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SMOOTHNESS OF ψ–DIRECT SUMS OF BANACH SPACES

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Abstract. Let $X_1, X_2, \cdots, X_n$ be Banach spaces and ψ a continuous convex function with some appropriate conditions on a certain convex subset of $\mathbb{R}^n$. Let $(X_1 \oplus X_2 \oplus \cdots \oplus X_n)_\psi$ be the direct sum of $X_1, X_2, \cdots, X_n$ equipped with the associated norm with ψ. Then we give the characterization of smoothness of $(X_1 \oplus X_2 \oplus \cdots \oplus X_n)_\psi$.

1. Introduction

A norm $\|\cdot\|$ on $\mathbb{C}^n$ is said to be absolute if

$$\|(x_1, x_2, \cdots, x_n)\| = \|(|x_1|, |x_2|, \cdots, |x_n|)\|$$

for all $(x_1, x_2, \cdots, x_n) \in \mathbb{C}^n$, and normalized if

$$\|(1, 0, \cdots, 0)\| = \|(0, 1, 0, \cdots, 0)\| = \cdots = \|(0, \cdots, 0, 1)\| = 1.$$

Let AN_n be the family of all absolute normalized norms on $\mathbb{C}^n$. In [9], Saito, Kato and Takahashi characterized absolute normalized norms on $\mathbb{C}^n$ by means of the corresponding convex function as follows. For each $n \in \mathbb{N}$ with $n \geqslant 2$, we put

$$\Delta_n = \left\{ (t_1, t_2, t_3, \cdots, t_{n-1}) \in \mathbb{R}^{n-1} : t_j \geqslant 0, \sum_{j=1}^{n-1} t_j \leqslant 1 \right\}$$

and define the set Ψ_n of all continuous convex functions on Δ_n satisfying the following conditions:

$$\psi(0, 0, \cdots, 0) = \psi(1, 0, 0, \cdots, 0) = \psi(0, 1, 0, \cdots, 0) = \cdots = \psi(0, \cdots, 0, 1) = 1 \qquad (A_0)$$

$$\psi(t_1, \ldots, t_{n-1}) \geqslant (t_1 + \cdots + t_{n-1})\psi\left(\frac{t_1}{t_1 + \cdots + t_{n-1}}, \cdots, \frac{t_{n-1}}{t_1 + \cdots + t_{n-1}}\right) \qquad (A_1)$$

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$$\psi(t_1,\dots,t_{n-1}) \geqslant (1-t_1)\psi\left(0,\frac{t_2}{1-t_1},\cdots,\frac{t_{n-1}}{1-t_1}\right) \qquad (A_2)$$

$$\psi(t_1,\dots,t_{n-1}) \geqslant (1-t_2)\psi\left(\frac{t_1}{1-t_2},0,\frac{t_3}{1-t_2},\cdots,\frac{t_{n-1}}{1-t_2}\right) \qquad (A_3)$$

$$\vdots \qquad\qquad \vdots$$

$$\psi(t_1,\dots,t_{n-1}) \geqslant (1-t_{n-1})\psi\left(\frac{t_1}{1-t_{n-1}},\cdots,\frac{t_{n-2}}{1-t_{n-1}},0\right). \qquad (A_n)$$

Then for each $n\in\mathbb{N}$ with $n\geqslant 2$, AN_n and Ψ_n are in a one-to-one correspondence under the following equation:

$$\psi(t_1,\cdots,t_{n-1}) = \left\|\left(1-\sum_{j=1}^{n-1}t_j, t_1,\cdots,t_{n-1}\right)\right\| \qquad (1)$$

for $(t_1,\cdots,t_{n-1})\in\Delta_n$. Indeed, for any $\psi\in\Psi_n$, we define

$$\left\|(x_1,x_2,\cdots,x_n)\right\|_\psi = \begin{cases} \left(|x_1|+\cdots+|x_n|\right)\psi\left(\dfrac{|x_2|}{|x_1|+\cdots+|x_n|},\cdots,\dfrac{|x_n|}{|x_1|+\cdots+|x_n|}\right), \\ \qquad\qquad \text{if } (x_1,\cdots,x_n)\neq(0,\cdots,0), \\ 0, \qquad\qquad \text{if } (x_1,\cdots,x_n)=(0,\cdots,0). \end{cases}$$

Then $\|\cdot\|_\psi\in AN_n$ and satisfies (1). From this, we can consider many non-ℓ_p type norms.

Mitani, Saito and Suzuki in [6] calculated all norming functionals of absolute normalized norms on $\mathbb{C}^n$ and they gave a necessary and sufficient condition of ψ that $(\mathbb{C}^n,\|\cdot\|_\psi)$ is smooth. That is, they showed that $(\mathbb{C}^n,\|\cdot\|_\psi)$ is smooth if and only if for each $t=(t_1,t_2,\cdots,t_{n-1})\in\Delta_n$, the following equalities hold:

1. $\psi'_-(t;p_j-t)=\psi'_+(t;p_j-t)$ for all $j\in I_n$ with $t_j>0$;
2. $\psi'_+(t;p_j-t)=-\psi(t)$ for all $j\in I_n$ with $t_j=0$

(see the notations of ψ'_-, ψ'_+, p_j and I_n in §2 or [6, Corollary 4]).

On the other hand, Kato, Saito and Tamura in [4, 8] introduced the ψ-direct sum $(X_1\oplus X_2\oplus\cdots\oplus X_n)_\psi$ as follows. Let $X_1,X_2,\cdots,X_n$ be Banach spaces and let $\psi\in\Psi_n$. Then the product space $X_1\times\cdots\times X_n$ with the norm

$$\left\|\left(x_1,x_2,\cdots,x_n\right)\right\|_\psi = \left\|\left(||x_1||,||x_2||,\cdots,||x_n||\right)\right\|_\psi \text{ with } x_i\in X_i \text{ for } 1\leqslant i\leqslant n,$$

is a Banach space which is denoted by $(X_1\oplus X_2\oplus\cdots\oplus X_n)_\psi$. They characterized the strict convexity, uniform convexity of $(X_1\oplus X_2\oplus\cdots\oplus X_n)_\psi$ using the convex function ψ.

In this paper, we give the necessary and sufficient condition that $(X_1\oplus X_2\oplus\cdots\oplus X_n)_\psi$ is smooth. Namely, we shall show that $(X_1\oplus X_2\oplus\cdots\oplus X_n)_\psi$ is smooth if and only if $(\mathbb{C}^n,\|\cdot\|_\psi)$ is smooth and X_i is smooth for all i. In §3, we consider the dual space of $\mathbb{C}^n$ with an absolute normalized norm. In §5,we calculate all norming functionals of $(X_1\oplus X_2\oplus\cdots\oplus X_n)_\psi$. It follows from this that we give the characterization of smoothness of $(X_1\oplus X_2\oplus\cdots\oplus X_n)_\psi$. Finally, in §6, we show a necessary and sufficient condition that $(X_1\oplus X_2\oplus\cdots\oplus X_n)_\psi$ is uniformly smooth.

2. Preliminaries

Throughout of this paper, we denote by $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ the set of positive integers, real numbers and complex numbers, respectively. Let X be a Banach space with norm $\|\cdot\|$ and let X^* be the dual space of X. Let $S_X = \{x \in X : \|x\| = 1\}$ be the unit sphere of X. A bounded linear functional $\alpha \in X^*$ is said to be a norming functional of $x \in X$ with $x \neq 0$ if $\alpha \in S_{X^*}$ and $\langle \alpha, x \rangle = \|x\|$ (see [1]). We denote by $D(X,x)$ the set of all norming functionals of x. The Hahn-Banach theorem yields that, for every $x \in X$ with $x \neq 0$, there exists at least one norming functional of x. A Banach space X is said to be smooth if for every $x \in X$ with $x \neq 0$, there exists a unique norming functional of x. We know that X is smooth if and only if $\|\cdot\|$ is Gâteaux differentiable at any $x \in X \setminus \{0\}$, that is,

$$\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists for every $x, y \in X$ with $x \neq 0$ (cf. [1]). Let f be a continuous convex function from a convex subset C of a real Banach space X into $\mathbb{R}$. As in [7], we denote by $\partial f(x)$ the subdifferential of f at $x \in C$;

$$\partial f(x) = \{a \in X^* : f(y) \geqslant f(x) + \langle a, y-x \rangle \text{ for } y \in C\}.$$

It is clear that $\partial f(x)$ is a closed convex subset of X^*. We know $\partial f(x) \neq \emptyset$ at every $x \in \overset{\circ}{C}$, where $\overset{\circ}{C}$ is the set of interior points of C. In particular, if C is the closed interval $[0,1]$ of $\mathbb{R}$, then the following equation holds:

$$\partial f(t) = \begin{cases} \big(-\infty, f'_R(t)\big], & \text{if } t = 0, \\ \big[f'_L(t), f'_R(t)\big], & \text{if } 0 < t < 1, \\ \big[f'_L(t), \infty\big), & \text{if } t = 1, \end{cases}$$

where $f'_L(t)$ is the left derivative of f at t and $f'_R(t)$ is the right derivative of f at t, respectively.

In this paper, we use the following notations. For $n \in \mathbb{N}$ with $n \geqslant 2$, we put $I_n = \{0, 1, 2, \cdots, n-1\}$. We also put

$$p_0 = (0,0,0,\cdots,0) \in \Delta_n$$

and

$$p_j = (0,0,\cdots,0,\overset{(j)}{1},0,0,\cdots,0) \in \Delta_n$$

for $j = 1, 2, \ldots, n-1$. We define the directional derivative $\psi'_+(t;s)$ of ψ at $t \in \Delta_n$ with respect to $s \in \mathbb{R}^{n-1}$ which satisfies $t + \lambda s \in \Delta_n$ for some $\lambda > 0$,

$$\psi'_+(t;s) = \lim_{\lambda \to +0} \frac{\psi(t+\lambda s) - \psi(t)}{\lambda}.$$

Similarly, if $t \in \Delta_n$ and $s \in \mathbb{R}^{n-1}$ satisfy $t + \lambda s \in \Delta_n$ for some $\lambda < 0$, we define $\psi'_-(t;s)$ by

$$\psi'_-(t;s) = \lim_{\lambda \to -0} \frac{\psi(t+\lambda s) - \psi(t)}{\lambda}.$$

It is clear that $\psi'_-(t;s) = -\psi'_+(t;-s)$ if there exists $\lambda > 0$ such that $t+\lambda s$ and $t-\lambda s$ belong to Δ_n.

3. Dual spaces of $\mathbb{C}^n$ with an absolute norm

In this section, we consider the dual space of $\mathbb{C}^n$ with an absolute normalized norm. Let $\psi \in \Psi_n$. Let $\|\cdot\|_\psi^*$ be the dual norm of $\|\cdot\|_\psi$. That is, for any $(x_1,x_2,\cdots,x_n) \in \mathbb{C}^n$,

$$\begin{aligned}&\left\|(x_1,x_2,\cdots,x_n)\right\|_\psi^* \\ &\qquad = \sup\left\{\left|\left\langle (x_1,x_2,\cdots,x_n),(y_1,y_2,\cdots,y_n)\right\rangle\right| : \left\|(y_1,y_2,\cdots,y_n)\right\|_\psi = 1\right\} \\ &\qquad = \sup\left\{\left|\sum_{j=1}^n x_j y_j\right| : \left\|(y_1,y_2,\cdots,y_n)\right\|_\psi = 1\right\}.\end{aligned}$$

Then $\|\cdot\|_\psi^* \in AN_n$ and the corresponding convex function is given by

$$\begin{aligned}&\psi^*(s_1,\cdots,s_{n-1}) \\ &\quad = \sup_{(t_1,\cdots,t_{n-1})\in\Delta_n} \frac{(1-t_1-\cdots-t_{n-1})(1-s_1-\cdots-s_{n-1})+t_1s_1+\cdots+t_{n-1}s_{n-1}}{\psi(t_1,\cdots,t_{n-1})}\end{aligned}$$

for $(s_1,\cdots,s_{n-1}) \in \Delta_n$. Note that $\|\cdot\|_\psi^* = \|\cdot\|_{\psi^*}$.

PROPOSITION 3.1. ([2] (Generalized Hölder inequality)) *Let $\psi \in \Psi_n$. Then we have*

$$|\langle x,y\rangle| \leqslant \|x\|_\psi \|y\|_{\psi^*}$$

for any $x,y \in \mathbb{C}^n$.

It is easy to see that $\psi^{**} = \psi$. Next we shall caluculate the convex function ψ^* of $\psi \in \Psi_n$.

EXAMPLE 3.2. For $1 \leqslant p \leqslant \infty$, the ℓ_p-norm $\|\cdot\|_p$ on $\mathbb{C}^n$ is an absolute normalized norm and the associated function ψ_p is defined by

$$\psi_p(s_1,s_2,\cdots,s_{n-1}) = \begin{cases} ((1-\sum_{i=1}^{n-1}s_i)^p + s_1^p+\cdots+s_{n-1}^p)^{1/p} & \text{if } 1\leqslant p<\infty, \\ \max(1-\sum_{i=1}^{n-1}s_i, s_1,\cdots,s_{n-1}) & \text{if } p=\infty.\end{cases}$$

Then we have $\psi_p^* = \psi_q$, where $\frac{1}{p}+\frac{1}{q}=1$.

In particular, we consider the absolute norm of $\mathbb{C}^2$ as in [9].

EXAMPLE 3.3. Let $1/2 \leqslant \alpha \leqslant 1$. We define $\|\cdot\|_\alpha \in AN_2$ by

$$\|(x_1,x_2)\|_\alpha = \max\{\|(x_1,x_2)\|_\infty, \alpha\|(x_1,x_2)\|_1\}$$

for $(x_1,x_2) \in \mathbb{C}^2$ and the corresponding convex function is given by $\psi_\alpha(s) = \max\{1-t,t,\alpha\}$. Then we have

$$\psi_\alpha^*(s) = \frac{1}{\alpha}\max\{(1-2\alpha)s+\alpha, (2\alpha-1)s+1-\alpha\}.$$

Next we consider the ψ-direct sum of Banach spaces. Let $X_1, X_2, \cdots, X_n$ be Banach spaces and let $\psi \in \Psi_n$. Then the product space $X_1 \times X_2 \times \cdots \times X_n$ with the norm

$$\left\|(x_1, x_2, \cdots, x_n)\right\|_\psi = \left\|(||x_1||, ||x_2||, \cdots, ||x_n||)\right\|_\psi$$

is a Banach space which is denoted by $(X_1 \oplus X_2 \oplus \cdots \oplus X_n)_\psi$.

PROPOSITION 3.4. *Let $\psi \in \Psi_n$. Then the dual of $(X_1 \oplus X_2 \oplus \cdots \oplus X_n)_\psi$ is isomorphic to $(X_1^* \oplus X_2^* \oplus \cdots \oplus X_n^*)_{\psi^*}$.*

Proof. Let $f = (f_1, f_2, \cdots, f_n) \in (X_1^* \oplus X_2^* \oplus \cdots \oplus X_n^*)_{\psi^*}$. The bounded linear functional $\langle f, \cdot\rangle$ on $(X_1 \oplus X_2 \oplus \cdots \oplus X_n)_\psi$ is defined by

$$\langle f, x\rangle = \sum_{i=1}^{n} f_i(x_i),\ x = (x_1, x_2, \cdots, x_n) \in (X_1 \oplus X_2 \oplus \cdots \oplus X_n)_\psi.$$

Then $||\langle f, \cdot\rangle|| = ||f||_{\psi^*}$. Indeed, using the generalized Hölder inequality, we have

$$\begin{aligned} |\langle f, x\rangle| &\leqslant \sum_{i=1}^{n} ||f_i||||x_i|| \\ &\leqslant \left\|(||f_1||, ||f_2||, \cdots, ||f_n||)\right\|_{\psi^*} \left\|(||x_1||, ||x_2||, \cdots, ||x_n||)\right\|_\psi = ||f||_{\psi^*}||x||_\psi \end{aligned}$$

Hence $||\langle f, \cdot\rangle|| \leqslant ||f||_{\psi^*}$. Also, for any $\varepsilon > 0$, there are $y_1, y_2, \cdots, y_n \in X$ such that $||f_i|| \leqslant (1+\varepsilon) f_i(y_i)$ and $||y_i|| = 1$ for all i. Hence we have

$$\begin{aligned} ||f||_{\psi^*} &= \left\|(||f_1||, ||f_2||, \cdots, ||f_n||)\right\|_{\psi^*} \\ &\leqslant (1+\varepsilon)||(f_1(y_1), f_2(y_2), \cdots, f_n(y_n))||_{\psi^*} \\ &= (1+\varepsilon) \sup\left\{\left|\sum_{i=1}^{n} \alpha_i f_i(y_i)\right| : \alpha_i \in \mathbb{C},\ ||(\alpha_1, \alpha_2, \cdots, \alpha_n)||_\psi = 1\right\} \\ &= (1+\varepsilon) \sup\left\{\left|\sum_{i=1}^{n} f_i(\alpha_i y_i)\right| : \alpha_i \in \mathbb{C},\ ||(\alpha_1 y_1, \alpha_2 y_2, \cdots, \alpha_n y_n)||_\psi = 1\right\} \\ &\leqslant (1+\varepsilon)||\langle f, \cdot\rangle||. \end{aligned}$$

Thus $||f||_{\psi^*} = ||\langle f, \cdot\rangle||$. On the other hand, it is easy to see that every bounded linear functional must be of the form $\langle f, \cdot\rangle$. This completes the proof. $\square$

4. Smoothness of absolute norms on $(X_1 \oplus X_2)_\psi$

In this section, we consider the smoothness of $(X_1 \oplus X_2)_\psi$. The reason for this is that the results for $(X_1 \oplus X_2)_\psi$ illustrate the mechanisms involved in the induction to follow.

Fix $\psi \in \Psi_2$. For each $t \in (0, 1]$, we denote by $\psi_L'(t)$ the left derivative of ψ at t. Similarly for each $t \in [0, 1)$, we denote by $\psi_R'(t)$ the right derivative of ψ at t. Mitani, Saito and Suzuki in [6] characterized the smoothness of $(\mathbb{C}^2, \|\cdot\|_\psi)$ as follows.

PROPOSITION 4.1. ([6]) *Fix $\psi \in \Psi_2$. Then $(\mathbb{C}^2, \|\cdot\|_\psi)$ is smooth if and only if ψ is differentiable at any $t \in (0, 1)$, $\psi_R'(0) = -1$ and $\psi_L'(1) = 1$.*

We first consider all norming functional of $(X_1 \oplus X_2)_\psi$.

THEOREM 4.2. *Let* $\psi \in \Psi_2$. *Let* $(x_1,x_2) \in (X_1 \oplus X_2)_\psi$ *with* $\|(x_1,x_2)\|_\psi = 1$. *Then we have*

$$D\big((X_1 \oplus X_2)_\psi,(x_1,x_2)\big) = \left\{ \big(a_1f_1,a_2f_2\big) : \begin{array}{l} (a_1,a_2) \in D\big(\mathbb{C}^2,(\|x_1\|,\|x_2\|)\big) \\ f_i \in S_{X_1^*} \text{ for } i \text{ with } x_i = 0 \\ f_i \in D(X_i,x_i) \text{ for } i \text{ with } x_i \neq 0 \end{array} \right\}. \tag{2}$$

Proof. We put B as the right hand side of (2). We first show that $D\big((X_1 \oplus X_2)_\psi,(x_1,x_2)\big) \subset B$. Fix $(f_1,f_2) \in D\big((X_1 \oplus X_2)_\psi,(x_1,x_2)\big)$. From

$$\|(f_1,f_2)\|_{\psi^*} = \langle (f_1,f_2),(x_1,x_2)\rangle = \|(x_1,x_2)\|_\psi = 1,$$

we have

$$\begin{aligned} 1 &= f_1(x_1)+f_2(x_2) \\ &\leqslant \|f_1\|\|x_1\| + \|f_2\|\|x_2\| \\ &= \big\langle (\|f_1\|,\|f_2\|),(\|x_1\|,\|x_2\|)\big\rangle \\ &\leqslant \big\|(\|f_1\|,\|f_2\|)\big\|_{\psi^*}\big\|(\|x_1\|,\|x_2\|)\big\|_\psi \\ &= \big\|(f_1,f_2)\big\|_{\psi^*}\big\|(x_1,x_2)\|_\psi = 1. \end{aligned}$$

So we obtain

$$f_i(x_i) = \|f_i\|\|x_i\| \tag{3}$$

for each $i = 1,2$, and

$$\big\langle (\|f_1\|,\|f_2\|),(\|x_1\|,\|x_2\|)\big\rangle = \big\|(\|x_1\|,\|x_2\|)\big\|_\psi = 1. \tag{4}$$

We take any $h_i \in D(X_i,x_i)$ for i with $x_i \neq 0$, and any $h_i \in S_{X_i^*}$ for i with $x_i = 0$. We also put $g_i \in S_{X_i^*}$ as

$$g_i = \begin{cases} \frac{f_i}{\|f_i\|}, & \text{for } i \text{ with } f_i \neq 0 \\ h_i, & \text{for } i \text{ with } f_i = 0. \end{cases}$$

Then we obtain $(f_1,f_2) = (\|f_1\|g_1,\|f_2\|g_2)$. By (4), we have $\big(\|f_1\|,\|f_2\|\big) \in D\big(\mathbb{C}^2,(\|x_1\|,\|x_2\|)\big)$. We also have $g_i \in D\big(X_i,x_i\big)$ for each i with $x_i \neq 0$. Indeed, by (3), we have

$$g_i(x_i) = \frac{f_i}{\|f_i\|}(x_i) = \|x_i\|$$

for i with $x_i \neq 0$ and $f_i \neq 0$. For i with $x_i = 0$, it is clear that $g_i \in S_{X_i^*}$. Thus we have $(f_1,f_2) \in B$ and so $D\big((X_1 \oplus X_2)_\psi,(x_1,x_2)\big) \subset B$.

We next show that $D\big((X_1 \oplus X_2)_\psi,x\big) \supset B$. We put $(a_1f_1,a_2f_2) \in B$ where $(a_1,a_2) \in D\big(\mathbb{C}^2,(\|x_1\|,\|x_2\|)\big)$, $f_i \in S_{X_1^*}$ for i with $x_i = 0$ and $f_i \in D(X_i,x_i)$ for i

with $x_i \neq 0$. Note that $f_i(x_i) = 0 = \|x_i\|$ for i with $x_i = 0$. Since

$$\begin{aligned}\big\langle (a_1 f_1, a_2 f_2), (x_1, x_2)\big\rangle &= a_1 f_1(x_1) + a_2 f_2(x_2) \\ &= a_1\|x_1\| + a_2\|x_2\| \\ &= \big\langle (a_1, a_2), (||x_1||, ||x_2||)\big\rangle \\ &= \big\|\big(||x_1||, ||x_2||\big)\big\|_\psi = \|(x_1, x_2)\|_\psi = 1\end{aligned}$$

and

$$\begin{aligned}||(a_1 f_1, a_2 f_2)||_{\psi^*} &= ||(a_1||f_1||, a_2||f_2||)||_{\psi^*} \\ &= ||(a_1, a_2)||_{\psi^*} = 1,\end{aligned}$$

we obtain $(a_1 f_1, a_2 f_2) \in D\big((X_1 \oplus X_2)_\psi, (x_1, x_2)\big)$. Thus we have $D\big((X_1 \oplus X_2)_\psi, (x_1, x_2)\big) \supset B$. This completes the proof. □

From Theorem 4.2, we obtain the following

THEOREM 4.3. *Let* $\psi \in \Psi_2$. *Then* $(X_1 \oplus X_2)_\psi$ *is smooth if and only if* X_i *is smooth for each* i *and* $(\mathbb{C}^2, \|\cdot\|_\psi)$ *is smooth.*

Proof. Let $(X_1 \oplus X_2)_\psi$ be smooth. By embedding X_1 and X_2 into $(X_1 \oplus X_2)_\psi$, X_1, X_2 are smooth. In the same way, $(\mathbb{C}^2, ||\cdot||_\psi)$ is smooth. Conversely, we suppose that $(\mathbb{C}^2, \|\cdot\|_\psi)$ is smooth and X_i is smooth for each i. Fix $(x_1, x_2) \in (X_1 \oplus X_2)_\psi$ with $||(x_1, x_2)||_\psi = 1$, and

$$(a_1, a_2) \in D\big(\mathbb{C}^2, (||x_1||, ||x_2||)\big). \tag{5}$$

Then

$$^{\#}D\big(\mathbb{C}^2, (||x_1||, ||x_2||)\big) = 1 \tag{6}$$

and

$$^{\#}D(X_i, x_i) = 1 \tag{7}$$

for i with $x_i \neq 0$. Assume that $x_1 = 0$. Then we have $a_1 = 0$. Indeed, since

$$\langle (0, a_2), (\|x_1\|, \|x_2\|)\rangle = \langle (a_1, a_2), (\|x_1\|, \|x_2\|)\rangle = \|(\|x_1\|, \|x_2\|)\|_\psi = 1$$

and

$$\|(0, a_2)\|_{\psi^*} \leqslant \|(a_1, a_2)\|_{\psi^*} = 1,$$

we obtain $(0, a_2) \in D\big(\mathbb{C}^2, (||x_1||, ||x_2||)\big)$. By (5) and (6), we have $a_1 = 0$. In general, for i with $x_i = 0$, we have $a_i = 0$. Using Theorem 4.2, (6) and (7), we obtain

$$^{\#}D\big((X_1 \oplus X_2)_\psi, (x_1, x_2)\big) = 1.$$

Thus $(X_1 \oplus X_2)_\psi$ is smooth. □

Combining Theorem 4.3 and Proposition 4.1, we obtain the following

THEOREM 4.4. *Let* $\psi \in \Psi_2$. *Then* $(X_1 \oplus X_2)_\psi$ *is smooth if and only if* X_i *is smooth for each* i, ψ *is differentiable at any* $t \in (0,1)$, $\psi_R'(0) = -1$ *and* $\psi_L'(1) = 1$.

5. Smoothness of absolute norms on $(X_1 \oplus X_2 \oplus \cdots \oplus X_n)_\psi$

Mitani, Saito and Suzuki in [6] calculated all norming functionals of $(\mathbb{C}^n, \|\cdot\|_\psi)$ as follows. We put $I_n = \{0, 1, 2, \cdots, n-1\}$.

PROPOSITION 5.1. *Let $\psi \in \Psi_n$ be fixed. Let $x = (x_0, x_1, x_2, \cdots, x_{n-1}) \in \mathbb{C}^n$ with $\|x\|_\psi = 1$. Put*

$$t_j = \frac{|x_j|}{\sum_{k=0}^{n-1} |x_k|}$$

for $j \in I_n$, and

$$t = (t_1, t_2, \cdots, t_{n-1}) \in \Delta_n.$$

Put $\rho_j = \arg x_j \in [0, 2\pi)$ for $j \in I_n$, where $\arg 0 = 0$. Then

$$D(\mathbb{C}^n, x) = \left\{ \begin{pmatrix} c_0\big(\psi(t) + \langle a, p_0 - t\rangle\big) \\ c_1\big(\psi(t) + \langle a, p_1 - t\rangle\big) \\ c_2\big(\psi(t) + \langle a, p_2 - t\rangle\big) \\ \vdots \\ c_{n-1}\big(\psi(t) + \langle a, p_{n-1} - t\rangle\big) \end{pmatrix} : \begin{array}{l} a \in \partial\psi(t), \\ \psi(t) + \langle a, p_j - t\rangle \geqslant 0, \\ \quad \text{for } j \in I_n, \\ |c_j| = 1 \\ \quad \text{for } j \in I_n \text{ with } t_j = 0, \\ c_j = e^{-i\rho_j} \\ \quad \text{for } j \in I_n \text{ with } t_j > 0 \end{array} \right\}.$$

From this result, they characterized the smoothness of $(\mathbb{C}^n, \|\cdot\|_\psi)$.

PROPOSITION 5.2. *Let $\psi \in \Psi_n$. Then $(\mathbb{C}^n, \|\cdot\|_\psi)$ is smooth if and only if for each $t = (t_1, t_2, \cdots, t_{n-1}) \in \Delta_n$, the following equalities hold:*

1. *$\psi'_-(t; p_j - t) = \psi'_+(t; p_j - t)$ for all $j \in I_n$ with $t_j > 0$;*
2. *$\psi'_+(t; p_j - t) = -\psi(t)$ for all $j \in I_n$ with $t_j = 0$.*

In this section, we shall characterize the smoothness of $(X_1 \oplus X_2 \oplus \cdots \oplus X_n)_\psi$. We first consider the set $D\big((X_1 \oplus X_2 \oplus \cdots \oplus X_n)_\psi, x\big)$ of all norming functional at x in $(X_1 \oplus X_2 \oplus \cdots \oplus X_n)_\psi$.

THEOREM 5.3. *Let $\psi \in \Psi_n$. Let $x = (x_1, x_2, \cdots, x_n) \in (X_1 \oplus X_2 \oplus \cdots \oplus X_n)_\psi$ with $||x||_\psi = 1$. Then*

$$D\big((X_1 \oplus X_2 \oplus \cdots \oplus X_n)_\psi, x\big) \tag{8}$$
$$= \left\{ (a_1 f_1, a_2 f_2, \cdots, a_n f_n) : \begin{array}{l} (a_1, a_2, \cdots, a_n) \in D\big(\mathbb{C}^n, (||x_1||, ||x_2||, \cdots, ||x_n||)\big) \\ f_i \in S_{X_i^*} \text{ for } i \text{ with } x_i = 0 \\ f_i \in D(X_i, x_i) \text{ for } i \text{ with } x_i \neq 0 \end{array} \right\}.$$

Proof. We put B as the right hand side of (8). We first show that $D\big((X_1 \oplus X_2 \oplus \cdots \oplus X_n)_\psi, x\big) \subset B$. Fix $f = (f_1, f_2, \cdots, f_n) \in D\big((X_1 \oplus X_2 \oplus \cdots \oplus X_n)_\psi, x\big)$. From

$||f||_{\psi^*} = \langle f, x\rangle = 1$, we have

$$\begin{aligned} 1 &= \sum_{i=1}^{n} f_i(x_i) \leqslant \sum_{i=1}^{n} ||f_i||||x_i|| \\ &= \langle (||f_1||, ||f_2||, \cdots, ||f_n||), (||x_1||, ||x_2||, \cdots, ||x_n||)\rangle \\ &\leqslant ||(||f_1||, ||f_2||, \cdots, ||f_n||)||_{\psi^*} ||(||x_1||, ||x_2||, \cdots, ||x_n||)||_{\psi} \\ &= ||f||_{\psi^*}||x||_{\psi} = 1. \end{aligned}$$

So we obtain $f_i(x_i) = ||f_i||||x_i||$ for each i, and

$$\big(||f_1||, ||f_2||, \cdots, ||f_n||\big) \in D\big(\mathbb{C}^n, (||x_1||, ||x_2||, \cdots, ||x_n||)\big).$$

We take an $h_i \in D(X_i, x_i)$, for i with $x_i \neq 0$, and take an $h_i \in S_{X_i^*}$ for i with $x_i = 0$. We also put g_i as

$$g_i = \begin{cases} \frac{f_i}{||f_i||}, & \text{for } i \text{ with } f_i \neq 0 \\ h_i, & \text{for } i \text{ with } f_i = 0. \end{cases}$$

Then we obtain $f = \big(f_1, f_2, \cdots, f_n\big) = \big(||f_1||g_1, ||f_2||g_2, \cdots, ||f_n||g_n\big)$. For i with $x_i \neq 0$ and $f_i \neq 0$, we have

$$g_i(x_i) = \frac{f_i}{||f_i||}(x_i) = ||x_i||$$

and hence $g_i \in D\big(X_i, x_i\big)$. For i with $x_i = 0$, it is clear that $g_i \in X_i^*$. Thus we have $f \in B$ and so $D\big((X_1 \oplus X_2 \oplus \cdots \oplus X_n)_\psi, x\big) \subset B$.

We next show that $D\big((X_1 \oplus X_2 \oplus \cdots \oplus X_n)_\psi, x\big) \supset B$. Fix $(a_1f_1, a_2f_2, \cdots, a_nf_n) \in B$, where $(a_1, a_2, \cdots, a_n) \in D\big(\mathbb{C}^n, (||x_1||, ||x_2||, \cdots, ||x_n||)\big)$, $f_i \in S_X^*$ for i with $x_i = 0$ and $f_i \in D(X_i, x_i)$ for i with $x_i \neq 0$. Note that if $x_i = 0$, we have $f_i(x_i) = 0 = ||x_i||$. Since

$$\begin{aligned} \big\langle (a_1f_1, a_2f_2 \cdots, a_nf_n), (x_1, x_2, \cdots, x_n)\big\rangle &= \sum_{i=1}^{n} a_i f_i(x_i) = \sum_{i=1}^{n} a_i ||x_i|| \\ &= \Big\langle (a_1, a_2, \cdots, a_n), (||x_1||, ||x_2||, \cdots, ||x_n||)\Big\rangle \\ &= \big\|\big(||x_1||, ||x_2||, \cdots, ||x_n||\big)\big\|_{\psi} \\ &= ||x||_{\psi^*} = 1 \end{aligned}$$

and

$$\begin{aligned} ||(a_1f_1, a_2f_2, \cdots, a_nf_n)||_{\psi^*} &= ||(a_1||f_1||, a_2||f_2||, \cdots, a_n||f_n||)||_{\psi^*} \\ &= ||(a_1, a_2, \cdots, a_n)||_{\psi^*} = 1, \end{aligned}$$

we obtain $(a_1f_1, a_2f_2, \cdots, a_nf_n) \in D\big((X_1 \oplus X_2 \oplus \cdots \oplus X_n)_\psi, x\big)$. Thus we have $D\big((X_1 \oplus X_2 \oplus \cdots \oplus X_n)_\psi, x\big) \supset B$. This completes the proof. $\square$

From Theorem 5.3, we obtain the following

THEOREM 5.4. *Let* $\psi \in \Psi_n$. *Then* $(X_1 \oplus X_2 \oplus \cdots \oplus X_n)_\psi$ *is smooth if and only if* $(\mathbb{C}^n, \|\cdot\|_\psi)$ *is smooth and* X_i *is smooth for all* i.

Proof. $(\Rightarrow)$ is clear. Suppose that $(\mathbb{C}^n, \|\cdot\|_\psi)$ is smooth and X_i is smooth for all i. Fix $x = (x_1, x_2, \cdots, x_n) \in (X_1 \oplus X_2 \oplus \cdots \oplus X_n)_\psi$ with $||x||_\psi = 1$. Let $(a_1, a_2, \cdots, a_n) \in D\big(\mathbb{C}^n, (||x_1||, ||x_2||, \cdots, ||x_n||)\big)$. Then we have $a_i = 0$ for i with $x_i = 0$, because

$$(a_1, \cdots, a_{i-1}, 0, a_{i+1} \cdots, a_n) \in D\big(\mathbb{C}^n, (||x_1||, ||x_2||, \cdots, ||x_n||)\big)$$

and ${}^{\#}D\big(\mathbb{C}^n, (||x_1||, ||x_2||, \cdots, ||x_n||)\big) = 1$. So, using Theorem 5.3, we have ${}^{\#}D\big((X_1 \oplus X_2 \oplus \cdots \oplus X_n)_\psi, x\big) = 1$. Thus $(X_1 \oplus X_2 \oplus \cdots \oplus X_n)_\psi$ is smooth. □

6. Uniform smoothness on $(X_1 \oplus X_2 \oplus \cdots \oplus X_n)_\psi$

In this section, we shall characterize the uniform smoothness of $(X_1 \oplus X_2 \oplus \cdots \oplus X_n)_\psi$

DEFINITION 6.1. The modulus of smoothness $\rho_X(\tau)$ of a Banach space X is defined by

$$\rho_X(\tau) = \sup\{(\|x - y\| + \|x + y\|)/2 - 1;\ x, y \in X, \|x\| = 1, \|y\| = \tau\}.$$

Then X is said to be uniformly smooth if $\lim_{\tau \to 0} \rho_X(\tau)/\tau = 0$.

It is well known that for every Banach space X, X is uniformly convex if and only if X^* is uniformly smooth.

In [10], Saito-Kato-Takahashi characterized the strict convexity of absolute norms on $\mathbb{C}^n$ by the corresponding convex function. Recall that a function ψ on Δ_n is called strictly convex if for all $s, t \in \Delta_n$ $(s \neq t)$ we have $\psi\left((s+t)/2\right) < \left(\psi(s) + \psi(t)\right)/2$.

PROPOSITION 6.2. ([10]) *Let* $\psi \in \Psi_n$. *Then* $(\mathbb{C}^n, \|\cdot\|_\psi)$ *is strictly convex if and only if* ψ *is strictly convex on* Δ_n.

Then Saito-Kato [8] and Kato-Saito-Tamura [4] characterized the uniform convexity of $(X_1 \oplus X_2 \oplus \cdots \oplus X_n)_\psi$.

PROPOSITION 6.3. ([8, 4]) *Let* $\psi \in \Psi_n$ *and let* $X_1, X_2, \cdots, X_n$ *be Banach spaces. Then* $(X_1 \oplus X_2 \oplus \cdots \oplus X_n)_\psi$ *is uniformly convex if and only if* $X_1, X_2, \cdots, X_n$ *are uniformly convex and* ψ *is strictly convex on* Δ_n.

We know that if $\dim X < \infty$, X is smooth (resp. strictly convex) if and only if X^* is strictly convex (resp. smooth). So $(\mathbb{C}^n, \|\cdot\|_\psi)$ is smooth (resp. strictly convex) if and only if $(\mathbb{C}^n, \|\cdot\|_{\psi^*})$ is strictly convex (resp. smooth). It is clear that $(X_1 \oplus X_2 \oplus \cdots \oplus X_n)_\psi$ is uniformly smooth if and only if $(X_1^* \oplus X_2^* \oplus \cdots \oplus X_n^*)_{\psi^*}$ is uniformly convex. Since uniform smoothness is the dual notion of uniform convexity, we have by Propositions 6.2 and 6.3.

THEOREM 6.4. *Let* $\psi \in \Psi_n$ *and let* $X_1, X_2, \cdots, X_n$ *be Banach spaces. Then* $(X_1 \oplus X_2 \oplus \cdots \oplus X_n)_\psi$ *is uniformly smooth if and only if* $(\mathbb{C}^n, \|\cdot\|_\psi)$ *is smooth and* X_i *is uniformly smooth for all* i.

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