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A NOTE ON δ–QUASI–MONOTONE AND ALMOST INCREASING SEQUENCES

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Abstract. In this paper by using an almost increasing sequence a general theorem on $\varphi-|C,\alpha|_k$ summability factors, which among others generalizes a result of Mazhar[9] on $|C,1|_k$ summability factors, has been proved under weaker conditions.

1. A sequence of (b_n) of positive numbers is said to be δ-quasi-monotone, if $b_n \to 0$, $b_n > 0$ ultimately and $\Delta b_n \geqslant -\delta_n$, where (δ_n) is a sequence of positive numbers (see [3]). Let (φ_n) be a sequence of complex numbers and let $\sum a_n$ be a given infinite series with partial sums (s_n). We denote by σ_n^α and t_n^α the n-th Cesàro means of order α, with $\alpha > -1$, of the sequence (s_n) and (na_n), respectively, i.e.,

$$\sigma_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v \tag{1}$$

$$t_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v, \tag{2}$$

where

$$A_n^\alpha = O(n^\alpha), \quad \alpha > -1, \quad A_0^\alpha = 1 \quad \text{and} \quad A_{-n}^\alpha = 0 \quad \text{for} \quad n > 0. \tag{3}$$

The series $\sum a_n$ is said to be summable $|C,\alpha|_k$, $k \geqslant 1$ and $\alpha > -1$, if (see [5])

$$\sum_{n=1}^{\infty} n^{k-1} |\sigma_n^\alpha - \sigma_{n-1}^\alpha|^k = \sum_{n=1}^{\infty} \frac{1}{n} |t_n^\alpha|^k < \infty \tag{4}$$

and it is said to be summable $|C,\alpha;\beta|_k$, $k \geqslant 1$, $\alpha > -1$ and $\beta \geqslant 0$, if (see [6])

$$\sum_{n=1}^{\infty} n^{\beta k+k-1} |\sigma_n^\alpha - \sigma_{n-1}^\alpha|^k = \sum_{n=1}^{\infty} n^{\beta k-1} |t_n^\alpha|^k < \infty. \tag{5}$$

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The series $\sum a_n$ is said to be summable $\varphi - \mid C,\alpha\mid_k$, $k\geqslant 1$ and $\alpha > -1$, if (see [2] and [7])

$$\sum_{n=1}^{\infty} n^{-k}\mid \varphi_n t_n^{\alpha}\mid^k < \infty. \tag{6}$$

In the special case when $\varphi_n = n^{1-\frac{1}{k}}$ (resp. $\varphi_n = n^{\beta+1-\frac{1}{k}}$), $\varphi - \mid C,\alpha\mid_k$ summability is the same as $\mid C,\alpha\mid_k$ (resp. $\mid C,\alpha;\beta\mid_k$) summability.

Mazhar [9] has proved the following theorem for $\mid C,1\mid_k$ summability factors of infinite series.

THEOREM A. *Let $\lambda_n \to 0$ as $n\to\infty$. Suppose that there exists a sequence of numbers (B_n) such that it is δ-quasi-monotone with $\sum n\delta_n \log n < \infty$, $\sum B_n \log n$ is convergent and $\mid\Delta\lambda_n\mid\leqslant\mid B_n\mid$ for all n. If*

$$\sum_{n=1}^{m}\frac{1}{n}\mid t_n\mid^k = O(\log m) \quad as \quad m\to\infty, \tag{7}$$

where (t_n) is the n-th (C,1) mean of the sequence (na_n), then the series $\sum a_n\lambda_n$ is summable $\mid C,1\mid_k, k\geqslant 1$.

2. The aim of this paper is to generalize Theorem A under weaker conditions for $\varphi - \mid C,\alpha\mid_k$ summability. For this we need the concept of almost increasing sequence. A positive sequence (b_n) is said to be almost increasing if there exists a positive increasing sequence c_n and two positive constants A and B such that $Ac_n \leqslant b_n \leqslant Bc_n$ (see [1]). Obviously every increasing sequence is almost increasing sequence but the converse need not be true as can be seen from the example $b_n = ne^{(-1)^n}$. So we are weakening the hypotheses of the theorem replacing the increasing sequence by an almost increasing sequence and use a more general summability.

Now, we shall prove following theorem.

THEOREM. *Let (X_n) be an almost increasing sequence such that $\mid\Delta X_n\mid = O(\frac{X_n}{n})$ and $\lambda_n \to 0$ as $n\to\infty$. Suppose that there exists a sequence of numbers (B_n) such that it is δ-quasi-monotone with $\sum nX_n\delta_n < \infty$, $\sum B_nX_n$ is convergent and $\mid\Delta\lambda_n\mid\leqslant\mid B_n\mid$ for all n. If there exists an $\epsilon > 0$ such that the sequence $(n^{\epsilon-k}\mid\varphi_n\mid^k)$ is non-increasing and if the sequence (w_n^{α}), defined by (see [10])*

$$w_n^{\alpha} = \begin{cases} \mid t_n^{\alpha}\mid, & \alpha = 1 \\ \max\limits_{1\leqslant v\leqslant n}\mid t_v^{\alpha}\mid, & 0<\alpha<1 \end{cases} \tag{8}$$

satisfies the condition

$$\sum_{n=1}^{m} n^{-k}(w_n^{\alpha}\mid\varphi_n\mid)^k = O(X_m) \quad as \quad m\to\infty, \tag{9}$$

then the series $\sum a_n\lambda_n$ is summable $\varphi - \mid C,\alpha\mid_k$, $k\geqslant 1$, $1/k\leqslant\alpha\leqslant 1$.

We need the following lemmas for the proof of our theorem.

LEMMA 1. ([4]) *If* $0<\alpha\leqslant 1$ *and* $1\leqslant v\leqslant n$, *then*

$$\Big|\sum_{p=0}^{v}A_{n-p}^{\alpha-1}a_p\Big|\leqslant\max_{1\leqslant m\leqslant v}\Big|\sum_{p=0}^{m}A_{m-p}^{\alpha-1}a_p\Big|. \tag{10}$$

LEMMA 2. *Under the conditions regarding* λ_n *and* (X_n) *of the Theorem, we have*

$$|\lambda_n|X_n=O(1)\quad as\quad n\to\infty. \tag{11}$$

Proof. Since $\lambda_n\to 0\quad as\quad n\to\infty$, we have that

$$\begin{aligned}|\lambda_n|X_n &= X_n\Big|\sum_{v=n}^{\infty}\Delta\lambda_v\Big|\leqslant X_n\sum_{v=n}^{\infty}|\Delta\lambda_v|\leqslant O(1)\sum_{v=o}^{\infty}X_v|\Delta\lambda_v|\\ &\leqslant O(1)\sum_{v=o}^{\infty}X_v|B_v|<\infty.\end{aligned}$$

Hence $|\lambda_n|X_n=O(1)\quad as\quad n\to\infty$.

LEMMA 3. *Under the conditions pertaining to* (X_n) *and* (B_n) *of the Theorem, we have*

$$nB_nX_n=O(1) \tag{12}$$

$$\sum_{n=1}^{\infty}nX_n|\Delta B_n|<\infty. \tag{13}$$

The statements proof of Lemma 3 are proved in Theorem 1 and Theorem 2 of Leindler [8] and hence is omitted.

3. *Proof of the Theorem.* Let (T_n^α) be the n-th (C,α), with $1/k\leqslant\alpha\leqslant 1$, mean of the sequence $(na_n\lambda_n)$. Then, by (2), we have

$$T_n^\alpha=\frac{1}{A_n^\alpha}\sum_{v=1}^{n}A_{n-v}^{\alpha-1}va_v\lambda_v. \tag{14}$$

Using Abel's transformation, we get

$$T_n^\alpha=\frac{1}{A_n^\alpha}\sum_{v=1}^{n-1}\Delta\lambda_v\sum_{p=1}^{v}A_{n-p}^{\alpha-1}pa_p+\frac{\lambda_n}{A_n^\alpha}\sum_{v=1}^{n}A_{n-v}^{\alpha-1}va_v,$$

so that making use of Lemma 1, we have

$$\begin{aligned}|T_n^\alpha| &\leqslant \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1}|\Delta\lambda_v|\Big|\sum_{p=1}^{v}A_{n-p}^{\alpha-1}pa_p\Big|+\frac{|\lambda_n|}{A_n^\alpha}\Big|\sum_{v=1}^{n}A_{n-v}^{\alpha-1}va_v\Big|\\ &\leqslant \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1}A_v^\alpha w_v^\alpha|\Delta\lambda_v|+|\lambda_n|w_n^\alpha\\ &= T_{n,1}^\alpha+T_{n,2}^\alpha,\quad\text{say.}\end{aligned}$$

Since

$$|T_{n,1}^{\alpha}+T_{n,2}^{\alpha}|^k \leqslant 2^k(|T_{n,1}^{\alpha}|^k+|T_{n,2}^{\alpha}|^k),$$

to complete the proof of the theorem, it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{-k}\mid \varphi_n T_{n,r}^{\alpha}\mid^k<\infty \quad \text{for} \quad r=1,2, \quad \text{by} \quad (6).$$

Now, when $k>1$, applying Hölder's inequality with indices k and k', where $\frac{1}{k}+\frac{1}{k'}=1$, we get that

$$\begin{aligned}
\sum_{n=2}^{m+1} n^{-k}\mid \varphi_n T_{n,1}^{\alpha}\mid^k &\leqslant \sum_{n=2}^{m+1} n^{-k}(A_n^{\alpha})^{-k}\mid \varphi_n\mid^k \{\sum_{v=1}^{n-1} A_v^{\alpha} w_v^{\alpha}\mid \Delta\lambda_v\mid\}^k \\
&= O(1)\sum_{n=2}^{m+1} n^{-k} n^{-\alpha k}\mid \varphi_n\mid^k \{\sum_{v=1}^{n-1} v^{\alpha k}(w_v^{\alpha})^k\mid B_v\mid\}\{\sum_{v=1}^{n-1}\mid B_v\mid\}^{k-1} \\
&= O(1)\sum_{v=1}^{m} v^{\alpha k}(w_v^{\alpha})^k\mid B_v\mid \sum_{n=v+1}^{m+1}\frac{n^{-k}\mid\varphi_n\mid^k}{n^{\alpha k}} \\
&= O(1)\sum_{v=1}^{m} v^{\alpha k}(w_v^{\alpha})^k\mid B_v\mid \sum_{n=v+1}^{m+1}\frac{n^{\epsilon-k}\mid\varphi_n\mid^k}{n^{\alpha k+\epsilon}} \\
&= O(1)\sum_{v=1}^{m} v^{\alpha k}(w_v^{\alpha})^k\mid B_v\mid v^{\epsilon-k}\mid\varphi_v\mid^k \sum_{n=v+1}^{m+1}\frac{1}{n^{\alpha k+\epsilon}} \\
&= O(1)\sum_{v=1}^{m} v^{\alpha k}(w_v^{\alpha})^k\mid B_v\mid v^{\epsilon-k}\mid\varphi_v\mid^k \int_v^{\infty}\frac{dx}{x^{\alpha k+\epsilon}} \\
&= O(1)\sum_{v=1}^{m} v\mid B_v\mid v^{-k}(w_v^{\alpha}\mid\varphi_v\mid)^k \\
&= O(1)\sum_{v=1}^{m-1}\Delta(v\mid B_v\mid)\sum_{r=1}^{v} r^{-k}(w_r^{\alpha}\mid\varphi_r\mid)^k \\
&\qquad +O(1)m\mid B_m\mid\sum_{v=1}^{m} v^{-k}(w_v^{\alpha}\mid\varphi_v\mid)^k \\
&= O(1)\sum_{v=1}^{m-1}\Big|(v+1)\mid\Delta B_v\mid-\mid B_v\mid\Big|X_v \\
&= O(1)\sum_{v=1}^{m-1} v\mid\Delta B_v\mid X_v+O(1)\sum_{v=1}^{m-1}\mid B_v\mid X_v \\
&= O(1)\sum_{v=1}^{m-1} v\mid\Delta B_v\mid X_v+O(1)\sum_{v=1}^{m-1}\mid B_{v+1}\mid X_{v+1} \\
&\qquad +O(1)m\mid B_m\mid X_m=O(1) \quad \text{as} \quad m\to\infty,
\end{aligned}$$

by virtue of the hypotheses of the Theorem and Lemma 3.

Again, since $|\lambda_n| = O(1/X_n) = O(1)$, by (11), we have that

$$\begin{aligned}
\sum_{n=1}^{m} n^{-k} \mid \varphi_n T_{n,2}^{\alpha} \mid^k &= \sum_{n=1}^{m} \mid \lambda_n \mid^{k-1} \mid \lambda_n \mid n^{-k} (w_n^{\alpha} \mid \varphi_n \mid)^k \\
&= O(1) \sum_{n=1}^{m} \mid \lambda_n \mid n^{-k} (w_n^{\alpha} \mid \varphi_n \mid)^k \\
&= O(1) \sum_{n=1}^{m-1} \Delta \mid \lambda_n \mid \sum_{v=1}^{n} v^{-k} (w_v^{\alpha} \mid \varphi_v \mid)^k + O(1) \mid \lambda_m \mid \\
&\quad \cdot \sum_{n=1}^{m} n^{-k} (w_n^{\alpha} \mid \varphi_n \mid)^k \\
&= O(1) \sum_{n=1}^{m-1} \mid \Delta\lambda_n \mid X_n + O(1) \mid \lambda_m \mid X_m \\
&= O(1) \sum_{n=1}^{m-1} \mid B_n \mid X_n + O(1) \mid \lambda_m \mid X_m = O(1) \quad \text{as} \quad m \to \infty,
\end{aligned}$$

by virtue of the hypotheses of the Theorem and Lemma 2.

Therefore, we get that

$$\sum_{n=1}^{m} n^{-k} \mid \varphi_n T_{n,r}^{\alpha} \mid^k = O(1) \quad \text{as} \quad m \to \infty, \quad \text{for} \quad r = 1, 2.$$

This completes the proof of the Theorem.

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