac{y}{a}} K(1,u)u^{qA_1+s-2}du\leqslant k(2-s-qA_1)-\varphi_2(2-s-qA_1,y)$, and the result follows from Theorem 2. Note also that from (28) we obtain conditions $A_1\in(\frac{1-s}{q},\frac{1}{q})$ and $A_2\in(\frac{1-s}{p},\frac{1}{p})$. □

REMARK 5. If the function $K(x,y)$ from previous theorem is symmetrical, then $k(2-s-qA_1)=k(qA_1)$. So, if $\max\{\frac{1}{p},\frac{1}{q}\}<s$, then we can put $A_1=A_2=\frac{1}{pq}$ in Theorem 5 and obtain Theorem D from the Introduction.

If $a=0$ and $b=\infty$ in the previous theorem, we obtain inequalities for arbitrary nonnegative homogeneous function of degree $-s$.

COROLLARY 4. *If* $\frac{1}{p}+\frac{1}{q}=1$ *with* $p>1$ *and* $K(x,y)$ *is homogeneous function of degree* $-s$*, then the following inequalities hold and are equivalent*

$$\int_0^\infty\int_0^\infty K(x,y)f(x)g(y)dxdy$$
$$< L\left(\int_0^\infty x^{1-s+p(A_1-A_2)}f(x)^p dx\right)^{\frac{1}{p}}\left(\int_0^\infty y^{1-s+q(A_2-A_1)}g(y)^q dy\right)^{\frac{1}{q}}, \tag{29}$$

and

$$\int_0^\infty y^{(p-1)(s-1)+p(A_1-A_2)}\left(\int_0^\infty K(x,y)f(x)dx\right)^p$$
$$< L^p\int_0^\infty x^{1-s+p(A_1-A_2)}f(x)^p dx, \tag{30}$$

where $A_1\in(\frac{1-s}{q},\frac{1}{q})$*,* $A_2\in(\frac{1-s}{p},\frac{1}{p})$ *and* $L=k(pA_2)^{\frac{1}{p}}k(2-s-qA_1)^{\frac{1}{q}}$*.*
If $0<p<1$*, then the reverse inequalities in (29) and (30) are valid for any* $A_1\in(\frac{1}{q},\frac{1-s}{q})$ *and* $A_2\in(\frac{1-s}{p},\frac{1}{p})$*, as well as the inequality*

$$\int_0^\infty x^{(q-1)(s-1)+q(A_2-A_1)}\left(\int_0^\infty K(x,y)g(y)dy\right)^q$$
$$< L^q\int_0^\infty y^{1-s+q(A_2-A_1)}g(y)^q dy.$$

Note that in previous Corollary all the inequalities are strict (see Remark 1).

Now, we shall make some generalizations of Theorem 5. If we use substitution $u=x+\lambda$ and $v=y+\lambda$ we have

THEOREM 6. *Let* $\frac{1}{p}+\frac{1}{q}=1$ *with* $p>1$ *and let* $K(x,y)$ *be homogeneous function of degree* $-s$*,* $s>0$*, strictly decreasing in both variables* x *and* y*. Then the following inequalities hold and are equivalent*

$$\int_a^b\int_a^b K(x+\lambda,y+\lambda)f(x)g(y)dxdy$$
$$\leqslant\left(\int_a^b\big(k(pA_2)-\psi_1(pA_2,x,\lambda)\big)(x+\lambda)^{1-s+p(A_1-A_2)}f(x)^p dx\right)^{\frac{1}{p}}$$
$$\cdot\left(\int_a^b\big(k(2-s-qA_1)-\psi_2(2-s-qA_1,y,\lambda)\big)(y+\lambda)^{1-s+q(A_2-A_1)}g(y)^q dy\right)^{\frac{1}{q}} \tag{31}$$

and

$$\int_a^b\big(k(2-s-qA_1)-\psi_2(2-s-qA_1,y,\lambda)\big)^{1-p}(y+\lambda)^{(p-1)(s-1)+p(A_1-A_2)}$$

$$\cdot\left(\int_a^b K(x+\lambda,y+\lambda)f(x)dx\right)^p dy$$

$$\leqslant \int_a^b \big(k(pA_2)-\psi_1(pA_2,x,\lambda)\big)(x+\lambda)^{1-s+p(A_1-A_2)}f(x)^p dx, \tag{32}$$

where $A_1 \in (\frac{1-s}{q},\frac{1}{q})$, $A_2 \in (\frac{1-s}{p},\frac{1}{p})$ *and*

$$\psi_1(\alpha,x,\lambda)=\left(\frac{a+\lambda}{x+\lambda}\right)^{1-\alpha}\int_0^1 K(1,u)u^{-\alpha}du+\left(\frac{x+\lambda}{b+\lambda}\right)^{s+\alpha-1}\int_0^1 K(u,1)u^{s+\alpha-2}du,$$

$$\psi_2(\alpha,y,\lambda)=\left(\frac{a+\lambda}{y+\lambda}\right)^{s+\alpha-1}\int_0^1 K(u,1)u^{s+\alpha-2}du+\left(\frac{y+\lambda}{b+\lambda}\right)^{1-\alpha}\int_0^1 K(1,u)u^{-\alpha}du.$$

If $0<p<1$, $b=\infty$ *and* $K(x,y)$ *is strictly decreasing in* x *and strictly increasing in* y, *then the reverses in (31) and (32) are valid for any* $A_1 \in (\frac{1}{q},\frac{1-s}{q})$ *and* $A_2 \in (\frac{1-s}{p},\frac{1}{p})$ *as well as the inequality*

$$\int_a^\infty \big(k(pA_2)-\psi_1(pA_2,x,\lambda)\big)^{1-q}(x+\lambda)^{(q-1)(s-1)+q(A_2-A_1)}$$

$$\cdot\left(\int_a^\infty K(x+\lambda,y+\lambda)g(y)dy\right)^q dx$$

$$\leqslant \int_a^\infty \big(k(2-s-qA_1)-\psi_2(2-s-qA_1,y,\lambda)\big)(y+\lambda)^{1-s+q(A_2-A_1)}g(y)^q dy.$$

REMARK 6. If the function $K(x,y)$ from Theorem 6 is symmetrical and $0<1-\frac{2\lambda}{p}<s$, $0<1-\frac{2\lambda}{q}<s$, then, by putting $A_1=A_2=\frac{2\lambda}{pq}$ in the theorem, we obtain results of Jichang and Rassias ([7]).

Another way of generalizing Theorem 5 arises from the substitution $u=Ax^\alpha$ and $v=By^\beta$. More precisely, we have the following

THEOREM 7. *Let* $\frac{1}{p}+\frac{1}{q}=1$ *with* $p>1$ *and let* $K(x,y)$ *be homogeneous function of degree* $-s$, $s>0$, *strictly decreasing in both variables* x *and* y. *Then the following inequalities hold and are equivalent*

$$\int_a^b\int_a^b K(Ax^\alpha,By^\beta)f(x)g(y)dxdy$$

$$\leqslant M\left(\int_a^b \big(k(pA_2)-\zeta_1(pA_2,x)\big)x^{\alpha(1-s)+\alpha p(A_1-A_2)-(\alpha-1)(p-1)}f(x)^p dx\right)^{\frac{1}{p}}$$

$$\cdot\left(\int_a^b \big(k(2-s-qA_1)-\zeta_2(2-s-qA_1,y)\big)y^{\beta(1-s)+\beta q(A_2-A_1)-(\beta-1)(q-1)}g(y)^q dy\right)^{\frac{1}{q}} \quad (33)$$

and

$$\int_a^b \big(k(2-s-qA_1)-\zeta_2(2-s-qA_1,y)\big)^{1-p}y^{\beta(p-1)(s-1)+\beta p(A_1-A_2)+\beta-1}$$

$$\cdot\left(\int_a^b K(Ax^\alpha,By^\beta)f(x)dx\right)^p dy$$

$$\leqslant M^p\int_a^b \big(k(pA_2)-\zeta_1(pA_2,x)\big)x^{\alpha(1-s)+\alpha p(A_1-A_2)-(\alpha-1)(p-1)}f(x)^p dx, \quad (34)$$

where $A_1\in(\frac{1-s}{q},\frac{1}{q})$, $A_2\in(\frac{1-s}{p},\frac{1}{p})$, $M=\alpha^{-\frac{1}{q}}\beta^{-\frac{1}{p}}A^{\frac{2-s}{p}+A_1-A_2-1}B^{\frac{2-s}{q}+A_2-A_1-1}$ *and*

$$\zeta_1(\gamma,x)=\left(\frac{a}{x}\right)^{\alpha(1-\gamma)}\int_0^1 K(1,u)u^{-\gamma}du+\left(\frac{x}{b}\right)^{\alpha(s+\gamma-1)}\int_0^1 K(u,1)u^{s+\gamma-2}du,$$

$$\zeta_2(\gamma,y)=\left(\frac{a}{y}\right)^{\beta(s+\gamma-1)}\int_0^1 K(u,1)u^{s+\gamma-2}du+\left(\frac{y}{b}\right)^{\beta(1-\gamma)}\int_0^1 K(1,u)u^{-\gamma}du.$$

If $0<p<1$, $b=\infty$ *and* $K(x,y)$ *is strictly decreasing in* x *and strictly increasing in* y, *then the reverses in (33) and (34) are valid for any* $A_1\in(\frac{1}{q},\frac{1-s}{q})$ *and* $A_2\in(\frac{1-s}{p},\frac{1}{p})$ *as well as the inequality*

$$\int_a^\infty \big(k(pA_2)-\zeta_1(pA_2,x)\big)^{1-q}x^{\alpha(s-1)(q-1)+\alpha q(A_2-A_1)+\alpha-1}$$

$$\cdot\left(\int_a^\infty K(Ax^\alpha,By^\beta)g(y)dy\right)^q dx$$

$$\leqslant M^q\int_a^\infty \big(k(2-s-qA_1)-\zeta_2(2-s-qA_1,y)\big)y^{\beta(1-s)+\beta q(A_2-A_1)-(\beta-1)(q-1)}g(y)^q dy.$$

Further, if $0<p<1$, $a=0$ *and* $K(x,y)$ *is strictly increasing in* x *and strictly decreasing in* y, *then the reverses in (33) and (34) are valid for any* $A_1\in(\frac{1}{q},\frac{1-s}{q})$ *and* $A_2\in(\frac{1-s}{p},\frac{1}{p})$ *as well as the inequality*

$$\int_0^b \big(k(pA_2)-\zeta_1(pA_2,x)\big)^{1-q}x^{\alpha(s-1)(q-1)+\alpha q(A_2-A_1)+\alpha-1}$$

$$\cdot\left(\int_0^b K(Ax^\alpha, By^\beta)g(y)dy\right)^q dx$$

$$\leqslant M^q\int_0^b \big(k(2-s-qA_1)-\zeta_2(2-s-qA_1,y)\big)y^{\beta(1-s)+\beta q(A_2-A_1)-(\beta-1)(q-1)}g(y)^q dy.$$

If $a=0$ and $b=\infty$, we have the inequalities for arbitrary nonnegative homogeneous function of degree $-s$

THEOREM 8. *Let $\frac{1}{p}+\frac{1}{q}=1$ with $p>1$, $A,B,\alpha,\beta>0$ and let $K(x,y)$ be homogeneous function of degree $-s$. Then the following inequalities hold and are equivalent*

$$\int_0^\infty\int_0^\infty K(Ax^\alpha,By^\beta)f(x)g(y)dxdy$$

$$< N\left(\int_0^\infty x^{\alpha(1-s)+\alpha p(A_1-A_2)-(\alpha-1)(p-1)}f(x)^p dx\right)^{\frac{1}{p}}$$

$$\cdot\left(\int_0^\infty y^{\beta(1-s)+\beta q(A_2-A_1)-(\beta-1)(q-1)}g(y)^q dy\right)^{\frac{1}{q}} \tag{35}$$

and

$$\int_0^\infty y^{\beta(p-1)(s-1)+\beta p(A_1-A_2)+\beta-1}\left(\int_0^\infty K(Ax^\alpha,By^\beta)f(x)dx\right)^p dy$$

$$< N^p\int_0^\infty x^{\alpha(1-s)+\alpha p(A_1-A_2)-(\alpha-1)(p-1)}f(x)^p dx, \tag{36}$$

where $A_1\in(\frac{1-s}{q},\frac{1}{q})$, $A_2\in(\frac{1-s}{p},\frac{1}{p})$, $N=L\cdot M$, and L is defined in Corollary 4 and M in Theorem 7.
If $0<p<1$, then the reverse inequalities in (35) and (36) are valid for any $A_1\in(\frac{1}{q},\frac{1-s}{q})$ and $A_2\in(\frac{1-s}{p},\frac{1}{p})$ as well as the inequality

$$\int_0^\infty x^{\alpha(q-1)(s-1)+\alpha q(A_2-A_1)+\alpha-1}\left(\int_0^\infty K(Ax^\alpha,By^\beta)g(y)dy\right)^q dx$$

$$< N^q\int_0^\infty y^{\beta(1-s)+\beta q(A_2-A_1)-(\beta-1)(q-1)}g(y)^q dy$$

We also give the results in discrete case.

THEOREM 9. *If $\{a_n\}$ and $\{b_n\}$ are nonnegative real sequences, $K(x,y)$ is homogeneous function of degree $-s$ strictly decreasing in both parameters x and y,*

$\frac{1}{p}+\frac{1}{q}=1$, $p>1$, $A,B,\alpha,\beta>0$, *then the following inequalities hold*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}K(Am^{\alpha},Bn^{\beta})a_mb_n$$

$$<N\left(\sum_{m=1}^{\infty}m^{\alpha(1-s)+\alpha p(A_1-A_2)+(p-1)(1-\alpha)}a_m{}^p\right)^{\frac{1}{p}}$$

$$\cdot\left(\sum_{n=1}^{\infty}n^{\beta(1-s)+\beta q(A_2-A_1)+(q-1)(1-\beta)}b_n{}^q\right)^{\frac{1}{q}}, \tag{37}$$

and

$$\sum_{n=1}^{\infty}n^{\beta(s-1)(p-1)+p\beta(A_1-A_2)+\beta-1}\left(\sum_{m=1}^{\infty}K(Am^{\alpha},Bn^{\beta})a_m\right)^p$$

$$<N^p\sum_{m=1}^{\infty}m^{\alpha(1-s)+\alpha p(A_1-A_2)+(p-1)(1-\alpha)}a_m{}^p, \tag{38}$$

where N *is defined in previous theorem,*
$A_1\in\big(\max\{\frac{1-s}{q},\frac{\alpha-1}{\alpha q}\},\frac{1}{q}\big), A_2\in\big(\max\{\frac{1-s}{p},\frac{\beta-1}{\beta p}\},\frac{1}{p}\big)$. *In particular, inequalities (37) and (38) are equivalent.*

Proof. We prove inequality (37). Put $\varphi(Am^{\alpha})=(Am^{\alpha})^{A_1+\frac{1}{q\alpha}-\frac{1}{q}}$ and $\psi(Bn^{\beta})=(Bn^{\beta})^{A_2+\frac{1}{p\beta}-\frac{1}{p}}$ in Theorem 1. Since $qA_1+\frac{1}{\alpha}-1\geqslant 0$ and $pA_2+\frac{1}{\beta}-1\geqslant 0$, the functions $F(Am^{\alpha})=\sum_{n=1}^{\infty}\frac{K(Am^{\alpha},Bn^{\beta})}{(Bn^{\beta})^{pA_2+\frac{1}{\beta}-1}}$ and $G(Bn^{\beta})=\sum_{n=1}^{\infty}\frac{K(Am^{\alpha},Bn^{\beta})}{(Ax^{\alpha})^{qA_1+\frac{1}{\alpha}-1}}$ are strictly decreasing, wherefrom we have $F(Am^{\alpha})\leqslant\int_0^{\infty}\frac{K(Am^{\alpha},By^{\beta})}{(By^{\beta})^{pA_2+\frac{1}{\beta}-1}}dy$ and $G(Bn^{\beta})\leqslant\int_0^{\infty}\frac{K(Ax^{\alpha},Bn^{\beta})}{(Ax^{\alpha})^{qA_1+\frac{1}{\alpha}-1}}dx$ and the result follows from Theorem 2. □

Theorem 9 is a generalization of our paper ([8]), where we considered the function $K(x,y)=\frac{1}{(x+y)^s}$. We'll explore such functions in the following section.

5. Examples

In this section we continue with some special homogeneous functions. We will use the Theorems 1 and 2 but shall use various methods to estimate the integrals of type

$$\int_{\frac{a}{x}}^{\frac{b}{x}}K(1,u)u^{-\alpha}du.$$

At first, we'll discuss the case when $K(x,y) = \dfrac{1}{(x+y)^s}$, symmetrical homogeneous function of degree $-s$.

COROLLARY 5. *Let* $\frac{1}{p}+\frac{1}{q}=1$ *with* $p>1$. *Then the following inequalities hold and are equivalent*

$$\int_a^b\int_a^b \frac{f(x)g(y)}{(x+y)^s}dxdy$$

$$< B(\frac{s}{2},\frac{s}{2})\left(\int_a^b \left(1-\frac{1}{2}(\frac{a}{x})^{\frac{s}{2}}-\frac{1}{2}(\frac{x}{b})^{\frac{s}{2}}\right)x^{-\frac{sp}{2}+p-1}f(x)^p dx\right)^{\frac{1}{p}}$$

$$\cdot\left(\int_a^b \left(1-\frac{1}{2}(\frac{a}{y})^{\frac{s}{2}}-\frac{1}{2}(\frac{y}{b})^{\frac{s}{2}}\right)y^{-\frac{sq}{2}+q-1}g(y)^q dy\right)^{\frac{1}{q}} \tag{39}$$

and

$$\int_a^b \left(1-\frac{1}{2}(\frac{a}{y})^{\frac{s}{2}}-\frac{1}{2}(\frac{y}{b})^{\frac{s}{2}}\right)^{1-p} y^{\frac{sp}{2}-1}\left(\int_a^b \frac{f(x)}{(x+y)^s}dx\right)^p dy$$

$$< B(\frac{s}{2},\frac{s}{2})^p\int_a^b \left(1-\frac{1}{2}(\frac{a}{x})^{\frac{s}{2}}-\frac{1}{2}(\frac{x}{b})^{\frac{s}{2}}\right)x^{-\frac{sp}{2}+p-1}f(x)^p dx. \tag{40}$$

Proof. We use the proof of Theorem 5, where $A_1=\dfrac{2-s}{2q}$, $A_2=\dfrac{2-s}{2p}$, and also the inequality

$$\int_a^\infty \frac{u^{\frac{s}{2}-1}}{(1+u)^s}du > \frac{1}{2}a^{-\frac{s}{2}}B(\frac{s}{2},\frac{s}{2}), \qquad a>1,$$

which can easily be proved. Now the result follows from the Theorem 2. □

REMARK 7. Furthermore, if we use A-G inequality

$$\frac{1}{2}(\frac{a}{x})^{\frac{s}{2}}+\frac{1}{2}(\frac{x}{b})^{\frac{s}{2}} \geqslant (\frac{a}{b})^{\frac{s}{4}},$$

then the inequalities (39) and (40) become

$$\int_a^b\int_a^b \frac{f(x)g(y)}{(x+y)^s}dxdy$$

$$< B(\frac{s}{2},\frac{s}{2})\left(1-(\frac{a}{b})^{\frac{s}{4}}\right)\left(\int_a^b x^{-\frac{sp}{2}+p-1}f(x)^p dx\right)^{\frac{1}{p}}\left(\int_a^b y^{-\frac{sq}{2}+q-1}g(y)^q dy\right)^{\frac{1}{q}}$$

and

$$\int_a^b y^{\frac{sp}{2}-1}\left(\int_a^b \frac{f(x)}{(x+y)^s}dx\right)^p dy$$

$$< \left(B(\frac{s}{2}, \frac{s}{2}) \left(1 - (\frac{a}{b})^{\frac{s}{4}} \right) \right)^p \int_a^b x^{-\frac{sp}{2}+p-1} f(x)^p dx,$$

which is the result from [13]. These inequalities also generalize [12], and if we put $p = q = 2$ in these inequalities, we obtain the result of Yang ([23]). Note also that the case $a = 0$ and $b = \infty$ was proved by Brnetić and Pečarić in [3]; it's Theorem C from the Introduction.

REMARK 8. It is interesting to consider another special case of Corollary 5, namely $s = 1$, $A_1 = \dfrac{1}{2q}$ and $A_2 = \dfrac{1}{2p}$. In this case we do not estimate integrals as in the proof, we can easily calculate them, so we have the following inequalities

$$\int_a^b \int_a^b \frac{f(x)g(y)}{x+y} dxdy$$

$$\leqslant \left(\int_a^b \left(\pi - 2 \operatorname{arc\,tg} \sqrt{\frac{x}{b}} - 2 \operatorname{arc\,tg} \sqrt{\frac{a}{x}} \right) x^{\frac{p}{2}-1} f(x)^p dx \right)^{\frac{1}{p}}$$

$$\cdot \left(\int_a^b \left(\pi - 2 \operatorname{arc\,tg} \sqrt{\frac{y}{b}} - 2 \operatorname{arc\,tg} \sqrt{\frac{a}{y}} \right) y^{\frac{q}{2}-1} g(y)^q dy \right)^{\frac{1}{q}} \tag{41}$$

and

$$\int_a^b \left(\pi - 2 \operatorname{arc\,tg} \sqrt{\frac{y}{b}} - 2 \operatorname{arc\,tg} \sqrt{\frac{a}{y}} \right)^{1-p} y^{\frac{p}{2}-1} \left(\int_a^b \frac{f(x)}{x+y} dx \right)^p dy$$

$$\leqslant \int_a^b \left(\pi - 2 \operatorname{arc\,tg} \sqrt{\frac{x}{b}} - 2 \operatorname{arc\,tg} \sqrt{\frac{a}{x}} \right) x^{\frac{p}{2}-1} f(x)^p dx \tag{42}$$

If $0 < p < 1$, then the reverse inequalities are also valid. Furthermore, it is easy to see that the function

$$f(x) = \operatorname{arc\,tg} \sqrt{\frac{x}{b}} + \operatorname{arc\,tg} \sqrt{\frac{a}{x}}, \quad a \leqslant x \leqslant b,$$

reaches the minimum value $\operatorname{arc\,tg} \sqrt[4]{\dfrac{a}{b}}$ in $\sqrt{ab}$, so the inequalities (41) and (42) become

$$\int_a^b \int_a^b \frac{f(x)g(y)}{x+y} dxdy$$

$$\leqslant \left(\pi - 4 \operatorname{arc\,tg} \sqrt[4]{\frac{a}{b}} \right) \left(\int_a^b x^{\frac{p}{2}-1} f(x)^p dx \right)^{\frac{1}{p}} \left(\int_a^b y^{\frac{q}{2}-1} g(y)^q dy \right)^{\frac{1}{q}}$$

and

$$\int_a^b y^{\frac{p}{2}-1}\left(\int_a^b \frac{f(x)}{x+y}dx\right)^p dy \leqslant \left(\pi - 4\operatorname{arc\,tg}\sqrt[4]{\frac{a}{b}}\right)^p \int_a^b x^{1-\frac{p}{2}} f(x)^p dx,$$

which are the generalizations of Yang's results in [23]. The second inequality can be found in [13].

On the other hand, we obtain another general types of inequalities for $K(x,y) = \dfrac{1}{(x+y)^s}$. It is the content of the following

THEOREM 10. *If* $\frac{1}{p}+\frac{1}{q}=1$ *with* $p>1$, *then the following inequalities hold and are equivalent*

$$\int_a^b \int_a^b \frac{f(x)g(y)}{(x+y)^s}dxdy$$
$$\leqslant Q\left(\int_a^b x^{1-s+p(A_1-A_2)} f(x)^p dx\right)^{\frac{1}{p}} \left(\int_a^b y^{1-s+q(A_2-A_1)} g(y)^q dy\right)^{\frac{1}{q}} \tag{43}$$

and

$$\int_a^b y^{(p-1)(s-1)+p(A_1-A_2)}\left(\int_a^b \frac{f(x)}{(x+y)^s}dx\right)^p dy$$
$$\leqslant Q^p \int_a^b x^{1-s+p(A_1-A_2)} f(x)^p dx \tag{44}$$

for $A_1 \in (\frac{1-s}{q}, \frac{1}{q})$, $A_2 \in (\frac{1-s}{p}, \frac{1}{p})$ *and* $Q = k_{l_1}(pA_2)^{\frac{1}{p}} k_{l_2}(qA_1)^{\frac{1}{q}}$, *where*

$$k_l(\alpha) = \int_{\frac{a^l-b^l}{b(b^{l-1}-a^{l-1})}}^{\frac{a^l-b^l}{a(b^{l-1}-a^{l-1})}} \frac{u^{-\alpha}}{(1+u)^s}du \quad \text{and} \quad l_1 = \frac{1-pA_2}{s}, l_2 = \frac{1-qA_1}{s}.$$

Proof. We use the proof of Theorem 5, but for an estimate of the integral

$$\int_{\frac{a}{x}}^{\frac{b}{x}} \frac{u^{-\alpha}}{(1+u)^s}du,$$

we use the fact that the function $f(x) = \int_{\frac{a}{x}}^{\frac{b}{x}} \frac{u^{-\alpha}}{(1+u)^s}du$, $x \in (0,\infty)$, reaches the maximum value for $x = \dfrac{ab^l - ba^l}{a^l - b^l}$, $l = \dfrac{1-\alpha}{s}$. □

Now, if we put $A_1 = A_2 = \dfrac{2-s}{pq}$ in the previous theorem, we have following

COROLLARY 6. *Let* $\frac{1}{p}+\frac{1}{q}=1$, $p>1$ *and* $s>2-\min\{p,q\}$. *Then the following inequalities hold and are equivalent*

$$\int_a^b \int_a^b \frac{f(x)g(y)}{(x+y)^s}dxdy \leqslant Q_1\left(\int_a^b x^{1-s}f(x)^p dx\right)^{\frac{1}{p}}\left(\int_a^b y^{1-s}g(y)^q dy\right)^{\frac{1}{q}}$$

and

$$\int_a^b y^{(s-1)(p-1)}\left(\int_a^b \frac{f(x)}{(x+y)^s}dx\right)^p dy < Q_1{}^p \int_a^b x^{1-s}f(x)^p dx,$$

where

$$Q_1 = k_{\frac{q+s-2}{qs}}\left(\frac{2-s}{q}\right)^{\frac{1}{p}} k_{\frac{p+s-2}{ps}}\left(\frac{2-s}{p}\right)^{\frac{1}{q}}.$$

REMARK 9. Similarly, if $A_1=\frac{2-s}{2q}$ and $A_2=\frac{2-s}{2p}$, the inequalities (43) and (44) from Theorem 10 become

$$\int_a^b \int_a^b \frac{f(x)g(y)}{(x+y)^s}dxdy$$

$$\leqslant k_{\frac{1}{2}}(\frac{2-s}{2})\left(\int_a^b x^{-\frac{sp}{2}+p-1}f(x)^p dx\right)^{\frac{1}{p}}\left(\int_a^b y^{-\frac{sq}{2}+q-1}g(y)^q dy\right)^{\frac{1}{q}}$$

and

$$\int_a^b y^{\frac{sp}{2}-1}\left(\int_a^b \frac{f(x)}{(x+y)^s}dx\right)^p dy < \left(k_{\frac{1}{2}}(\frac{2-s}{2})\right)^p \int_a^b x^{-\frac{sp}{2}+p-1}f(x)^p dx,$$

which are the main results in [11].

Further, we shall consider some other types of homogeneous functions. Observe that $K(x,y)=\frac{\ln\frac{y}{x}}{y-x}$ is symmetrical homogeneous function of degree -1, so we have the following

COROLLARY 7. *If* $\frac{1}{p}+\frac{1}{q}=1$ *with* $p>1$, *then the following inequalities hold and are equivalent*

$$\int_0^\infty \int_0^\infty \frac{\ln\frac{y}{x}}{y-x}f(x)g(y)dxdy$$

$$< R\left(\int_0^\infty x^{p(A_1-A_2)}f(x)^p dx\right)^{\frac{1}{p}}\left(\int_0^\infty y^{q(A_2-A_1)}g(y)^q dy\right)^{\frac{1}{q}}$$

and

$$\int_0^\infty y^{p(A_1-A_2)}\left(\int_0^\infty \frac{\ln\frac{y}{x}}{y-x}f(x)dx\right)^p dy < R^p \int_0^\infty x^{p(A_1-A_2)}f(x)^p dx,$$

where $A_1 \in (0, \frac{1}{q})$, $A_2 \in (0, \frac{1}{p})$ *and* $R = \frac{\pi^2}{(\sin pA_2\pi)^{\frac{2}{p}}(\sin qA_1\pi)^{\frac{2}{q}}}$.

If $0 < p < 1$, *then the reverses are valid for any* $A_1 \in (\frac{1}{q}, 0)$ *and* $A_2 \in (0, \frac{1}{p})$.

REMARK 10. Let us put $p(A_1 - A_2) = \varepsilon$ and observe the constant R from the previous corollary as the function of A_1. We obtain that, for $p > 1$, R reaches the minimum value if $A_1 = \frac{\pi+\varepsilon}{\pi pq}$. If $\varepsilon = 0$, then the minimum value is $\frac{\pi^2}{\sin\frac{\pi}{p}}$, so we obtain the inequality from [7] and R is the best possible constant.

Finally, if $K(x,y) = \frac{1}{\max\{x,y\}^s}$, we again don't use other estimates except Hölder's inequality because we can calculate all the integrals that we have estimated before.

COROLLARY 8. *Let* $\frac{1}{p} + \frac{1}{q} = 1$ *with* $p > 1$. *Then the following inequalities hold and are equivalent*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x,y\}^s}dxdy$$

$$\leqslant T\left(\int_0^\infty x^{1-s+p(A_1-A_2)}f(x)^p dx\right)^{\frac{1}{p}}\left(\int_0^\infty y^{1-s+q(A_2-A_1)}g(y)^q dy\right)^{\frac{1}{q}}$$

and

$$\int_0^\infty y^{(p-1)(s-1)+p(A_1-A_2)}\left(\int_0^\infty \frac{f(x)}{\max\{x,y\}^s}dx\right)^p dy$$

$$\leqslant T^p \int_0^\infty x^{1-s+p(A_1-A_2)}f(x)^p dx,$$

where $A_1 \in (\frac{1-s}{q}, \frac{1}{q})$, $A_2 \in (\frac{1-s}{p}, \frac{1}{p})$ *and* $T = k(pA_2)^{\frac{1}{p}}k(qA_1)^{\frac{1}{q}}$ *where* $k(\alpha) = \frac{s}{(1-\alpha)(s+\alpha-1)}$. *If* $0 < p < 1$, *then the reverses of these inequalities are valid for any* $A_1 \in (\frac{1}{q}, \frac{1-s}{q})$ *and* $A_2 \in (\frac{1-s}{p}, \frac{1}{p})$.

REMARK 11. If we put $p(A_1 - A_2) = \varepsilon$ and observe the constant T from Corollary 8 as the function in A_1, we obtain that, for $p > 1$, T reaches the minimum value if $A_1 = \frac{2-s-\varepsilon}{pq}$. If $\varepsilon = 0$, then the minimum value is $\frac{pqs}{(p+s-2)(q+s-2)}$, so we obtain the inequality from [7] and T is the best possible constant.

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