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RATE OF CONVERGENCE OF A KANTOROVICH VARIANT OF THE MEYER–KÖNIG AND ZELLER OPERATORS

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(*communicated by Ram N. Mohapatra*)

Abstract. This paper is concerned with a Kantorovich variant of the Meyer-König and Zeller operators which was defined by Maier, Müller and Swetits. We derive sharp bounds for the first and second central moments yielding estimates for the rate of convergence in terms of the modulus of continuity. Finally, we study the asymptotic behaviour of these operators.

1. Introduction

The Kantorovich variant $\hat{M}_n$ of the Meyer-König and Zeller operators, as defined by Müller [11] and studied by Maier, Müller and Swetits [10], is given by

$$\hat{M}_n(f;x) = (n+1)(1-x)^n \sum_{k=0}^{\infty} \binom{n+k+1}{k} x^k \int_{I_k} f(t)\, dt, \tag{1}$$

where $I_k = \left[\frac{k}{n+k}, \frac{k+1}{n+k+1}\right]$. The operators (1) were treated by several authors. Guo [7, Lemma 5] and Love e.a. [8] studied the convergence for functions f of bounded variation. To this end they gave estimates for the second central moment of the operators $\hat{M}_n$. Throughout the paper, for each real x, put $\psi_x(t) = t - x$. Guo [7, Lemma 5] showed, that for each fixed $x \in [0,1]$, there holds the asymptotic relation

$$\hat{M}_n\left(\psi_x^2; x\right) = \frac{x(1-x)^2}{n-1} + o\left(n^{-1}\right) \qquad (n \to \infty).$$

However, the latter result does not imply his Eq. (2.10)

$$\frac{x(1-x)^2}{2n} \leqslant \hat{M}_n\left(\psi_x^2; x\right) \leqslant \frac{2x(1-x)^2}{n}$$

uniformly in $[0,1]$ which is crucial for his main result. This was pointed out in [8]. Love e.a. [8, Lemma 7] proved the estimate

$$\hat{M}_n\left(\psi_x^2; x\right) \leqslant \frac{4x(1-x)}{n-1} + \frac{(1-x)^2}{3(n-1)^2} \qquad (0 \leqslant x \leqslant 1).$$

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In this note we derive estimates for the first and second central moments of the operators $\hat{M}_n$, where the main order terms are optimal. As a consequence we obtain estimates for the rate of convergence by the modulus of continuity. Finally, we treat the asymptotic behaviour of the operators $\hat{M}_n$. In particular we give a Voronovskaja type result. Analogous results for the ordinary Meyer-König and Zeller operators [9] in the slight modification of Cheney and Sharma [5]

$$M_n(f;x) = (1-x)^{n+1} \sum_{k=0}^{\infty} \binom{n+k}{k} x^k f\left(\frac{k}{n+k}\right)$$

(see also [6]) are contained in [1] and [2].

2. A sharp estimate of the first and second central moment

In this section we focus on the first and second central moment of the operators $\hat{M}_n$.

PROPOSITION 2.1. *For the first central moment $\hat{M}_n(\psi_x;x)$, we have the estimate*

$$\hat{M}_n(\psi_x;x) = \frac{(1-x)(1-3x)}{2(n-1)} + \frac{(1-x)^2(4x-1)}{(n-1)^{\underline{2}}} + r_n^{[1]}(x) \qquad (n>5), \qquad (2)$$

where the remainder $r_n^{[1]}(x)$ can be estimated by

$$-\frac{12(1-x)^3}{(n-1)^{\underline{3}}} \leqslant r_n^{[1]}(x) \leqslant \frac{3(1-x)^4}{(n-1)^{\underline{3}}}\left(1+\frac{20}{n-4}\right) \qquad (x\in[0,1]).$$

Since $-1 \leqslant (1-x)^2(4x-1) \leqslant 1/4$ on $[0,1]$, we have the

COROLLARY 2.2. *For each choice of constants $C_1 < -1$ and $C_2 > 1/4$, there exists an integer $n_0 \in \mathbb{N}$, such that the estimate*

$$\frac{C_1}{(n-1)(n-2)} \leqslant \hat{M}_n(\psi_x;x) - \frac{(1-x)(1-3x)}{2(n-1)} \leqslant \frac{C_2}{(n-1)(n-2)} \qquad (0\leqslant x\leqslant 1)$$

is valid, for all $n > n_0$. In particular, for each constant $C > 0$, there holds

$$\frac{(1-x)(1-3x)-C}{2(n-1)} \leqslant \hat{M}_n(\psi_x;x) \leqslant \frac{(1-x)(1-3x)+C}{2(n-1)} \qquad (0\leqslant x\leqslant 1),$$

for all sufficiently large integers n.

PROPOSITION 2.3. *For the second central moment $\hat{M}_n\left(\psi_x^2;x\right)$, we have the estimate*

$$\hat{M}_n\left(\psi_x^2;x\right) = \frac{x(1-x)^2}{n-1} + \frac{a_2(x)}{(n-1)^{\underline{2}}} + r_n^{[2]}(x) \qquad (n>5), \qquad (3)$$

where the remainder $r_n(x)$ *can be estimated by*

$$\left|r_n^{[2]}(x)\right| \leqslant \frac{K}{n^3} \qquad (x \in [0,1])$$

with a constant K *independent of* n. *More precisely, there holds*

$$\frac{a(x)}{(n-1)^{\underline{3}}} \leqslant r_n^{[2]}(x) \leqslant \sum_{r=3}^{5} \frac{a_r(x)}{(n-1)^{\underline{r}}} \qquad (0 \leqslant x \leqslant 1),$$

where the polynomials $a_r(x)$ *are given by*

$$\begin{aligned} a_2(x) &= \frac{1}{3}(1-x)^2\left(1-20x+31x^2\right), \\ a_3(x) &= \frac{1}{3}(1-x)^3\left(367-547x+72x^2\right), \\ a_4(x) &= \frac{8}{3}(1-x)^4(227-72x), \\ a_5(x) &= 480(1-x)^5 \end{aligned}$$

and $a(x)$ *is a certain polynomial independent of* n.

COROLLARY 2.4. *For each choice of constants* $C_1 < -0.4$ *and* $C_2 > 1/3$, *there exists an integer* $n_0 \in \mathbb{N}$, *such that the estimate*

$$\frac{C_1}{(n-1)(n-2)} \leqslant \hat{M}_n\left(\psi_x^2; x\right) - \frac{x(1-x)^2}{n-1} \leqslant \frac{C_2}{(n-1)(n-2)} \qquad (0 \leqslant x \leqslant 1)$$

is valid, for all $n > n_0$. *In particular, for each constant* $C > 0$, *there holds*

$$\frac{x(1-x)^2 - C}{n-1} \leqslant \hat{M}_n\left(\psi_x^2; x\right) \leqslant \frac{x(1-x)^2 + C}{n-1} \qquad (0 \leqslant x \leqslant 1),$$

for all sufficiently large integers n.

The corollary is an immediate consequence of Proposition 2.3, since, for all $x \in [0,1]$,

$$\begin{aligned} -0.387877 \approx \frac{55 - 7\sqrt{993}}{3813248}&\left(63 + \sqrt{993}\right)^2 \\ &= a_2\left(\frac{1}{124}\left(61 - \sqrt{993}\right)\right) \leqslant a_2(x) \leqslant a_2(0). \end{aligned}$$

The next two lemmas follow by direct computation.

LEMMA 2.5. *For* $r = 0, 1, 2, \ldots$, *and* $n > r$, *there holds*

$$(1-x)^n \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(n+k-1)^{\underline{r}}} = \frac{(1-x)^r}{(n-1)^{\underline{r}}}.$$

LEMMA 2.6. *The identities*

$$\begin{aligned} \frac{1}{w} &= \frac{1}{w-1} - \frac{1!}{(w-1)^{\underline{2}}} + \frac{2!}{w^{\underline{3}}} \\ &= \frac{1}{w-1} - \frac{1!}{(w-1)^{\underline{2}}} + \frac{2!}{(w-1)^{\underline{3}}} - \frac{3!}{(w-1)^{\underline{4}}} + \frac{4!}{w^{\underline{5}}}, \\ \frac{1}{w+1} &= \frac{1}{w-1} - \frac{2!}{(w-1)^{\underline{2}}} + \frac{3!}{(w-1)^{\underline{3}}} - \frac{4!}{(w+1)\,(w-1)^{\underline{3}}} \end{aligned}$$

are valid, for all w, for which the denominators do not vanish.

Proof of Proposition 2.1. After a short calculation we have

$$\begin{aligned} (n+1)\,\frac{(n+k+1)^{\underline{2}}}{(n+1)^{\underline{2}}}\int_{I_k} t\,dt &= \frac{(n+k+1)^{\underline{2}}}{n}\left(\frac{n}{(n+k+1)^{\underline{2}}} - n^2\frac{2\,(n+k)+1}{2\left[(n+k+1)^{\underline{2}}\right]^2}\right) \\ &= 1 - \frac{n}{2}\left(\frac{1}{n+k+1} + \frac{1}{n+k}\right). \end{aligned}$$

By definition (1), we obtain

$$\hat{M}_n\,(\psi_x;x) = 1 - x - \frac{n}{2}\,(1-x)^n \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k \left(\frac{1}{n+k+1} + \frac{1}{n+k}\right).$$

Application of Lemma 2.6 yields

$$\frac{1}{w+1} + \frac{1}{w} = \frac{2}{w-1} - \frac{3}{(w-1)^{\underline{2}}} + \frac{8}{(w-1)^{\underline{3}}} - \frac{24}{(w+1)\,(w-1)^{\underline{3}}} - \frac{6}{(w-1)^{\underline{4}}} + \frac{24}{w^{\underline{5}}}$$

and we obtain, for $w > 4$,

$$-\frac{30\,(w-3)}{(w+1)\,(w-1)^{\underline{4}}} \leqslant \left(\frac{1}{w+1} + \frac{1}{w}\right) - \left(\frac{2}{w-1} - \frac{3}{(w-1)^{\underline{2}}} + \frac{8}{(w-1)^{\underline{3}}}\right) \leqslant \frac{-6}{w^{\underline{4}}}.$$

Putting $w = n+k$ we conclude that

$$\begin{aligned} \hat{M}_n\,(\psi_x;x) = 1 - x - \frac{n}{2}\,(1-x)^n \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k \\ \left(\frac{2}{n+k-1} - \frac{3}{(n+k-1)^{\underline{2}}} + \frac{8}{(n+k-1)^{\underline{3}}} - \frac{30\xi_k}{(n+k-1)^{\underline{4}}}\right) \end{aligned}$$

with $0 < \xi_k < 1$ $(k = 0, 1, 2, \ldots)$. Using Lemma 2.5 we obtain with a certain number $\xi = \xi\,(n,x) \in (0,1)$

$$\begin{aligned} \hat{M}_n\,(\psi_x;x) &= 1 - x - \frac{n}{2}\left(\frac{2\,(1-x)}{n-1} - \frac{3\,(1-x)^2}{(n-1)^{\underline{2}}} + \frac{8\,(1-x)^3}{(n-1)^{\underline{3}}} - \frac{30\xi\,(1-x)^4}{(n-1)^{\underline{4}}}\right) \\ &= \frac{(1-x)\,(1-3x)}{2\,(n-1)} + \frac{(1-x)^2\,(4x-1)}{(n-1)^{\underline{2}}} + r_n^{[1]}\,(x), \end{aligned}$$

where

$$\frac{-12(1-x)^3}{(n-1)^{\underline{3}}} \leqslant r_n^{[1]}(x) = \frac{-12(1-x)^3 + 15\xi(1-x)^4}{(n-1)^{\underline{3}}} + \frac{60\xi(1-x)^4}{(n-1)^{\underline{4}}}$$
$$\leqslant \frac{(1-x)^3(3-15x) + 60(1-x)^4/(n-4)}{(n-1)^{\underline{3}}},$$

which implies the inclusion of $r_n^{[1]}(x)$. □

Proof of Proposition 2.3. Using the obvious equation

$$\int_{I_k} \psi_x^2(t)\ dt = \int_{I_k}(t-x)^2\ dt = \frac{1}{3}\left(\frac{k+1}{n+k+1}-x\right)^3 - \frac{1}{3}\left(\frac{k}{n+k}-x\right)^3$$

some easy calculations yield

$$\begin{aligned}&(n+1)\frac{(n+k+1)^{\underline{2}}}{(n+1)^{\underline{2}}}\int_{I_k}\psi_x^2(t)\ dt\\ &\quad = x^2 + \left(n\frac{1+2(n+k)}{(n+k)(n+k+1)} - 2\right)x\\ &\qquad + \frac{1}{3}\left[\left(\frac{k+1}{n+k+1}\right)^2 + \frac{k+1}{n+k+1}\frac{k}{n+k} + \left(\frac{k}{n+k}\right)^2\right].\end{aligned}$$

By definition (1), this implies

$$\hat{M}_n\left(\psi_x^2;x\right) = (1-x)^n \sum_{k=0}^{\infty}\binom{n+k-1}{k}\left[b_2(n,k)x^2 + b_1(n,k)x + b_0(n,k)\right]x^k, \tag{4}$$

where the coefficients can be written in the form

$$\begin{aligned} b_2(n,k) &= 1,\\ b_1(n,k) &= \left(\frac{n}{n+k} + \frac{n}{n+k+1}\right) - 2,\\ b_0(n,k) &= 1 - \frac{n^2+n}{n+k+1} + \frac{n^2-n}{n+k} + \frac{1}{3}\frac{n^2}{(n+k+1)^2(n+k)^2}.\end{aligned}$$

By application of Lemma 2.6 with $w = n+k$, we obtain

$$b_1(n,k) \leqslant -2 + \frac{2n}{n+k-1} - \frac{3n}{(n+k-1)^{\underline{2}}} + \frac{8n}{(n+k-1)^{\underline{3}}},$$

$$
\begin{aligned}
& b_0(n,k) \\
& \quad \leqslant 1-\left(n^2+n\right)\left[\frac{1}{n+k-1}-\frac{2!}{(n+k-1)^{\underline{2}}}+\frac{3!}{(n+k-1)^{\underline{3}}}-\frac{4!}{(n+k-1)^{\underline{4}}}\right] \\
& \qquad +\left(n^2-n\right)\left[\frac{1}{n+k-1}-\frac{1!}{(n+k-1)^{\underline{2}}}+\frac{2!}{(n+k-1)^{\underline{3}}}-\frac{3!}{(n+k-1)^{\underline{4}}}+\frac{4!}{(n+k-1)^{\underline{5}}}\right] \\
& \qquad +\frac{n^2}{3}\frac{1}{(n+k-1)^{\underline{4}}} \\
& \quad =1+\frac{-2n}{n+k-1}+\frac{n^2+3n}{(n+k-1)^{\underline{2}}}+\frac{-4n^2-8n}{(n+k-1)^{\underline{3}}}+\frac{(55/3)\,n^2+30n}{(n+k-1)^{\underline{4}}}+\frac{24\left(n^2-n\right)}{(n+k-1)^{\underline{5}}}.
\end{aligned}
$$

By Lemma 2.5, we conclude

$$
(1-x)^n \sum_{k=0}^{\infty}\binom{n+k-1}{k} b_2(n,k)\, x^{k+2}=x^2,
$$

$$
\begin{aligned}
& (1-x)^n \sum_{k=0}^{\infty}\binom{n+k-1}{k} b_1(n,k)\, x^{k+1} \\
& \quad \leqslant -2x+\frac{2nx(1-x)}{n-1}-\frac{3nx(1-x)^2}{(n-1)^{\underline{2}}}+\frac{8nx(1-x)^3}{(n-1)^{\underline{3}}} \\
& \quad =-2x^2+\frac{x(1-x)(3x-1)}{n-1}-\frac{2x(4x-1)(1-x)^2}{(n-1)^{\underline{2}}}+\frac{24x(1-x)^3}{(n-1)^{\underline{3}}},
\end{aligned}
$$

and

$$
\begin{aligned}
& (1-x)^n \sum_{k=0}^{\infty}\binom{n+k-1}{k} b_0(n,k)\, x^{k} \\
& \quad \leqslant 1-2n\frac{1-x}{n-1}+\left(n^2+3n\right)\frac{(1-x)^2}{(n-1)^{\underline{2}}}-\left(4n^2+8n\right)\frac{(1-x)^3}{(n-1)^{\underline{3}}} \\
& \qquad +\left(\frac{55}{3}n^2+30n\right)\frac{(1-x)^4}{(n-1)^{\underline{4}}}+24\left(n^2-n\right)\frac{(1-x)^5}{(n-1)^{\underline{5}}} \\
& \quad =x^2+\frac{2x(1-x)(1-2x)}{n-1}+\frac{(1-x)^2\left(1-26x+55x^2\right)}{3(n-1)^{\underline{2}}} \\
& \qquad +\frac{(1-x)^3\left(367-619x+72x^2\right)}{3(n-1)^{\underline{3}}} \\
& \qquad +\frac{8(1-x)^4(227-72x)}{3(n-1)^{\underline{4}}}+\frac{480(1-x)^5}{(n-1)^{\underline{5}}}.
\end{aligned}
$$

Collecting these estimates yields, by Eq. (4),

$$\begin{aligned}\hat{M}_n\left(\psi_x^2;x\right) &\leqslant x^2-2x+1-\frac{2n(1-x)^2}{n-1}+\frac{\left(n^2+3n(1-x)\right)(1-x)^2}{(n-1)^{\underline{2}}}\\&\quad-\frac{\left(4n^2+8n(1-x)\right)(1-x)^3}{(n-1)^{\underline{3}}}+\left(\frac{55}{3}n^2+30n\right)\frac{(1-x)^4}{(n-1)^{\underline{4}}}\\&\quad+24\left(n^2-n\right)\frac{(1-x)^5}{(n-1)^{\underline{5}}}.\end{aligned}$$

and after a short calculation it results representation (3) with the estimate

$$r_n^{[2]}(x)\leqslant\sum_{r=3}^{5}\frac{a_r(x)}{(n-1)^{\underline{r}}}.$$

The estimate of $r_n(x)$ from below follows by a refinement of the identities gathered in Lemma 2.6. The very technical proof is omitted here. □

3. The rate of convergence

The estimates of the first and second central moment enable us to obtain estimations for the rate of convergence by the first modulus of continuity. The corollaries of the preceding section imply that for each $\varepsilon>0$, there exists an integer $n_0\in\mathbb{N}$, such that, for all $n>n_0$ there holds

$$\left|\hat{M}_n(\psi_x;x)\right|\leqslant\frac{(1-x)|1-3x|+\varepsilon}{2(n-1)}\qquad(0\leqslant x\leqslant 1),$$

$$\left|\hat{M}_n\left(\psi_x^2;x\right)\right|\leqslant\frac{x(1-x)^2+\varepsilon}{n-1}\qquad(0\leqslant x\leqslant 1).$$

By standard arguments (see, e.g., [3, Theorem 5.1.2]) we obtain the following results.

THEOREM 3.1. *Let $f\in C[0,1]$ and $\delta>0$. For each $\varepsilon>0$, there exists an integer $n_0\in\mathbb{N}$, such that*

$$\left|\hat{M}_n(f;x)-f(x)\right|\leqslant\left(1+\frac{1}{\delta}\sqrt{\frac{x(1-x)^2+\varepsilon}{n-1}}\right)\omega(f;\delta)\qquad(x\in[0,1],\quad n>n_0).$$

Moreover, if f is differentiable on $[0,1]$ with f' bounded on $[0,1]$, we also have

$$\begin{aligned}\left|\hat{M}_n(f;x)-f(x)\right| &\leqslant \frac{(1-x)|1-3x|+\varepsilon}{2(n-1)}\left|f'(x)\right|\\&\quad+\sqrt{\frac{x(1-x)^2+\varepsilon}{n-1}}\left(1+\frac{1}{\delta}\sqrt{\frac{x(1-x)^2+\varepsilon}{n-1}}\right)\omega\left(f';\delta\right).\end{aligned}$$

Theorem 3.1 applied to $\delta = \sqrt{\left(x(1-x)^2+\varepsilon\right)/(n-1)}$ implies the following corollary.

COROLLARY 3.2. *Let $f \in C[0,1]$. For each $\varepsilon > 0$, there exists an integer $n_0 \in \mathbb{N}$, such that*

$$\left|\hat{M}_n(f;x) - f(x)\right| \leqslant 2\omega\left(f;\sqrt{\frac{x(1-x)^2+\varepsilon}{n-1}}\right) \qquad (x \in [0,1], \quad n > n_0).$$

Moreover, if f is differentiable on $[0,1]$ with f' bounded on $[0,1]$, we also have

$$\left|\hat{M}_n(f;x) - f(x)\right|$$
$$\leqslant \frac{(1-x)|1-3x|+\varepsilon}{2(n-1)}|f'(x)| + 2\sqrt{\frac{x(1-x)^2+\varepsilon}{n-1}} \quad \omega\left(f';\sqrt{\frac{x(1-x)^2+\varepsilon}{n-1}}\right).$$

4. The asymptotic behaviour

Fix $x \in (0,1)$. Put $g_r(t) = (1-t)^r$. Then we have, for $r = 0,1,2,\ldots,$

$$\int_{I_k} g_r(t)\ dt = \frac{n^{r+1}}{r+1}\left[(n+k)^{-r-1} - (n+k+1)^{-r-1}\right] \qquad (k = 0,1,2,\ldots).$$

Application of the formula

$$\frac{r!}{z^{r+1}} = \int_0^\infty t^r e^{-zt}\, dt \qquad (z > 0)$$

leads to

$$\int_{I_k} g_r(t)\ dt = \frac{n^{r+1}}{(r+1)!}\int_0^\infty t^r\left[e^{-(n+k)t} - e^{-(n+k+1)t}\right]\ dt \qquad (k = 0,1,2,\ldots)$$

and we obtain

$$\begin{aligned}\hat{M}_n(g_r;x) &= \frac{n^{r+1}(n+1)}{(r+1)!}(1-x)^n\sum_{k=0}^\infty\binom{n+k+1}{k}x^k\int_0^\infty t^r e^{-kt}\left[e^{-nt} - e^{-(n+1)t}\right] dt \\ &= \frac{n^{r+1}(n+1)}{(r+1)!}\int_0^\infty t^r\frac{(1-x)^n}{(1-xe^{-t})^{n+2}}\left[e^{-nt} - e^{-(n+1)t}\right]\ dt \\ &= \frac{n^{r+1}(n+1)}{(r+1)!(1-x)}\int_0^\infty t^r\left(\frac{1-x}{e^t-x}\right)^{n+1}\frac{e^t-1}{e^t-x}e^t\ dt.\end{aligned}$$

A change of variable replacing $e^t - x$ by $(1-x)e^t$ yields the Laplace integral

$$\hat{M}_n(g_r;x) = \frac{n^{r+1}(n+1)}{(r+1)!}\int_0^\infty \log^r(1+(1-x)(e^t-1))\ e^{-(n+1)t}\ (e^t-1)\ dt.$$

Thus, $\hat{M}_n(g_r;x)$ can be written as the Laplace transform of the function

$$w_{r,x}(t) = (e^t - 1)\ \log^r(1 + (1-x)(e^t - 1)).$$

Note that $w_{r,x}$ is holomorphic in a neighborhood of the origin $t = 0$ and satisfies the growth condition $w_{r,x}(t) = O(t^r e^t)$ as $t \to +\infty$. Thus, we can apply Watson's Lemma (see, e.g., [4, C.3, p. 614]), obtaining the complete asymptotic expansion of the latter integral

$$\int_0^\infty \log^r(1 + (1-x)(e^t - 1))\ e^{-(n+1)t}\ (e^t - 1)\ dt \sim \sum_{k=0}^\infty \frac{k!\ a_k^{[r]}}{(n+1)^{k+1}} \qquad (n \to \infty),$$

where the coefficients $a_k^{[r]}$ are determined by the power series expansion

$$w_{r,x}(t) = \sum_{k=0}^\infty a_k^{[r]} t^k.$$

It remains to calculate the coefficients $a_k^{[r]}$, which depend on r and x. We make use of the well-known formulas

$$(e^t - 1)^r = r! \sum_{k=r}^\infty S_k^r \frac{t^k}{k!} \qquad (t \in \mathbb{R})$$

and

$$\log^r(1+t) = r! \sum_{k=r}^\infty \sigma_k^r \frac{t^k}{k!} \qquad (|t| < 1),$$

where the quantities S_k^r and σ_k^r denote the Stirling numbers of the first and second kinds, respectively. We obtain

$$\begin{aligned} w_{r,x}(t) &= r! \sum_{\mu=r}^\infty \sigma_\mu^r \frac{1}{\mu!} (1-x)^\mu (e^t - 1)^{\mu+1} \\ &= r! \sum_{\mu=r}^\infty \sigma_\mu^r (1-x)^\mu (\mu+1) \sum_{\nu=\mu+1}^\infty S_\nu^{\mu+1} \frac{t^\nu}{\nu!} \\ &= r! \sum_{\nu=r+1}^\infty \frac{t^\nu}{\nu!} \sum_{\mu=r}^{\nu-1} (\mu+1) S_\nu^{\mu+1} \sigma_\mu^r (1-x)^\mu \end{aligned}$$

and, therefore, $a_k^{[r]} = 0 \ \ (0 \leqslant k \leqslant r)$, and

$$a_k^{[r]} = \frac{r!}{k!} \sum_{\mu=r}^{k-1} (\mu+1) S_k^{\mu+1} \sigma_\mu^r (1-x)^\mu \qquad (k \geqslant r+1).$$

Combining the latter results yields

$$\hat{M}_n(g_r;x) \sim \frac{n^{r+1}}{r+1} \sum_{k=r+1}^\infty \frac{1}{(n+1)^k} \sum_{\mu=r}^{k-1} (\mu+1) S_k^{\mu+1} \sigma_\mu^r (1-x)^\mu \qquad (n \to \infty).$$

For the first central moments, we obtain

$$\begin{aligned}
\hat{M}_n\left(\psi_x^0;x\right) &= 1,\\
\hat{M}_n\left(\psi_x;x\right) &= \frac{(1-x)(1-3x)}{2(n+1)}+\frac{2x(1-x)(1-2x)}{(n+1)^2}+O\left(n^{-3}\right),\\
\hat{M}_n\left(\psi_x^2;x\right) &= \frac{x(1-x)^2}{n+1}+\frac{(1-x)^2\left(1-14x+31x^2\right)}{3(n+1)^2}+O\left(n^{-3}\right),\\
\hat{M}_n\left(\psi_x^3;x\right) &= \frac{x(1-x)^3(5-19x)}{2(n+1)^2}+O\left(n^{-3}\right),\\
\hat{M}_n\left(\psi_x^4;x\right) &= \frac{3x^2(1-x)^4}{(n+1)^2}+O\left(n^{-3}\right) \qquad (n\to\infty).
\end{aligned}$$

We mention that $\hat{M}_n\left(\psi_x^s;x\right)=O\left(n^{-3}\right)$ $(s=5,6)$ and $\hat{M}_n\left(\psi_x^s;x\right)=O\left(n^{-4}\right)$ $(s=7,8)$.

In order to derive an asymptotic expansion of the operators $\hat{M}_n$ we use the following general approximation theorem for positive linear operators due to Sikkema [12, Theorem 3] (cf. [13, Theorems 1 and 2]).

LEMMA 4.1. *Let I be an interval. For $q\in\mathbb{N}$ and fixed $x\in I$, let $A_n:L_\infty(I)\to C(I)$ be a sequence of positive linear operators with the property*

$$A_n(\psi_x^s;x)=O(n^{-\lfloor (s+1)/2\rfloor}) \qquad (n\to\infty) \qquad\qquad (s=0,1,\ldots,2q+2).$$

Then, we have, for each $f\in L_\infty(I)$ which is $2q$ times differentiable at x, the asymptotic relation

$$A_n(f;x)=\sum_{s=0}^{2q}\frac{f^{(s)}(x)}{s!}A_n(\psi_x^s;x)+o(n^{-q}) \qquad (n\to\infty). \tag{5}$$

If, in addition, $f^{(2q+2)}(x)$ exists, the term $o(n^{-q})$ in (5) can be replaced by $O(n^{-(q+1)})$.

THEOREM 4.2. *Let $f\in L_\infty[0,1]$ and assume that $f^{(4)}(x)$ exists at a point $x\in(0,1)$. Then, we have the asymptotic relation*

$$\hat{M}_n(f;x)=f(x)+\frac{c_1(f;x)}{n+1}+\frac{c_2(f;x)}{(n+1)^2}+o\left(n^{-2}\right) \qquad (n\to\infty),$$

where the coefficients $c_k(f;x)$ $(k=1,2)$ are given by

$$\begin{aligned}
c_1(f;x) &= \frac{1}{2}\left[(1-x)(1-3x)f'(x)+x(1-x)^2f''(x)\right],\\
c_2(f;x) &= 2x(1-x)(1-2x)f'(x)+\frac{1}{6}(1-x)^2\left(1-14x+31x^2\right)f''(x)\\
&\quad+\frac{1}{12}x(1-x)^3(5-19x)f^{(3)}(x)+\frac{1}{8}x^2(1-x)^4f^{(4)}(x).
\end{aligned}$$

COROLLARY 4.3. (Voronovskaja-type theorem) *Let $f \in L_\infty[0,1]$ be twice differentiable in $x \in (0,1)$. Then, we have*

$$\lim_{n\to\infty} n\left(\hat{M}_n(f;x) - f(x)\right) = \frac{1}{2}\left(x(1-x)^2 f'(x)\right)'.$$

REMARK. For the first and second central moment, we obtain after a short calculation the asymptotic relations

$$\hat{M}_n(\psi_x;x) = \frac{(1-x)(1-3x)}{2(n-1)} + \frac{(1-x)^2(4x-1)}{(n-1)(n-2)} + O\left(n^{-3}\right),$$

$$\hat{M}_n\left(\psi_x^2;x\right) = \frac{x(1-x)^2}{n-1} + \frac{(1-x)^2\left(1-20x+31x^2\right)}{3(n-1)(n-2)} + O\left(n^{-3}\right) \qquad (n\to\infty),$$

which is a consequence of Propositions 2.1 and 2.3.

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