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A PURELY ALGEBRAIC PROOF OF AG INEQUALITY

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Dedicated to the memory
of prof. Mladen Alić

(*communicated by P. S. Bullen*)

Abstract. We give a purely algebraic proof of AG inequality. We also give some examples.

0. Introduction

AG inequality states that

$$\frac{x_1 + \ldots + x_n}{n} \geqslant \sqrt[n]{x_1 \cdot \ldots \cdot x_n} \tag{1}$$

where $x_1, \ldots, x_n$ are positive real numbers. The equality in (1) holds if and only if $x_1 = \ldots = x_n$.
A form of (1) that can be viewed as an algebraic version of AG inequality is the following:

$$(x_1 + \ldots + x_n)^n \geqslant n^n x_1 \cdot \ldots \cdot x_n \tag{2}$$

For $n = 2$, AG inequality follows from the identity

$$(X_1 + X_2)^2 - 2^2 \cdot X_1 X_2 = (X_1 - X_2)^2$$

It is easy to check the identity

$$\begin{aligned}(X_1 + X_2 + X_3)^3 &- 3^3 \cdot X_1 X_2 X_3 \\ &= \frac{1}{2}((X_1 + X_2 + 7X_3)(X_1 - X_2)^2 + (X_1 + X_3 + 7X_2)(X_1 - X_3)^2 \\ &\quad + (X_2 + X_3 + 7X_1)(X_2 - X_3)^2)\end{aligned}$$

from which AG inequality for $n = 3$ follows directly.

To generalize these identities we introduce the concept of quasi-sum of squares.

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DEFINITION 1. *Let f be a homogenous symmetric n-degree polynomial in n variables. We say that f is a quasi-sum of squares if*

$$f = \sum_{1 \leqslant i < j \leqslant n} f_{i,j}(X_i - X_j)^2$$

where $f_{i,j}$ (for each i,j) is a homogenous polynomial of degree $n-2$ that is a linear combination of monomials with nonnegative coefficients.

For example,

$$f := X^2Y + X^2Z + Y^2X + Y^2Z + Z^2X + Z^2Y - 6XYZ$$

is a quasi-sum of squares since

$$f = Z(X-Y)^2 + Y(X-Z)^2 + X(Y-Z)^2.$$

It is easy to see that if f is a quasi-sum of squares then $f(x_1, x_2, ..., x_n) \geqslant 0$ for all nonnegative $x_1, x_2, ..., x_n$.

In this note we prove the following

THEOREM 1.

$$(X_1 + ... + X_n)^n - n^n X_1 \cdot ... \cdot X_n \tag{3}$$

is a quasi-sum of squares, for all natural numbers n.

Note that Hurwitz [3] gave an explicit expression of

$$X_1^n + X_2^n ... + X_n^n - nX_1X_2...X_n$$

as a quasi-sum of squares (see [1], p.87), which may be understood as a purely algebraic proof of AG inequality for positive numbers $x_1^n, ..., x_n^n$. However, the Hurwitz result is insufficient for purely algebraic proof of (2). Note also that Theorem 1 provides a proof of a noncommutative version of AG inequality (see [2]).

The organization of the paper is the following. In section 1 we give a characterization of the standard ordering on the partitions of a fixed natural number n in terms of quasi-sum of squares, and in section 2 we use this characterization to prove AG inequality.

1. A characterization of the standard ordering of partitions

The vector $\lambda = (\lambda_1, ..., \lambda_n)$ where λ_i are nonnegative integers is called a partition of a natural number n if $\lambda_1 \geqslant ... \geqslant \lambda_n \geqslant 0$ and $\lambda_1 + ... + \lambda_n = n$ (see, for example [4]). We define the monomial X^λ and the symmetric polynomial m_λ as follows:

$$X^\lambda := X_1^{\lambda_1} \cdot ... \cdot X_n^{\lambda_n}$$

where $X_1, ..., X_n$ are commutative variables; m_λ is the sum of monomials obtained from X^λ by different permutations of $(\lambda_1, ..., \lambda_n)$. For example, if $\lambda = (3,1,0,0)$, a partition of 4, then $X^\lambda = X_1^3X_2$ (we omit variables with zero exponents), and $m_\lambda = X_1^3X_2 + X_1^3X_3 + X_1^3X_4 + X_2^3X_1 + X_2^3X_3 + X_2^3X_4 + X_3^3X_1 + X_3^3X_2 + X_3^3X_4 + X_4^3X_1 + X_4^3X_2 + X_4^3X_3$.

We denote by n_λ the number of monomials contained in m_λ (in our example $n_\lambda = 12$).

Let us recall the definitions of the standard ordering and the lexicographic ordering on the set of partitions of a fixed natural number n. A partition λ is said to be larger than a partition μ (in the standard ordering) if

$$\sum_{1 \leqslant i \leqslant k} \lambda_i \geqslant \sum_{1 \leqslant i \leqslant k} \mu_i$$

for all $k = 1, ..., n$. A partition λ is said to be larger than a partition μ (in the lexicographic ordering) if $\lambda_i \geqslant \mu_i$ for the first index i such that $\lambda_i \neq \mu_i$.
Note that the standard ordering is not linear for $n \geqslant 6$ and that the lexicographic ordering is linear for all n.

Assume that $\lambda \geqslant \mu$ in the standard ordering. Then $\lambda \geqslant \mu$ in the lexicographic ordering. In order to describe the paths joining λ and μ we have to describe the set of successors of a fixed partition. Here, we say that μ is a successor of λ (in an ordering) if

(i) $\lambda > \mu$ and

(ii) if $\lambda > \pi \geqslant \mu$ for a partition π then $\pi = \mu$.

We have two types of successors according to the following two types of situations:

1. type. If $\lambda_i - \lambda_{i+1} \geqslant 2$ for some i then the corresponding successor is the partition $(\lambda_1, ..., \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1} + 1, \lambda_{i+2}, ..., \lambda_n)$.

2. type. If $\lambda_i - 1 = \lambda_{i+1} = ... = \lambda_{i+m} = \lambda_{i+m+1} + 1$ then the corresponding successor is the partitions $(\lambda_1, ..., \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1}, ..., \lambda_{i+m}, \lambda_{i+m+1} + 1, \lambda_{i+m+2}, ..., \lambda_n)$.

LEMMA 1. *Assume that $\lambda \geqslant \mu$ in the standard ordering. Then there exists a chain in the standard ordering such that:*

(i) it consists of successive partitions

(ii) it joins λ and μ.

Proof. (induction on the number of steps between λ and μ in the lexicographic ordering). If μ is the successor of λ in the lexicographic ordering then it is a successor of λ in the standard ordering (since $\lambda \geqslant \mu$). Assume now that i is the first index such that $\lambda_i > \mu_i$, and, further, that $j > i$ is the first index such that $\lambda_i > \lambda_j$. There are two possibilities.
(i) $\lambda_i - \lambda_j \geqslant 2$. Then there is π, a successor of λ of the 1. type, such that $\lambda > \pi \geqslant \mu$.
(ii) $\lambda_i - \lambda_j = 1$. Then $\lambda_j \neq 0$, and we look at the first index $k > j$ such that $\lambda_j - \lambda_k \geqslant 1$. Therefore, there is π, a successor of λ either of the 1. type or of the 2. type, such that $\lambda > \pi \geqslant \mu$.

THEOREM 2. *Let λ and μ be partitions of a natural number n. Then $\lambda \geqslant \mu$ in the standard ordering if and only if*

$$\frac{m_\lambda(x_1, ..., x_n)}{n_\lambda} \geqslant \frac{m_\mu(x_1, ..., x_n)}{n_\mu}$$

for all positive $x_1, ..., x_n$. *The equality holds if and only if* $x_1 = ... = x_n$.
Moreover

$$\frac{m_\lambda}{n_\lambda} - \frac{m_\mu}{n_\mu}$$

is a quasi-sum of squares.

Note that the Hurwitz's result shows that the theorem is valid for $\lambda = (n, 0, ..., 0)$ and $\mu = (1, ..., 1)$. Note also that the first part of the theorem is well-known and that it is a special case of the Muirhead theorem (see [1], p.357-359).

EXAMPLE 1. Assume that $n = 4$, $m_\lambda = X_1^4 + X_2^4 + X_3^4 + X_4^4$, with $n_\lambda = 4$; and $m_\mu = X_1^3X_2 + X_2^3X_1 + X_1^3X_3 + X_3^3X_1 + X_1^3X_4 + X_4^3X_1 + X_2^3X_3 + X_3^3X_2 + X_2^3X_4 + X_4^3X_2 + X_3^3X_4 + X_4^3X_3$, with $n_\mu = 12$.

It is easy to see that

$$\frac{m_\lambda}{n_\lambda} - \frac{m_\mu}{n_\mu} = \frac{1}{12}\Big(\sum_{1\leqslant<i<j\leqslant 4}(X_i^2 + X_iX_j + X_j^2)(X_i - X_j)^2\Big)$$

A step in our proof of Theorem 2 is a proof of an analogous result for neighboring partitions.

LEMMA 2. *Let* λ *be a partition of a natural number* n *and let* μ *be a successor of* λ *in the standard ordering. Then*

$$\frac{m_\lambda(x_1, ..., x_n)}{n_\lambda} \geqslant \frac{m_\mu(x_1, ..., x_n)}{n_\mu}$$

for all positive $x_1, ..., x_n$. *The equality holds if and only if* $x_1 = ... = x_n$.
Moreover

$$\frac{m_\lambda}{n_\lambda} - \frac{m_\mu}{n_\mu}$$

is a quasi-sum of squares.

Proof. However we only have to prove the second statement. Let us denote $\lambda = (a_1, a_2, ..., a_n)$. There are three possibilities.

(i) $X^\lambda = X_1^{a_1} \cdot ... \cdot X_l^{a_l} \cdot X_{l+1}^{a_k} \cdot ... \cdot X_k^{a_k} \cdot X_{k+1}^{a_k-2} \cdot ... \cdot X_{k+m}^{a_k-2} \cdot X_{k+m+1}^{a_{k+m+1}} \cdot ...$

$X_\mu = X_1^{a_1} \cdot ... \cdot X_l^{a_l} \cdot X_{l+1}^{a_k} \cdot ... \cdot X_{k-1}^{a_k} X_k^{a_k-1} \cdot X_{k+1}^{a_k-1} \cdot X_{k+2}^{a_k-2} \cdot ... \cdot X_{k+m}^{a_k-2} \cdot X_{k+m+1}^{a_{k+m+1}} \cdot ...$

with $a_k - a_{k+1} = 2$, $a_{k+m+1} < a_k - 2$ or $a_k = 2$, and $a_l > a_k$ or $l = 0$.

We see that $n_\lambda = \binom{n}{l}\binom{n-l}{n-k-m}\binom{k+m-l}{k-l}$, $n_\mu = \binom{n}{l}\binom{n-l}{n-k-m}\binom{k+m-l}{k-l-1}\binom{m+1}{2}$, hence $n_\lambda : n_\mu = 2 : (k-l)m$, so we have to prove that $(k-l)mm_\lambda - 2m_\mu$ is a quasi-sum of squares. We fix l variables with dominant exponents, say $X_1^{a_1}, ..., X_l^{a_l}$ and $n-k-m$ variables with lower exponents, say $X_{k+m+1}^{a_{k+m+1}}, ...$ and permutate the remaining variables. Let us denote by I the subsets of $\{l+1, ..., k+m\}$ with cardinality $k-l$ and by J the subsets of $\{l+1, ..., k+m\}$ with cardinality $k-l-1$. We get (after neglecting $X_1^{a_1}, ..., X_l^{a_l}$ and $X_{k+m+1}^{a_{k+m+1}}, ...$):

$$(k-l)m\sum_I \prod_{i\in I} X_i^{a_k} \prod_{i\notin I} X_i^{a_k-2} - 2\sum_J \prod_{j\in J} X_j^{a_k} \sum_{r<s;r,s\notin J} X_r^{a_k-1}X_s^{a_k-1} \prod_{j\notin J; j\neq r,s} X_j^{a_k-2}$$

$$= \sum_J \prod_{j\in J} X_j^{a_k} \sum_{r<s;r,s\notin J} \prod_{j\notin J} X_j^{a_k-2}(X_r - X_s)^2$$

For example, if $\lambda = (4,3,3,1,0,...)$ we have $n = 11$, $l = 1$, $k = 3$, $m = 1$; $|I| = 2$, $|J| = 1$, $I,J \subset \{2,3,4\}$; $X^\lambda = X_1^4X_2^3X_3^3X_4$, $X_\mu = X_1^4X_2^3X_3^2X_4^2$. After neglecting X_1^4 we have

$$2\cdot 1(X_2^3X_3^3X_4 + X_2^3X_4^3X_3 + X_3^3X_4^3X_2) - 2(X_2^3X_3^2X_4^2 + X_2^3X_4^2X_3^2 + X_3^3X_4^2X_2^2)$$
$$= X_2^3X_3X_4((X_3-X_4)^2 + X_3^3X_2X_4((X_2-X_4)^2 + X_4^3X_2X_3((X_2-X_3)^2$$

Therefore,

$$n_\mu m_\lambda - n_\lambda m_\mu = (X_3^4X_4^3X_1X_2 + X_4^4X_3^3X_1X_2)(X_1-X_2)^2 + ...$$

(ii) $X^\lambda = X_1^{a_1}\cdot ...\cdot X_l^{a_l}\cdot X_{l+1}^{a_k}\cdot ...\cdot X_k^{a_k}\cdot X_{k+1}^{a_{k+1}}\cdot ...\cdot X_{k+m}^{a_{k+1}}\cdot X_{k+m+1}^{a_{k+m+1}}\cdot ...$

$X_\mu = X_1^{a_1}\cdot ...\cdot X_l^{a_l}\cdot X_{l+1}^{a_k}\cdot ...\cdot X_{k-1}^{a_k}X_k^{a_k-1}\cdot X_{k+1}^{a_{k+1}+1}\cdot X_{k+2}^{a_{k+1}}\cdot ...\cdot X_{k+m}^{a_{k+1}}\cdot X_{k+m+1}^{a_{k+m+1}}\cdot ...$

with $a_k - a_{k+1} = t \geqslant 3$, $a_{k+m+1} < a_{k+1}$ or $a_{k+1} = 0$, and $a_l > a_k$ or $l = 0$.

Similarly as in (i) we see that $n_\lambda : n_\mu = 1 : (k-l)m$, so we have to prove that $(k-l)mm_\lambda - m_\mu$ is a quasi-sum of squares. Similarly as in (i) we get:

$$(k-l)m\sum_I \prod_{i\in I} X_i^{a_k} \prod_{i\notin I} X_i^{a_{k+1}}$$
$$-\sum_J \prod_{j\in J} X_j^{a_k} \sum_{r<s;r,s\notin J} (X_r^{a_k-1}X_s^{a_{k+1}+1} + X_s^{a_k-1}X_r^{a_{k+1}+1}) \prod_{j\notin J;j\neq r,s} X_j^{a_{k+1}}$$
$$= \sum_J \prod_{j\in J} X_j^{a_k} \prod_{j\notin J} X_j^{a_{k+1}} \sum_{r<s;r,s\notin J} (X_r^{t-2} + X_r^{t-3}X_s + ... + X_s^{t-2})(X_r - X_s)^2$$

(iii) $X^\lambda = X_1^{a_1}\cdot ...\cdot X_l^{a_l}\cdot X_{l+1}^{a_k}\cdot ...\cdot X_k^{a_k}\cdot X_{k+1}^{a_k-1}\cdot ...\cdot X_{k+m}^{a_k-1}\cdot X_{k+m+1}^{a_k-2}\cdot ...\cdot X_{k+p}^{a_k-2}\cdot X_{k+p+1}^{a_{k+p+1}}\cdots$

$X_\mu = X_1^{a_1}\cdot ...\cdot X_l^{a_l}\cdot X_{l+1}^{a_k}\cdot ...\cdot X_{k-1}^{a_k}\cdot X_k^{a_k-1}\cdot X_{k+1}^{a_k-1}\cdot ...\cdot X_{k+m}^{a_k-1}\cdot X_{k+m+1}^{a_k-1}\cdot X_{k+m+2}^{a_k-2}\cdots$

with $a_k - a_{k+1} = 1$, $a_{k+m} - a_{k+m+1} = 1$, $a_{k+p+1} < a_{k+p}$ or $a_{k+p} = 0$, and $a_l > a_k$ or $l = 0$.

We see that $n_\lambda : n_\mu = (m+1)(m+2) : (k-l)(p-m)$, so we have to prove that $(k-l)(p-m)m_\lambda - (m+1)(m+2)m_\mu$ is a quasi-sum of squares. We fix $X_1^{a_1},...,X_l^{a_l}$ and $n-k-p$ variables $X_{k+p+1}^{a_{k+p+1}},...$ and permutate the remaining variables. Denote by I the subsets of $\{l+1,...,k+p\}$ with cardinality $k-l$ and by J the subsets with cardinality $k-l-1$, by V the subsets with cardinality m and by W the subsets with cardinality $m+2$. We get:

$$(k-l)(p-m)\sum_I \prod_{i\in I} X_i^{a_k} \sum_{V;V\cap I=\emptyset} \prod_{i\in V} X_i^{a_k-1} \prod_{i\notin I\cup V} X_i^{a_k-2}$$
$$-(m+1)(m+2)\sum_J \prod_{j\in J} X_j^{a_k} \sum_{W;W\cap J=\emptyset} \prod_{j\in W} a_j^{a_k-1} \prod_{j\notin J\cup W} X_j^{a_k-2}$$

$$= \sum_J \prod_{j\in J} X_j^{a_k} \sum_{V:V\cap J=\emptyset} \prod_{j\in V} X_j^{a_k-1} \sum_{r<s; r,s\notin J\cup V} \prod_{j\notin J\cup V} X_j^{a_k-2}(X_r - X_s)^2$$

Proof of Theorem 2. If $\lambda > \mu$ then, by Lemma 1, there exists a chain $\lambda > \pi > \rho > ... > \chi > \mu$ consisting of successors. Therefore

$$\frac{m_\lambda}{n_\lambda} - \frac{m_\mu}{n_\mu} = (\frac{m_\lambda}{n_\lambda} - \frac{m_\pi}{n_\pi}) + (\frac{m_\pi}{n_\pi} - \frac{m_\rho}{n_\rho}) + ... + (\frac{m_\chi}{n_\chi} - \frac{m_\mu}{n_\mu})$$

By Lemma 2, each summand on the right is a quasi sum of squares, so that $\frac{m_\lambda}{n_\lambda} - \frac{m_\mu}{n_\mu}$ is a quasi-sum of squares, too. The converse is well-known.

COROLLARY 1. *Let P be a nonempty subset of the set of all partitions of a fixed natural number n and let μ be a partition of n such that $\lambda > \mu$ for all $\lambda \in P$. Define $f := \sum_{\lambda\in P} b_\lambda m_\lambda$, with positive real numbers b_λ. Then there exists one and only one positive real number d such that $f(x_1,...,x_n) \geqslant d \cdot m_\mu(x_1,...,x_n)$ for all positive real numbers $x_1,...,x_n$. The equality holds if and only if $x_1 = ... = x_n$. Moreover $f(X_1,...,X_n) - d \cdot m_\mu$ is a quasi-sum of squares.*

Proof. By Theorem 2, for each $\lambda \in P$ there exists unique positive real number d_λ such that $m_\lambda - d_\lambda m_\mu$ is a quasi-sum of squares. It is easy to see that $d := \sum_{\lambda\in P} b_\lambda d_\lambda$ satisfies the statement of the Corollary.

Similarly, one may prove a slightly more general assertion.

COROLLARY 2. *Let P, Q be two disjoint nonempty subsets of the set of all partitions of a fixed natural number n such that each element of P is greater than each element of Q (in the standard ordering). Define $f := \sum_{\lambda\in P} b_\lambda m_\lambda$, $g := \sum_{\lambda\in Q} c_\lambda m_\lambda$, with positive real numbers b_λ, c_λ. Then there exists one and only one positive real number d such that $f(x_1,...,x_n) \geqslant d \cdot g(x_1,...,x_n)$ for all positive real numbers $x_1,...,x_n$. The equality holds if and only if $x_1 = ... = x_n$. Moreover $f(X_1,...,X_n) - d \cdot g(X_1,...,X_n)$ is a quasi-sum of squares.*

Note that a similar procedure as in the proof of Theorem 2 leads to an elegant proof of the Hurwitz result (see [1], p. 359). The simplicity of expression therein is a consequence of a choice of a special chain joining partitions $(n,0,...,0)$ and $(1,1,...,1)$.

EXAMPLE 2. Since

$$X^3 + Y^3 + Z^3 + 3XYZ - (X^2Y + Y^2X + X^2Z + Z^2X + Y^2Z + Z^2Y)$$

$$= \frac{1}{2}((X+Y-Z)(X-Y)^2 + (X+Z-Y)(X-Z)^2 + (Y+Z-X)(Y-Z)^2)$$

$$= ((X+Y)^3 + Z^3 - Z(X+Y)^2 - Z^2(X+Y)) + XY(5Z - 4X - 4Y)$$

etc., we see that $x^3 + y^3 + z^3 + 3xyz \geqslant x^2y + y^2x + x^2z + z^2x + y^2z + z^2y$ for all positive x, y, z.

However, $X^3 + Y^3 + Z^3 + 3XYZ - (X^2Y + Y^2X + X^2Z + Z^2X + Y^2Z + Z^2Y)$ is not a quasi-sum of squares.

EXAMPLE 3. We have

$$3(X^4+Y^4+Z^4+W^4)+2(X^2Y^2+X^2Z^2+X^2W^2+Y^2Z^2+Y^2W^2+Z^2W^2)$$

$-2(X^3Y+Y^3X+X^3Z+Z^3X+X^3W+W^3X+Y^3Z+Z^3Y+Y^3W+W^3Y+Z^3W+W^3Z)=$ $(X^2+Y^2)(X-Y)^2+(X^2+Z^2)(X-Z)^2+(X^2+W^2)(X-W)^2+(Y^2+Z^2)(Y-Z)^2+$ $(Y^2+W^2)(Y-W)^2+(Z^2+W^2)(Z-W)^2$. This identity shows that the converse of the statement from Corollary 1. is not valid, generally.

2. AG inequality

Proof of Theorem 1. $(X_1+...+X_n)^n-n!X_1\cdot...\cdot X_n$ is a linear combination with positive coefficients of m_λ where λ pass through all partitions of n, apart the minimal partition. Since $X_1\cdot...\cdot X_n$ corresponds to the minimal partition we conclude, by Corollary 1, that $(X_1+...+X_n)^n-n!X_1\cdot...\cdot X_n-d\cdot X_1\cdot...\cdot X_n$ is a quasi-sum of squares for some positive d, and, by the uniqueness of d, we conclude that $d=n^n-n!$.

As a consequence we have that

$$(x_1+...+x_n)^n\geqslant n^nx_1\cdot...\cdot x_n$$

for all positive real numbers $x_1,...,x_n$ with equality if and only if $x_1=...=x_n$.

In the following example we use procedure from Lemma 2 to represent

$$(X_1+...+X_n)^n-n^nX_1\cdot...\cdot X_n \tag{4}$$

as a quasi-sum of squares, for some n.

EXAMPLE 4. In this example m_4 denotes m_λ for $\lambda=(4,0,0,0)$, $m_{3,1}$ denotes m_λ for $\lambda=(3,1,00)$ etc. We have:
(for $n=3$)

$$\frac{1}{2}(2m_3-m_{2,1})=\frac{1}{2}((X_1+X_2(X_1-X_2)^2+...)(3+\frac{1}{2})(m_{2,1}-m_{1,1,1})=\frac{7}{2}(X_3(X_1-X_2)^2+...).$$

Combining these two identities we obtain

$$(X_1+X_2+X_3)^3-3^3\cdot X_1X_2X_3=\frac{1}{2}((X_1+X_2+7X_3)(X_1-X_2)^2+...)$$

(for $n=4$) Similarly as for $n=3$ we obtain

$$\begin{aligned}&(X_1+X_2+X_3+X_4)^4-4^4\cdot X_1X_2X_3X_4\\&\qquad=\frac{1}{3}((X_1^2+X_2^2+11X_3^2+11X_4^2+14X_1X_2+58X_3X_4)(X_1-X_2)^2+...)\end{aligned}$$

(for $n=5$)

$$\begin{aligned}&(X_1+X_2+X_3+X_4+X_5)^5-5^5\cdot X_1X_2X_3X_4X_5\\&=\frac{1}{24}(6(X_1^3+X_2^3)+122(X_3^3+X_4^3+X_5^3)\\&\quad+132(X_1^2X_2+X_2^2X_1)+361(X_3^2X_4+X_4^2X_3+...)\\&\qquad+362(X_1X_2X_3+X_1X_2X_4+X_1X_2X_5)+3606X_3X_4X_5))(X_1-X_2)^2+...)\end{aligned}$$

(for $n = 6$). In this case the standard ordering is not linear, so we have to modify our procedure. We first follow the path $(6,0,0,0,0,0) > (5,1,0,0,0,0) > (4,2,0,0,0,0) > (4,1,1,0,0,0) > (3,2,1,0,0,0)$. Here we add the contribution of the path $(3,3,0,0,0,0) > (3,2,1,0,0,0)$ and continue through the path $(3,2,1,0,0,0) > (3,1,1,1,0,0) > (2,2,1,1,0,0)$. Here we add the contribution of the path $(2,2,2,0,0,0) > (2,2,1,1,0,0)$ and continue. Finally, we get:

$$
\begin{aligned}
&(X_1+X_2+\ldots+X_6)^6 - 6^6\cdot X_1X_2\cdot\ldots\cdot X_6\\
&=\frac{1}{10}((2(X_1^4+X_2^4)+53(X_3^4+X_4^4+X_5^4+X_6^4)\\
&+64(X_1^3X_2+X_2^3X_1)+276(X_3^3X_4+X_4^3X_3+\ldots)\\
&+25(X_1X_3^3+X_2X_3^3+\ldots)+64X_1^2X_2^2+100(X_3^2X_4^2+X_3^2X_5^2+\ldots)\\
&+203(X_1^2X_2X_3+X_2^2X_1X_3+\ldots)+976(X_3^2X_4X_5+X_3^2X_4X_6+\ldots)\\
&+952(X_1X_2X_3X_4+X_1X_2X_3X_5+\ldots)+15312X_3X_4X_5X_6)(X_1-X_2)^2+\ldots)
\end{aligned}
$$

At present we do not know any closed general formula for the difference (4).

In the following example we show that the representation as a quasi-sum of squares is not unique, generally.

EXAMPLE 5. Let notation be as in Example 4. Then,

$$
\begin{aligned}
&4m_{4,2}-m_{3,2,1}\\
&=\sum_{1\leqslant i<j\leqslant 6}\Big(\sum_{r\neq i,j}X_r^4+(X_i^2X_j+X_j^2X_i)X_r\Big)(X_i-X_j)^2\\
&=\sum_{1\leqslant i<j\leqslant 6}\Big(\sum_{r\neq i,j}X_i^2X_j^2+(X_i+X_j)X_r^3\Big)(X_i-X_j)^2.
\end{aligned}
$$

These two representations correspond to the two paths that join the partition $(4,2,0,0,0,0)$ and the partition $(3,2,1,0,0,0)$.

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