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ON FIRST ORDER DIFFERENTIAL INCLUSIONS WITH PERIODIC BOUNDARY CONDITIONS

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Abstract. In this paper a fixed point theorem for condensing maps combined with upper and lower solutions are used to investigate the existence of solutions for first order differential inclusions with periodic boundary conditions.

1. Introduction

This note is concerned with the existence of solutions for the periodic multivalued problem:

$$y'(t) \in F(t, y(t)), \quad \text{for almost each (a.e.) } t \in J = [0, T] \tag{1}$$

$$y(0) = y(T), \tag{2}$$

where $F : J \times \mathbb{R} \longrightarrow 2^{\mathbb{R}}$ is a nonempty compact and convex valued multivalued map.

The method of upper and lower solutions has been successfully applied to study the existence of multiple solutions for initial and boundary value problems of the first and second order. This method has been used only in the context of single-valued differential equations. We refer to the books of Bernfeld-Lakshmikantham [2], Heikkila-Lakshmikantham [10], Ladde-Lakshmikantham-Vatsala [13], to the thesis of De Coster [5], to the papers of Carl-Heikkila-Kumpulainen [4], Cabada [3], Frigon [7], Frigon-O'Regan [8], Heikkila-Cabada [9], Lakshmikantham-Leela [14], Nkashama [17] and the references therein.

In this note we establish an existence result for the problem (1)–(2). Our approach is based on the existence of upper and lower solutions and on a fixed point theorem for condensing maps due to Martelli [16]. The result of this note extends to the multivalued case some existence results from the above literature.

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2. Preliminaries

We will briefly recall some basic definitions and facts from multivalued analysis that we will use in the sequel.
$AC(J,\mathbb{R})$ is the space of all absolutely continuous functions $y : J \longrightarrow \mathbb{R}$.
Condition

$$y \leqslant \bar{y} \quad \text{if and only if} \quad y(t) \leqslant \bar{y}(t) \quad \text{for all} \quad t \in J$$

defines a partial ordering in $AC(J,\mathbb{R})$. If α, $\beta \in AC(J,\mathbb{R})$ and $\alpha \leqslant \beta$, we denote

$$[\alpha,\beta] = \{y \in AC(J,\mathbb{R}) : \alpha \leqslant y \leqslant \beta\}.$$

Let $(X, \|\cdot\|)$ be a normed space. A multivalued map $G : X \longrightarrow 2^X$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(B) = \cup_{x\in B} G(x)$ is bounded in X for all bounded subsets B of X (i.e. $\sup_{x\in B}\{\sup\{\|y\| : y \in G(x)\}\} < \infty$). G is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$ the set $G(x_0)$ is a nonempty, closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighbourhood M of x_0 such that $G(M) \subseteq N$.

G is said to be completely continuous if $G(B)$ is relatively compact for every bounded subset $B \subset X$. If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e. $x_n \longrightarrow x_*$, $y_n \longrightarrow y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$). G has a fixed point if there is $x \in X$ such that $x \in G(x)$.

In the following $CC(X)$ denotes the set of all nonempty compact and convex subsets of X. An upper semi-continuous map $G : X \longrightarrow 2^X$ is said to be condensing [16] if for any bounded subset $B \subseteq X$, we have $\alpha(G(B)) < \alpha(B)$, where α denotes the Kuratowski measure of noncompacteness [1]. We remark that a compact map is the easiest example of a condensing map. The multivalued map $F : J \longrightarrow CC(\mathbb{R})$ is said to be measurable, if for every $y \in \mathbb{R}$, the function $t \longmapsto d(y, F(t)) = \inf\{|y - z| : z \in F(t)\}$ is measurable. For more details on multivalued maps see the books of Deimling [6] and Hu and Papageorgiou [12].

DEFINITION 1. A multivalued map $F : J \times \mathbb{R} \longrightarrow 2^{\mathbb{R}}$ is said to be an L^1-Carathéodory if

- (i) $t \longmapsto F(t,y)$ is measurable for each $y \in \mathbb{R}$;
- (ii) $y \longmapsto F(t,y)$ is upper semicontinuous for almost all $t \in J$;
- (iii) For each $k > 0$, there exists $h_k \in L^1(J,\mathbb{R}_+)$ such that

$$\|F(t,y)\| = \sup\{|v| : v \in F(t,y)\} \leqslant h_k(t), \text{ for all } |y| \leqslant k \text{ and for almost all } t \in J.$$

So let us start by defining what we mean by a solution of problem (1)–(2).

DEFINITION 2. A function $y \in AC(J,\mathbb{R})$ is said to be a solution of (1)–(2) if there exists a function $v \in L^1(J,\mathbb{R})$ such that $v(t) \in F(t,y(t))$ a.e. on J, $y'(t) = v(t)$ a.e. on J and $y(0) = y(T)$.

The following concept of lower and upper solutions for (1)–(2) has been introduced by Halidias and Papageorgiou in [11] for second order multivalued boundary value problems. It will be the basic tools in the approach that follows.

DEFINITION 3. A function $\alpha \in AC(J,\mathbb{R})$ is said to be a lower solution of (1)–(2) if there exists $v_1 \in L^1(J,\mathbb{R})$ such that $v_1(t) \in F(t,\alpha(t))$ a.e. on J, $\alpha'(t) \leqslant v_1(t)$ a.e. on J and $\alpha(0) \leqslant \alpha(T)$.
Similarly a function $\beta \in AC(J,\mathbb{R})$ is said to be an upper solution of (1)–(2) if there exists $v_2 \in L^1(J,\mathbb{R})$ such that $v_2(t) \in F(t,\beta(t))$ a.e. on J, $\beta'(t) \geqslant v_2(t)$ a.e. on J and $\beta(0) \geqslant \beta(T)$.

For the multivalued map F and for each $y \in C(J,\mathbb{R})$ we define $S^1_{F,y}$ by

$$S^1_{F,y} = \{v \in L^1(J,\mathbb{R}) : v(t) \in F(t,y(t)) \text{ for a.e. } t \in J\}.$$

Our main result is based on the following:

LEMMA 2.1. [15]. *Let I be a compact real interval and X be a Banach space. Let $F : I \times X \longrightarrow CC(X); (t,y) \rightarrow F(t,y)$ measurable with respect to t for any $y \in X$ and u.s.c. with respect to y for almost each $t \in I$ and $S^1_{F,y} \neq \emptyset$ for any $y \in C(I,X)$ and let Γ be a linear continuous mapping from $L^1(I,X)$ to $C(I,X)$, then the operator*

$$\Gamma \circ S^1_F : C(I,X) \longrightarrow CC(C(I,X)),\ y \longmapsto (\Gamma \circ S^1_F)(y) := \Gamma(S^1_{F,y})$$

is a closed graph operator in $C(I,X) \times C(I,X)$.

LEMMA 2.2. [16]. *Let $G : X \longrightarrow CC(X)$ be an u.s.c. and condensing map. If the set*

$$M := \{v \in X : \lambda v \in G(v) \text{ for some } \lambda > 1\}$$

is bounded, then G has a fixed point.

In the proof of our main result we need the following auxiliary result. The proof of this result is very easy, therefore we shall not present it.

LEMMA 2.3. *Let $g \in L^1(J,\mathbb{R})$. Then the periodic value problem*

$$y'(t) + y(t) = g(t), \quad t \in J \tag{3}$$

subjected to the condition (2) has a unique solution y given by

$$y(t) = \int_0^T G(t,s)g(s)ds$$

where $G(t,s)$ is the Green function defined by

$$G(t,s) = \frac{1}{e^T - 1}\begin{cases} e^{T+s-t}, & \text{if } 0 \leqslant s \leqslant t \leqslant T; \\ e^{s-t}, & \text{if } 0 \leqslant t \leqslant s \leqslant T. \end{cases}$$

3. Main Result

We are now in a position to state and prove our existence result for the problem (1)–(2).

THEOREM 3.1. *Suppose* $F: J\times\mathbb{R}\longrightarrow CC(\mathbb{R})$ *is an* L^1 *-Carathéodory multivalued map. In addition assume that the following condition is satisfied:*

(H) there exist α *and* β *in* $AC(J,\mathbb{R})$ *lower and upper solutions respectively for the problem (1)–(2) such that* $\alpha\leqslant\beta$.

Then the problem (1)–(2) has at least one solution $y\in AC(J,\mathbb{R})$ *such that*

$$\alpha(t)\leqslant y(t)\leqslant\beta(t)\ \textit{for all}\ t\in J.$$

Proof. Transform the problem into a fixed point problem. Consider the following modified problem

$$y'(t)+y(t)\in F_1(t,y(t)),\ \text{ a.e. } t\in J, \tag{4}$$

with the condition (2), where $F_1(t,y)=F(t,(\tau y)(t))+(\tau y)(t)$ and $\tau: C(J,\mathbb{R})\longrightarrow C(J,\mathbb{R})$ is the truncation operator defined by

$$(\tau y)(t)=\begin{cases}\alpha(t), & \text{if } y(t)<\alpha(t);\\ y(t), & \text{if } \alpha(t)\leqslant y\leqslant\beta(t);\\ \beta(t), & \text{if } \beta(t)<y(t).\end{cases}$$

REMARK 1. Notice that F_1 is an L^1 -Carathéodory multivalued map with nonempty compact convex values and there exists $\phi\in L^1(J,\mathbb{R}_+)$ such that

$$\|F_1(t,y(t))\|\leqslant\phi(t)+\max(\sup_{t\in J}|\alpha(t)|,\sup_{t\in J}|\beta(t)|)\ \text{ for a.e. } t\in J \text{ and all } y\in C(J,\mathbb{R}).$$

From Lemma 2.3 it follows that a solution to (4)–(2) is a fixed point of the operator $N: C(J,\mathbb{R})\longrightarrow 2^{C(J,\mathbb{R})}$ defined by

$$N(y):=\left\{h\in C(J,\mathbb{R}): h(t)=\int_0^T G(t,s)[v(s)+(\tau y)(s)]ds:\ v\in\tilde{S}^1_{F,y}\right\}$$

where

$$\begin{aligned}&\tilde{S}^1_{F,y}=\{v\in S^1_{F,\tau y}: v(t)\geqslant v_1(t)\text{ a.e. on }A_1\text{ and }v(t)\leqslant v_2(t)\text{ a.e. on }A_2\},\\ &S^1_{F,\tau y}=\{v\in L^1(J,\mathbb{R}): v(t)\in F(t,(\tau y)(t))\text{ for a.e. }t\in J\},\\ &A_1=\{t\in J: y(t)<\alpha(t)\leqslant\beta(t)\},\ A_2=\{t\in J:\alpha(t)\leqslant\beta(t)<y(t)\}.\end{aligned}$$

REMARK 2.

(i) For each $y\in C(J,\mathbb{R})$ the set $S^1_{F,y}$ is nonempty (see Lasota and Opial [15]).

(ii) For each $y\in C(J,\mathbb{R})$ the set $\tilde{S}^1_{F,y}$ is nonempty. Indeed, by (i) there exists $v\in S^1_{F,y}$. Set

$$w=v_1\chi_{A_1}+v_2\chi_{A_2}+v\chi_{A_3},$$

where

$$A_3 = \{t \in J : \alpha(t) \leqslant y(t) \leqslant \beta(t)\}.$$

Then by decomposability $w \in \tilde{S}^1_{F,y}$.

We shall show that N has a fixed point, by applying Lemma 2.2. The proof will be given in several steps. We first shall show that N is a completely continuous multivalued map, u.s.c. with convex closed values.

Step 1: $N(y)$ is convex for each $y \in C(J,\mathbb{R})$.

Indeed, if h, $\overline{h}$ belong to $N(y)$, then there exist $v \in \tilde{S}^1_{F,y}$ and $\overline{v} \in \tilde{S}^1_{F,y}$ such that

$$h(t) = \int_0^T G(t,s)[v(s) + (\tau y)(s)]ds, \quad t \in J$$

and

$$\overline{h}(t) = \int_0^T G(t,s)[\overline{v}(s) + (\tau y)(s)]ds, \quad t \in J.$$

Let $0 \leqslant k \leqslant 1$. Then for each $t \in J$ we have

$$[kh + (1-k)\overline{h}](t) = \int_0^T G(t,s)[kv(s) + (1-k)\overline{v}(s) + (\tau y)(s)]ds.$$

Since $\tilde{S}^1_{F,y}$ is convex (because F has convex values) then

$$kh + (1-k)\overline{h} \in N(y).$$

Step 2: N is completely continuous.

Let $B_r := \{y \in C(J,\mathbb{R}) : \|y\|_\infty \leqslant r\}$ be a bounded set in $C(J,\mathbb{R})$ and $y \in B_r$, then for each $h \in N(y)$ there exists $v \in \tilde{S}^1_{F,y}$ such that

$$h(t) = \int_0^T G(t,s)[v(s) + (\tau y)(s)]ds, \quad t \in J.$$

Thus for each $t \in J$ we get

$$\begin{aligned} |h(t)| &\leqslant \int_0^T \|G(t,s)\| |v(s) + (\tau y)(s)| ds \\ &\leqslant \max_{(t,s)\in J\times J} |G(t,s)| [\|\phi_R\|_{L^1} + T \max(r, \sup_{t\in J}|\alpha(t)|, \sup_{t\in J}|\beta(t)|)] = K. \end{aligned}$$

Furthermore

$$\begin{aligned} |h'(t)| &\leqslant \int_0^T \|G'_t(t,s)\| |v(s) + (\tau y)(s)| ds \\ &\leqslant \max_{(t,s)\in J\times J} |G'_t(t,s)| [\|\phi_R\|_{L^1} + T \max(r, \sup_{t\in J}|\alpha(t)|, \sup_{t\in J}|\beta(t)|)] = K_1. \end{aligned}$$

We note that $G(t,s)$ and $G'_t(t,s)$ are continuous on $J \times J$. Thus N maps bounded set of $C(J,\mathbb{R})$ into uniformly bounded and equicontinuous set of $C(J,\mathbb{R})$.

Step 3: N has a closed graph.

Let $y_n \longrightarrow y_*$, $h_n \in N(y_n)$, and $h_n \longrightarrow h_*$. We shall prove that $h_* \in N(y_*)$. $h_n \in N(y_n)$ means that there exists $v_n \in \tilde{S}_{F,y_n}$ such that

$$h_n(t) = \int_0^T G(t,s)[v_n(s) + (\tau y_n)(s)]ds, \quad t \in J.$$

We must prove that there exists $v_* \in \tilde{S}_{F,y_*}$ such that

$$h_*(t) = \int_0^T G(t,s)[v_*(s) + (\tau y_*)(s)]ds, \quad t \in J.$$

Since $y_n \to y_*$, $h_n \to h_*$ and τ is a continuous function we have that

$$\left\| \left(h_n - \int_0^T G(t,s)(\tau y_n)(s)ds \right) - \left(h_* - \int_0^T G(t,s)(\tau y_*)(s)ds \right) \right\|_\infty \longrightarrow 0,$$

as $n \longrightarrow \infty$.

Now, we consider the linear continuous operator

$$\Gamma : L^1(J,R) \longrightarrow C(J,R)$$

$$v \longmapsto \Gamma(v)(t) = \int_0^T G(t,s)v(s)ds.$$

From Lemma 2.1, it follows that $\Gamma \circ \tilde{S}_F$ is a closed graph operator. Moreover, we have that

$$\left(h_n(t) - \int_0^T G(t,s)(\tau y_n)(s)ds \right) \in \Gamma(\tilde{S}_{F,y_n}).$$

Since $y_n \longrightarrow y_*$, it follows from Lemma 2.1 that

$$h_*(t) = \int_0^T G(t,s)(\tau y_*)(s)ds + \int_0^T G(t,s)v_*(s)ds$$

for some $v_* \in \tilde{S}_{F,y_*}$.

Therefore N is a completely continuous multivalued map, u.s.c. with convex closed values.

Step 4: The set

$$M := \{y \in C(J,\mathbb{R}) : \lambda y \in N(y) \textit{ for some } \lambda > 1\}$$

is bounded.

Let $\lambda y \in N(y)$ for some $\lambda > 1$. Then there exists $v \in \tilde{S}^1_{F,y}$ such that

$$y(t) = \lambda^{-1} \int_0^T G(t,s)[v(s) + (\tau y)(s)]ds, \quad t \in J.$$

Thus

$$|y(t)| \leqslant \|G(t,s)\| \int_0^T |v(s) + (\tau y)(s)|ds, \qquad t \in J.$$

Thus we obtain

$$\|y\|_\infty \leqslant \frac{e^T}{e^T-1}[\|\phi\|_{L^1} + T\max(\sup_{t\in J}|\alpha(t)|, \sup_{t\in J}|\beta(t)|)].$$

This shows that M is bounded. Hence, Lemma 2.2 applies and N has a fixed point which is a solution to problem (3.1)-(3.2).

Step 5: The solution y of (4)–(2) satisfies

$$\alpha(t) \leqslant y(t) \leqslant \beta(t) \ \textit{for all}\ \ t \in J.$$

Let y be a solution to (4)–(2). We prove that

$$y(t) \leqslant \beta(t) \ \text{ for all } \ t \in J.$$

Assume that $y-\beta$ attains a positive maximum on J at t_0, that is

$$(y-\beta)(t_0) = \max\{y(t)-\beta(t) : t \in J\} > 0.$$

If $t_0 \in (0,T)$ there exists $t_0^* \in (0,t_0)$ such that

$$0 < y(t) - \beta(t) \leqslant y(t_0) - \beta(t_0) \ \text{ for all } \ t \in [t_0^*, t_0].$$

Thus for $t = t_0^*$ we get

$$\beta(t_0) - \beta(t_0^*) \leqslant y(t_0) - y(t_0^*).$$

By the definition of τ there exists $v \in L^1(J,\mathbb{R})$ with $v(t) \leqslant v_2(t)$ a.e. on (t_0^*, t_0) and $v(t) \in F(t,\beta(t))$ a.e. on (t_0^*, t_0) such that

$$\begin{aligned} y(t_0) - y(t_0^*) &= \int_{t_0^*}^{t_0} (v(s) - y(s) + \beta(s))ds \\ &\leqslant \int_{t_0^*}^{t_0} (v_2(s) - (y(s) - \beta(s)))ds. \end{aligned}$$

Using the fact that β is an upper solution to (1)–(2) the above inequality yields

$$\begin{aligned} y(t_0) - y(t_0^*) &\leqslant \beta(t_0) - \beta(t_0^*) - \int_{t_0^*}^{t_0} (y(s) - \beta(s))ds \\ &< \beta(t_0) - \beta(t_0^*). \end{aligned}$$

Thus we obtain the contradiction

$$\beta(t_0) - \beta(t_0^*) < \beta(t_0) - \beta(t_0^*).$$

Finally, $t_0 \neq 0$ for if $t_0 = 0$ then

$$\beta(T) \leqslant \beta(0) < y(0) = y(T)$$

which is a contradiction.

Analogously, we can prove that $y(t) \geqslant \alpha(t)$ for all $t \in J$. This shows that the problem (4)–(2) has a solution in the interval $[\alpha,\beta]$. Since $\tau(y) = y$ for all $y \in [\alpha,\beta]$, then y is a solution to (4)–(2).

REFERENCES

[1] J. BANAS AND K. GOEBEL, *Measures of Noncompactness in Banach Spaces*, Marcel Dekker, New York, 1980.
[2] S. BERNFELD AND V. LAKSHMIKANTHAM, *An Introduction to Nonlinear Boundary Value Problems*, Academic Press, New York, 1974.
[3] A. CABADA, *The monotone method for first order problems with linear and nonlinear boundary conditions*, Appl. Math. Comp. **63** (1994), 163–186.
[4] S. CARL, S. HEIKKILA AND M. KUMPULAINEN, *On solvability of first order discontinuous scalar differential equations*, Nonlinear Times Digest **2** 1 (1995), 11–27.
[5] C. DE COSTER, *The method of lower and upper solutions in boundary value problems*, PhD thesis, Université Catholique de Louvain, 1994.
[6] K. DEIMLING, *Multivalued Differential Equations*, De Gruyter, Berlin, 1992.
[7] M. FRIGON, *Application de la théorie de la transversalité topologique à des problèmes non linéaires pour des équations différentielles ordinaires*, Dissertationes Mathemacae, **296** Warszawa, 1990.
[8] M. FRIGON AND D. O'REGAN, *Existence results for some initial and boundary value problems without growth restriction*, Proc. Amer. Math. Soc. **123** 1 (1995), 207–216.
[9] S. HEIKKILA AND A. CABADA, *On first order discontinuous differential equations with nonlinear boundary conditions*, Nonlinear Word **3** (1996), 487–503.
[10] S. HEIKKILA AND V. LAKSHMIKANTHAM, *Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations*, Marcel Dekker, New York, 1994.
[11] N. HALIDIAS AND N.S. PAPAGEORGIOU, *Second order multivalued boundary value problems*, Arch. Math. (Brno) **34** (1998), 267–284.
[12] SH. HU AND N. PAPAGEORGIOU, *Handbook of Multivalued Analysis*, Kluwer, Dordrecht, Boston, London, 1997.
[13] G. S. LADDE, V. LAKSHMIKANTHAM AND A. S. VATSALA, *Monotone Iterative Techniques for Nonlinear Differential Equations*, Pitman, Boston M. A., 1985.
[14] V. LAKSHMIKANTHAM AND S. LEELA, *Remarks on first and second order periodic boundary value problems*, Nonlinear Anal. **8** 3 (1984), 281–287.
[15] A. LASOTA AND Z. OPIAL, *An application of the Kakutani-Ky Fan theorem in the theory of ordinary differential equations*, Bull. Acad. Pol. Sci. Ser. Sci. Math. Astronom. Phys. **13** (1965), 781–786.
[16] M. MARTELLI, *A Rothe's type theorem for non compact acyclic-valued maps*, Boll. Un. Mat. Ital. **4** (Suppl. Fasc.) (1975), 70–76.
[17] M. N. NKASHAMA, *A generalized upper and lower solutions method and multiplicity results for nonlinear first order ordinary differential equations*, J. Math. Anal. Appl. **140** (1989), 381–395.

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