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COMMUTATORS OF IMAGINARY POWERS OF LAPLACE OPERATORS

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Abstract. We obtain continuity properties of commutators generated by the imaginary powers of the Laplace operator and BMO functions on Hardy spaces and certain Hardy type spaces.

1. Introduction and statements

Let $\widehat{f}(\xi) = \int f(y)\, e^{-i<y,\xi>}\, dy$ be the Fourier transform in $\mathbb{R}^n$. For each $u \in \mathbb{R} \setminus \{0\}$, let K_u be the tempered distribution on $\mathbb{R}^n$ such that $\widehat{K_u}(\xi) = |\xi|^{-iu}$. Here $\widehat{K_u}$ is defined via $\langle \widehat{K_u}, f \rangle = \langle K_u, \widehat{f} \rangle$ for all $f \in \mathscr{S}(\mathbb{R}^n)$. Explicitly, K_u may be given by

$$K_u(x) = C(u)\, |x|^{-n+iu}$$

where $C(u) = \pi^{-\frac{n}{2}+iu}\Gamma(n-iu/2)/\Gamma(iu/2)$. Then define the singular integral operators T_u on $\mathscr{S}(\mathbb{R}^n)$ by

$$T_u f = K_u * f,$$

where K_u is the kernel of T_u. For every $k \in \mathbb{N}$, K_u is of class C^k away from the origin and satisfies the following properties:

$$(a) \quad |K_u(x)| \leqslant C\,(1+|u|)^{\frac{n}{2}}\, |x|^{-n}, \quad x \neq 0,$$

$$(b) \quad |K_u(x-y) - K_u(x)| \leqslant C\,(1+|u|)^{\frac{n}{2}+1}\, |y||x|^{-n-1}, \quad |x| > 2|y| > 0,$$

$$(c) \quad |D^\gamma K_u(x)| \leqslant C\,(1+|u|)^{\frac{n}{2}+k}\, |x|^{-n-|\gamma|}, \quad x \neq 0,$$

for every multi index γ with $|\gamma| \leqslant k$ (see [8, 15] for more details).

The singular integral operators that are imaginary powers of the Laplace operator T_u in $\mathbb{R}^n$ have been studied by many authors [5, 6, 8, 11, 13]. Recently, Gunawan and Wright [8, 9] showed the following sharp estimates for T_u:

$$||T_u f||_{L^p(\mathbb{R}^n)} \leqslant C_p\,(1+|u|)^{|\frac{n}{p}-\frac{n}{2}|}\, ||f||_{L^p(\mathbb{R}^n)}$$

for $1 < p < \infty$, and

$$||T_u f||_{L^p(\mathbb{R}^n)} \leqslant C_p\,(1+|u|)^{\frac{n}{p}-\frac{n}{2}}\, ||f||_{H^p(\mathbb{R}^n)} \tag{1.1}$$

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for $0 < p \leqslant 1$.

We consider the commutator $T_{u,b}$ defined by

$$T_{u,b}f(x) = p.v. \int_{\mathbb{R}^n} K_u(x-y)[b(x)-b(y)]f(y)\,dy,$$

where $b \in BMO(\mathbb{R}^n)$, that is b is locally integrable and

$$||b||_{BMO} := \frac{1}{|Q|}\int_Q |b(x)-b_Q|\,dx \leqslant A < \infty$$

holds for all balls $Q \subset \mathbb{R}^n$. Here $b_Q = |Q|^{-1}\int_Q b(x)\,dx$ and $||b||_{BMO}$ denotes the BMO norm of b.

In [4], Coifman, Rochberg and Weiss proved that the commutator generated by Calderón-Zygmund singular integral operator and a BMO function is bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$. C. Pérez [14] proved that this commutator is not weak type $(1,1)$, but satisfies a weak type $L\log L$ inequality. It is known that a Calderón-Zygmund singular integral operator maps $H^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ under certain assumptions. However, it was observed in [12] that a corresponding result for the commutator generated by Calderón-Zygmund singular integral operator and a BMO function is false. As for $p < 1$, the commutator of Calderón-Zygmund singular integral operator and a BMO function is not bounded from $H^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$.

The purpose of this note is to study the continuity properties of the commutator $T_{u,b}$ on Hardy spaces $H^1(\mathbb{R}^n)$ and Hardy type spaces $H^p_b(\mathbb{R}^n)$, $0 < p \leqslant 1$. What we are interested in here is how the norm of the commutator $T_{u,b}$, especially from $H^p_b(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ where $0 < p \leqslant 1$, depends on the imaginary power.

DEFINITION 1. Let $0 < p \leqslant 1$ and d be an integer that satisfies $d \geqslant n(1/p-1)$. Let Q be a cube in $\mathbb{R}^n$. We say that a is a (p,d)-atom associated with Q if a is supported on $Q \subset \mathbb{R}^n$ and satisfies

$$(i) \quad ||a||_{L^\infty(\mathbb{R}^n)} \leqslant |Q|^{-1/p};$$
$$(ii) \quad \int_{\mathbb{R}^n} a(x)x^\beta\,dx = 0$$

where $\beta = (\beta_1, \beta_2, \cdots, \beta_n)$ is an n-tuple of non-negative integers satisfying $|\beta| = \beta_1+\beta_2+\cdots+\beta_n \leqslant d$, and $x^\beta = x^{\beta_1}x^{\beta_2}\cdots x^{\beta_n}$.

If $\{a_j\}$ is a collection of (p,d)-atoms and $\{\lambda_j\}$ is a sequence of complex numbers with $\sum_{j=1}^\infty |\lambda_j|^p < \infty$, then the series $f = \sum_{j=1}^\infty \lambda_j a_j$ converges in the sense of distributions, and its sum belongs to H^p with the quasinorm

$$||f||_{H^p} = \inf_{\sum_{j=1}^\infty \lambda_j a_j = f} \Big(\sum_{j=1}^\infty |\lambda_j|^p\Big)^{1/p}$$

(see [16]).

THEOREM 1. *Let* $b \in BMO(\mathbb{R}^n)$. *Then* $T_{u,b}$ *maps* $H^1(\mathbb{R}^n)$ *boundedly into weak-*$L^1(\mathbb{R}^n)$*; i.e., there exists a constant* $C = C(n)$ *such that for all* $f \in H^1(\mathbb{R}^n)$

$$|\{x \in \mathbb{R}^n : |T_{u,b}f(x)| > \alpha\}| \leqslant \frac{C}{\alpha}(1+|u|)^{\frac{n}{2}}\,||f||_{H^1(\mathbb{R}^n)}$$

for all $\alpha > 0$.

REMARK 1. *It was shown in [13] that the norm for the imaginary power* u *of the Laplace operator, from* $L^1(\mathbb{R}^n)$ *to* $L^{1,\infty}(\mathbb{R}^n)$*, depends on the imaginary power as* $(1+|u|)^{\frac{n}{2}}$.

DEFINITION 2. Let $0 < p \leqslant 1$ and b be a locally integrable function. Let Q be a cube in $\mathbb{R}^n$. A function a is said to be a $(p,\infty;b)$-atom, if

(i) $\operatorname{supp}(a) \subset Q$;

(ii) $||a||_{L^\infty(\mathbb{R}^n)} \leqslant |Q|^{-1/p}$;

(iii) $\displaystyle\int_{\mathbb{R}^n} a(x)\,x^\beta\,dx = \int_{\mathbb{R}^n} a(x)\,b(x)\,x^\beta\,dx = 0,\ \ |\beta| \leqslant [n(1/p-1)],$

where $[x]$ denotes the integer part of x.

A tempered distribution f is said to belong to the atomic Hardy space $H^p_b(\mathbb{R}^n)$, if it can be written $f = \sum_{j=1}^\infty \lambda_j a_j$, in the sense of distributions, where a_j's are $(p,\infty;b)$-atoms, $\lambda_j \in \mathbb{C}$ and $\sum_{j=1}^\infty |\lambda_j|^p < \infty$. Moreover, we define the quasinorm on H^p_b by

$$||f||_{H^p_b} = \inf_{\sum_{j=1}^\infty \lambda_j a_j = f} \Big(\sum_{j=1}^\infty |\lambda_j|^p\Big)^{1/p}$$

(see [1, 14]).

The following theorem states that (1.1) is true for the commutator $T_{u,b}$ on Hardy type spaces.

THEOREM 2. *Let* $b \in BMO(\mathbb{R}^n)$ *and* $0 < p \leqslant 1$. *The inequality*

$$||T_{u,b}f||_{L^p(\mathbb{R}^n)} \leqslant C_p\,(1+|u|)^{\frac{n}{p}-\frac{n}{2}}\,||f||_{H^p_b(\mathbb{R}^n)}$$

holds for all $f \in H^p_b(\mathbb{R}^n)$.

REMARK 2. Alvarez and Pérez [1, 14] defined the Hardy type space $H^p_b(\mathbb{R}^n)$ (for the case $\beta = 0$ in Definition 2), and showed that the commutator of a Calderón-Zygmund singular integral operator and a BMO function is a bounded operator from $H^p_b(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ where $n/(n+1) < p \leqslant 1$. It was pointed out in Y. Komori [10] that this commutator does not map $H^p_b(\mathbb{R}^n)$ into $L^{p,\infty}(\mathbb{R}^n)$ for the critical value $p = n/(n+1)$.

2. The Proof of Theorems

In this section, we prove the theorems 1 and 2. To begin with, we employ the following lemma due to M. Christ [3].

LEMMA 1. *For any* $\alpha > 0$ *and any finite collection of dyadic cubes* Q *and associated positive scalars* λ_Q*, there exists a collection of pairwise disjoint dyadic cubes* S *such that*

$$(1)\quad \sum_{Q\subset S}\lambda_Q \leqslant 8\,\alpha|S| \quad \textit{for all } S,$$

$$(2)\quad \sum|S| \leqslant \alpha^{-1}\sum|\lambda_Q|,$$

$$(3)\quad \Big\|\sum_{Q\not\subset \text{ any } S}\lambda_Q|Q|^{-1}\chi_Q\Big\|_\infty \leqslant \alpha.$$

Proof of Theorem 1. Suppose that $f(x) = \sum\lambda_Q\,a_Q(x)$ is an element of H^1, chosen arbitrarily except that the sum has finitely many terms, that $\sum\lambda_Q \leqslant 2\,||f||_{H^1}$ and that $\alpha > 0$ is given. Applying Lemma 1, set $h = \sum_S\sum_{Q\subset S}\lambda_Q a_Q$ and $g = f - h$. Then, we have $||g||_\infty \leqslant C\,\alpha$ and

$$\{x\in\mathbb{R}^n : |T_{u,b}f(x)| > \alpha\} \subset \Omega_1\cup\Omega_2\cup\Omega_3$$

where Ω_1 is the union of the double cubes S^* and

$$\Omega_2 = \Big\{x\in\mathbb{R}^n\setminus\Omega_1 : |T_{u,b}g(x)| > \frac{\alpha}{3}\Big\},$$
$$\Omega_3 = \Big\{x\in\mathbb{R}^n\setminus\Omega_1 : |T_{u,b}h(x)| > \frac{\alpha}{3}\Big\}.$$

We now assume that $|u| > 2$ (so that the property (b) of K_u holds for $|x| > |u||y| > 0$, see [8]). By the disjointness of the cubes S we have

$$|\Omega_1| \leqslant \sum_S|S^*| \leqslant \frac{C}{\alpha}\sum_Q|\lambda_Q| \leqslant \frac{C}{\alpha}(1+|u|)^{\frac{n}{2}}\,||f||_{H^1(\mathbb{R}^n)},$$

and Chebyshev's inequality and the L^2-boundedness of $T_{u,b}$ (see [2, 7]) imply

$$|\Omega_2| \leqslant C\,\frac{||g||_2^2}{\alpha^2} \leqslant \frac{C}{\alpha}\sum_Q|\lambda_Q| \leqslant \frac{C}{\alpha}(1+|u|)^{\frac{n}{2}}||f||_{H^1(\mathbb{R}^n)}.$$

In order to estimate the measure of Ω_3, fix the cube Q with radius R centered at x_Q. Applying (ii) of Definition 1, we split the integral

$$\begin{aligned} T_{u,b}a_Q(x) &= \int_{\mathbb{R}^n}(K_u(x-y)-K_u(x-x_Q))(b(x)-b_Q)\,a_Q(y)\,dy \\ &\quad + \int_{\mathbb{R}^n}K_u(x-y)(b_Q-b(y))\,a_Q(y)\,dy \\ &:= I_Q(x) + T_u((b_Q-b)a_Q)(x). \end{aligned}$$

We first estimate $I_Q(x)$. By applying the property (b) of K_u, we see that

$$|I_Q(x)| \leqslant C\,(1+|u|)^{\frac{n}{2}+1}\,|y-x_Q|\,|x-x_Q|^{-n-1}\,|b(x)-b_Q|.$$

Let $Q^{|u|}$ be the $|u|$ dilation of Q. Then

$$\begin{aligned}
&\Big|\Big\{x\in\mathbb{R}^n\setminus\Omega_1 : \sum_S\sum_{Q\subset S}\lambda_Q|I_Q(x)|>\frac{\alpha}{6}\Big\}\Big| \qquad (2.1)\\
&\leqslant \frac{C}{\alpha}\sum_S\sum_{Q\subset S}\lambda_Q\int_{\mathbb{R}^n\setminus Q^{|u|}}|I_Q(x)|\,dx\\
&\leqslant \frac{C}{\alpha}\sum_S\sum_{Q\subset S}\lambda_Q\,(1+|u|)^{\frac{n}{2}+1}R\sum_{j=1}^{\infty}(2^j|u|R)^{-n-1+n}\frac{1}{|Q^{2^{j+1}|u|}|}\int_{Q^{2^{j+1}|u|}}|b(x)-b_Q|\,dx\\
&\leqslant \frac{C}{\alpha}(1+|u|)^{\frac{n}{2}}\,||b||_{BMO}\sum_S\sum_{Q\subset S}\lambda_Q\sum_{j=1}^{\infty}2^{-j}.
\end{aligned}$$

Next, we observe that

$$\begin{aligned}
|T_u((b_Q-b)a_Q)(x)| &\leqslant C\int_Q|K_u(x-y)|\,|b_Q-b(y)|\,|a_Q(y)|\,dy\\
&\leqslant C\,(1+|u|)^{\frac{n}{2}}\int_Q|x-y|^{-n}\,|b_Q-b(y)|\,|a_Q(y)|\,dy\\
&\leqslant C\,(1+|u|)^{\frac{n}{2}}\,|x-x_Q|^{-n}\frac{1}{|Q|}\int_Q|b_Q-b(y)|\,dy\\
&\leqslant C\,(1+|u|)^{\frac{n}{2}}\,|x-x_Q|^{-n}\,||b||_{BMO}.
\end{aligned}$$

Thus, we clearly have

$$\begin{aligned}
&\Big|\Big\{x\in\mathbb{R}^n\setminus\Omega_1 : \sum_S\sum_{Q\subset S}\lambda_Q|T_u((b_Q-b)a_Q)(x)|>\frac{\alpha}{6}\Big\}\Big| \qquad (2.2)\\
&\qquad\leqslant \frac{C}{\alpha}(1+|u|)^{\frac{n}{2}}\,||b||_{BMO}\sum_S\sum_{Q\subset S}\lambda_Q.
\end{aligned}$$

Combining (2.1) and (2.2), we get the measure of Ω_3. This finishes the proof. □

We turn to the proof of Theorem 2.

Proof of Theorem 2. In view of Definition 2, it suffices to show that

$$||T_{u,b}a||_{L^p(\mathbb{R}^n)}\leqslant C\,(1+|u|)^{\frac{n}{p}-\frac{n}{2}} \qquad (2.3)$$

for any $(p,\infty;b)$ atom a.

By translation-invariance, we may assume that Q with radius R is centered at the origin and we write

$$\begin{aligned}
\int_{\mathbb{R}^n}|T_{u,b}a(x)|^p\,dx &= \int_{|x|<|u|R}|T_{u,b}a(x)|^p\,dx+\int_{|x|>|u|R}|T_{u,b}a(x)|^p\,dx\\
&:= L_1+L_2.
\end{aligned}$$

(We note that it is different from the usual technique: instead of splitting the integral at $2R$, we split at $|u|R$, just as in [8, 9]). For the integral L_1, we assume that $0 < p < 2$ and $p/2 + 1/q = 1$. Then L^2-boundedness of $T_{u,b}$ and Hölder's inequality, we have

$$L_1 \leqslant C\,(|u|R)^{n-\frac{np}{2}}\,||a||_2^p \;\leqslant\; C\,|u|^{n-\frac{np}{2}}.$$

Consider the integral L_2. Since $T_{u,b}a(x) = (b(x) - b_Q)T_u a(x) - T_u(b(x) - b_Q)a(x)$ and $0 < p \leqslant 1$, we have

$$\begin{aligned} L_2 &= \int_{|x|>|u|R} |T_{u,b}a(x)|^p\,dx \\ &\leqslant \sum_{j=1}^{\infty}\Big[\int_{Q^{2^{j+1}|u|}\setminus Q^{2^j|u|}} |b(x) - b_Q|^p\,|T_u a(x)|^p\,dx \\ &\qquad + \int_{Q^{2^{j+1}|u|}\setminus Q^{2^j|u|}} |T_u((b_Q - b)a)(x)|^p\,dx\Big] \\ &:= \sum_{j=1}^{\infty}[A_j + B_j]. \end{aligned}$$

In order to estimate each A_j and B_j, suppose first that $n/(n+1) < p \leqslant 1$. By using the property (b) of K_u and (ii) of Definition 1, we observe that

$$\begin{aligned} |T_u a(x)| &\leqslant \int_{|y|<R} |K_u(x-y) - K_u(x)|\,|a(y)|\,dy \\ &\leqslant C\,(1+|u|)^{\frac{n}{2}+1}\,|x|^{-n-1}\int_{|y|<R} |y|\,|a(y)|\,dy \\ &\leqslant C\,(1+|u|)^{\frac{n}{2}+1}\,R^{n+1-\frac{n}{p}}\,|x|^{-n-1}, \end{aligned}$$

whenever $|x| > |u|R$.

Hence, we have

$$\begin{aligned} A_j &= \int_{Q^{2^{j+1}|u|}\setminus Q^{2^j|u|}} |b(x) - b_Q|^p\,|T_u a(x)|^p\,dx \qquad (2.4)\\ &\leqslant C\,(1+|u|)^{\frac{np}{2}+p}\,R^{np+p-n}\int_{Q^{2^{j+1}|u|}\setminus Q^{2^j|u|}} |b(x) - b_Q|^p\,|x|^{-np-p}\,dx \\ &\leqslant C\,(1+|u|)^{\frac{np}{2}+p}\,R^{np+p-n}\,(2^j|u|R)^{-np-p+n}\,\frac{1}{|Q^{2^{j+1}|u|}|}\int_{Q^{2^{j+1}|u|}} |b(x) - b_Q|^p\,dx \\ &\leqslant C\,2^{j(n-np-p)}\,(1+|u|)^{n-\frac{np}{2}}\,||b||_{BMO}^p, \end{aligned}$$

by taking Hölder's inequality to $\frac{1}{|Q^{2^{j+1}|u|}|}\int_{Q^{2^{j+1}|u|}} |b(x) - b_Q|^p\,dx$ when $0 < p < 1$ with $p + 1/q = 1$. We now proceed to the estimate B_j. Similarly as before, by the property

(b) of K_u, (ii) of Definition 1 and (iii) of Definition 2, we consider

$$\begin{aligned} |T_u((b_Q-b)a)(x)| &\leqslant C\int_{|y|<R}|K_u(x-y)-K_u(x)|\,|b_Q-b(y)|\,|a(y)|\,dy \\ &\leqslant C\,(1+|u|)^{\frac{n}{2}+1}\,|x|^{-n-1}\,R^{1-\frac{n}{p}+n}\,\frac{1}{|Q|}\int_Q|b_Q-b(y)|\,dy \\ &\leqslant C\,(1+|u|)^{\frac{n}{2}+1}\,|x|^{-n-1}\,R^{n+1-\frac{n}{p}}\,||b||_{BMO}. \end{aligned}$$

Moreover, we have

$$\begin{aligned} B_j &= \int_{Q^{2^{j+1}|u|}\setminus Q^{2^j|u|}}|T_u((b_Q-b)a)(x)|^p\,dx \qquad (2.5)\\ &\leqslant C\,(1+|u|)^{\frac{np}{2}+p}\,R^{pn+p-n}\,||b||^p_{BMO}\int_{Q^{2^{j+1}|u|}\setminus Q^{2^j|u|}}|x|^{-np-p}\,dx \\ &\leqslant C\,2^{j(n-np-p)}\,(1+|u|)^{n-\frac{np}{2}}\,||b||^p_{BMO}. \end{aligned}$$

Thus combining (2.4) and (2.5), we have

$$L_2 = \sum_{j=1}^{\infty}[A_j+B_j] \leqslant C\sum_{j=1}^{\infty}2^{j(n-np-p)}\,(1+|u|)^{n-\frac{np}{2}}\,||b||^p_{BMO}$$

since $n/(n+1)<p\leqslant 1$. Then taking the p-th root, we obtain (2.3) as desired.

Suppose now that $n/(n+k)<p\leqslant n/(n+k-1)$ for some $k\in\mathbb{N}$. For the estimates, we use the fact that K_u is of C^k class away from the origin and satisfies (c). Let $P_{u,x}(y)$ denote the $(k-1)$-th order polynomial of the function $y\to K_u(x-y)$ expanded about the origin of the cube Q. Then using the cancellation property (ii) of Definition 1,

$$T_ua(x) = \int_{|y|<R}[K_u(x-y)-P_{u,x}(y)]\,a(y)\,dy, \qquad (2.6)$$

and a straightforward calculation shows that

$$\begin{aligned} |T_ua(x)| &\leqslant C\int_{|y|<R}\sum_{|\gamma|=k}|D^\gamma K_u(x)|\,|y|^k\,|a(y)|\,dy \qquad (2.7)\\ &\leqslant C\,(1+|u|)^{\frac{n}{2}+k}\,R^{n+k-\frac{n}{p}}\,|x|^{-n-k}, \quad |x|>|u|R. \end{aligned}$$

Following the same arguments with (2.4) and using (2.7), the estimate

$$A_j \leqslant C\,2^{j(n-np-kp)}\,(1+|u|)^{n-\frac{np}{2}}\,||b||^p_{BMO}$$

holds.

Finally, for the estimate B_j, likewise (2.6), using the property (c) of K_u and (iii) of Definition 2, we have

$$\begin{aligned}|T_u((b_Q-b)a)(x)| &\leqslant \int_{|y|<R} |K_u(x-y)-P_{u,x}(y)|\,|b_Q-b(y)|\,|a(y)|\,dy\\ &\leqslant C\int_{|y|<R}\sum_{|\gamma|=k}|D^\gamma K_u(x)|\,|y|^k\,|b_Q-b(y)|\,|a(y)|\,dy\\ &\leqslant C\,(1+|u|)^{\frac{n}{2}+k}\,|x|^{-n-k}\,R^{n+k-\frac{n}{p}}\,||b||_{BMO}.\end{aligned}$$

Then it follows that

$$B_j \leqslant C\,2^{j(n-np-kp)}\,(1+|u|)^{n-\frac{np}{2}}\,||b||^p_{BMO}.$$

Since $n/(n+k)<p\leqslant n/(n+k-1)$ for some $k\in\mathbb{N}$, we thus have

$$\begin{aligned}L_2=\sum_{j=1}^\infty[A_j+B_j] &\leqslant C\sum_{j=1}^\infty 2^{j(n-np-kp)}\,(1+|u|)^{n-\frac{np}{2}}\,||b||^p_{BMO}\\ &\leqslant C\,(1+|u|)^{n-\frac{np}{2}}\,||b||^p_{BMO}.\end{aligned}$$

The proof is complete. □

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