Mathematical
Inequalities
& Applications
Volume 8, Number 2 (2005), 223–231

# UNIFIED TREATMENT OF COMPLEMENTED SCHWARZ AND GRÜSS INEQUALITIES IN INNER PRODUCT SPACES

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*Dedicated to the memory*
*of prof. Mladen Alić*

(*communicated by C. Pearce*)

*Abstract.* There is a lots of known complemented Cauchy-Bunyakowsky-Schwarz' inequalities in the literature. In the first part of this paper we shall deduce many of them using very simple technique of inner product space. The similar technique is applied in the second part to complemented Grüss' inequality.

## 1. Complemented Cauchy-Bunyakowsky-Schwarz inequalities in inner product spaces

The idea for this work is given in papers by S. S. Dragomir [2]–[4], where similar technique was used. But, Dragomir obtained only some inequalities which are not so strong to imply all of classical inequalities given in this paper.

Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over $\mathbf{K}$ $(\mathbf{K} = \mathbf{C}, \mathbf{R})$. In this section we shall assume that for $x, y \in H$ and complex numbers $c, C$ the following is satisfied:

$$\text{(A1)} \qquad \mathrm{Re}(\overline{c}C) > 0$$

and

$$\text{(A2)} \qquad \mathrm{Re}\langle Cy - x, x - cy \rangle \geqslant 0.$$

LEMMA 1. *If* (A2) *is satisfied, then it holds*

$$\|x\|^2 + \mathrm{Re}(\overline{c}C)\|y\|^2 \leqslant |c + C||\langle x, y \rangle| \tag{1}$$

*Proof.* Let us note first that for complex numbers $C$, $c$ and $z$ we have

$$\begin{aligned}\mathrm{Re}(C\,\overline{z} + \overline{c}\,z) &= \mathrm{Re}(\overline{C\,\overline{z}} + \overline{c}\,z) = \mathrm{Re}(\overline{C}\,z + \overline{c}\,z)\\ &= \mathrm{Re}[(\overline{C} + \overline{c})z] \leqslant |C + c||z|.\end{aligned} \tag{2}$$

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*Mathematics subject classification* (2000): 26D15.
*Key words and phrases*: Schwarz inequality, Grüss inequality, inner product.

Condition (A2) is equivalent to

$$\|x\|^2+\operatorname{Re}(\overline{c}C)\|y\|^2 \leqslant \operatorname{Re}[(\overline{C}+\overline{c})\langle x,y\rangle]. \tag{3}$$

From this inequality and (2), it follows (1).

Let us note also that (A2) is equivalent to

$$\left\|x-\frac{C+c}{2}\cdot y\right\| \leqslant \frac{1}{2}|C-c|\,\|y\|, \tag{4}$$

see [2] for details.

REMARK 1. *Diaz-Metcalf's inequality* [1]. Let us suppose that $x=(x_1,\dots,x_n)$ and $y=(y_1,\dots,y_n)$ are two real $n$-tuples with

(B1) $$0<m_1\leqslant y_i\leqslant M_1<\infty \quad\text{and}\quad 0<m_2\leqslant x_i\leqslant M_2<\infty;$$

for each $i\in\{1,\dots,n\}$, and some constants $m_1,m_2,M_1,M_2$. Let us choose

(B2) $$C=\frac{M_2}{m_1} \qquad\text{and}\qquad c=\frac{m_2}{M_1}.$$

Then for all $i\in\{1,\dots,n\}$,

$$Cy_i-x_i=\frac{M_2}{m_1}y_i-x_i\geqslant M_2-x_i\geqslant 0,$$

and

$$x_i-cy_i=x_i-\frac{m_2}{M_1}y_i\geqslant x_i-m_2\geqslant 0.$$

Therefore, (A2) is satisfied. Hence, (1) implies *Diaz-Metcalf's* inequality

$$\sum_{k=1}^n x_k^2+\frac{m_2M_2}{m_1M_1}\sum_{k=1}^n y_k^2\leqslant\left(\frac{M_2}{m_1}+\frac{m_2}{M_1}\right)\sum_{k=1}^n x_ky_k. \tag{5}$$

The following theorem is proved in [5], Theorem 2. We give a simpler proof.

THEOREM 1. *If* (A1) *and* (A2) *are satisfied, then*

$$\frac{\|x\|^2\|y\|^2}{|\langle x,y\rangle|^2}\leqslant\frac{|c+C|^2}{4\operatorname{Re}(\overline{c}C)}. \tag{6}$$

*Proof.* From Lemma 1, we obtain

$$\begin{aligned}\|x\|^2\|y\|^2&\leqslant|C+c|\,|\langle x,y\rangle|\,\|y\|^2-\operatorname{Re}(\overline{c}C)\|y\|^4\\&=\frac{|C+c|^2}{4\operatorname{Re}(\overline{c}C)}|\langle x,y\rangle|^2-\left[\frac{|C+c||\langle x,y\rangle|}{2[\operatorname{Re}(\overline{c}C)]^{1/2}}-[\operatorname{Re}(\overline{c}C)]^{1/2}\|y\|^2\right]^2\end{aligned}$$

and (6) follows.

REMARK 2. *Pólya-Szegö's inequality* [8]. Suppose that (B1) are satisfied. Then from (6) it follows directly

$$\frac{\sum_{k=1}^{n} x_k^2 \sum_{k=1}^{n} y_k^2}{\left(\sum_{k=1}^{n} x_k y_k\right)^2} \leqslant \frac{1}{4}\left(\sqrt{\frac{M_1M_2}{m_1m_2}}+\sqrt{\frac{m_1m_2}{M_1M_2}}\right)^2. \tag{7}$$

In the sequel, let $w_1,\ldots,w_n$ be a sequence of positive real numbers.

REMARK 3. The weighted version of Pólya-Szegö's inequality is known in the literature as *Greub-Reinboldt's inequality* [6]:

$$\left(\sum_{k=1}^{n} w_k x_k^2\right)\left(\sum_{k=1}^{n} w_k y_k^2\right) \leqslant \frac{(M_1M_2+m_1m_2)^2}{4m_1m_2M_1M_2}\left(\sum_{k=1}^{n} w_k x_k y_k\right)^2. \tag{8}$$

If condition (B1) is satisfied, then (A1) and (A2) hold with inner product defined with weight $(w_1,\ldots,w_n)$, and with the choice of $C$ and $c$ given by (B2). Hence, (6) implies (8).

REMARK 4. *Cassel's inequality* [11]. Suppose that $x_1,\ldots,x_n$ and $y_1,\ldots,y_n$ are sequences of positive real numbers satisfying the condition

$$\text{(C1)} \qquad 0 < m \leqslant \frac{x_k}{y_k} \leqslant M < \infty, \qquad \text{for each } k \in \{1,\ldots,n\}$$

for some constants $m$ and $M$. Let us choose $C = M$ and $c = m$. Then for each $k$ it holds $Cy_k - x_k \geqslant 0$ and $x_k - cy_k \geqslant 0$. Hence, the condition (A2) is satisfied in inner product space, so from (6) we obtain another clone of Pólya-Szegö's inequality

$$\frac{\left(\sum_{k=1}^{n} w_k x_k^2\right)\left(\sum_{k=1}^{n} w_k y_k^2\right)}{\left(\sum_{k=1}^{n} w_k x_k y_k\right)^2} \leqslant \frac{(M+m)^2}{4mM}. \tag{9}$$

THEOREM 2. *If* (A1) *and* (A2) *are satisfied, then it holds*

$$\frac{\|x\|^2}{|\langle x,y\rangle|} - \frac{|\langle x,y\rangle|}{\|y\|^2} \leqslant |C+c| - 2[\mathrm{Re}(\overline{c}C)]^{1/2}. \tag{10}$$

*Proof.* We shall start again with inequality (1):

$$\frac{\|x\|^2}{|\langle x,y\rangle|} \leqslant |C+c| - \mathrm{Re}(\overline{c}C)\frac{\|y\|^2}{|\langle x,y\rangle|}.$$

Now we have

$$\begin{aligned}
&\frac{\|x\|^2}{|\langle x,y\rangle|} - \frac{|\langle x,y\rangle|}{\|y\|^2} \\
&\quad \leqslant |C+c| - \mathrm{Re}(\overline{c}C)\frac{\|y\|^2}{|\langle x,y\rangle|} - \frac{|\langle x,y\rangle|}{\|y\|^2} \\
&\quad = |C+c| - 2[\mathrm{Re}(\overline{c}C)]^{1/2} - \left([\mathrm{Re}(\overline{c}C)]^{1/2}\frac{\|y\|}{|\langle x,y\rangle|^{1/2}} - \frac{|\langle x,y\rangle|^{1/2}}{\|y\|}\right)^2 \\
&\quad \leqslant |C+c| - 2[\mathrm{Re}(\overline{c}C)]^{1/2}.
\end{aligned}$$

REMARK 5. Suppose that (C1) is satisfied, and choose $C = M$, $c = m$. Then (10) implies *Klamkin-McLenaghan's inequality*

$$\frac{\sum_{k=1}^n w_k x_k^2}{\sum_{k=1}^n w_k x_k y_k} - \frac{\sum_{k=1}^n w_k x_k y_k}{\sum_{k=1}^n w_k y_k^2} \leqslant (\sqrt{M} - \sqrt{m})^2.$$

Another version of the same inequality is known in the literature as *Shisha-Mond's inequality*. Suppose that conditions (B1) are satisfied, and choose $C = M_2/m_1$, $c = m_2/M_1$. Then it holds:

$$\frac{\sum_{k=1}^n x_k^2}{\sum_{k=1}^n x_k y_k} - \frac{\sum_{k=1}^n x_k y_k}{\sum_{k=1}^n y_k^2} \leqslant \left[\left(\frac{M_2}{m_1}\right)^{\frac{1}{2}} - \left(\frac{m_2}{M_1}\right)^{\frac{1}{2}}\right]^2.$$

Let us prove now some inequalities which have no known counterparts in the classical case. The following theorem is given in [5]. We offer a simpler proof.

THEOREM 3. *If* (A1) *and* (A2) *are satisfied, then it holds*

$$\|x\|^2\|y\|^2 - |\langle x, y\rangle|^2 \leqslant \frac{|C-c|^2}{4\mathrm{Re}(\overline{c}C)}|\langle x, y\rangle|^2. \tag{11}$$

*Proof.* We shall use (6):

$$\|x\|^2\|y\|^2 - |\langle x, y\rangle|^2 \leqslant \left(\frac{|C+c|^2}{4\mathrm{Re}(\overline{c}C)} - 1\right)|\langle x, y\rangle|^2.$$

The expression inside parentheses is

$$\frac{(C+c)(\overline{C}+\overline{c}) - 4\mathrm{Re}(\overline{c}C)}{4\mathrm{Re}(\overline{c}C)} = \frac{|C|^2 + |c|^2 - 2\mathrm{Re}(\overline{c}C)}{4\mathrm{Re}(\overline{c}C)} = \frac{|C-c|^2}{4\mathrm{Re}(\overline{c}C)},$$

which proves the theorem.

THEOREM 4. *If* (A2) *is satisfied, then it holds*

$$\|x\|\|y\| - |\langle x, y\rangle| \leqslant \frac{|C-c|^2}{4|C+c|}\|y\|^2. \tag{12}$$

*Proof.* Using basic inequality (1), we can write

$$\begin{aligned}\|x\|\|y\| - |\langle x, y\rangle| &\leqslant \|x\|\|y\| - \frac{1}{|C+c|}\|x\|^2 - \frac{\mathrm{Re}(\overline{c}C)}{|c+C|}\|y\|^2 \\ &= \left[\frac{|C+c|}{4} - \frac{\mathrm{Re}(\overline{c}C)}{|C+c|}\right]\|y\|^2 - \frac{1}{|c+C|}\left(\|x\| - \frac{|C+c|}{2}\|y\|\right)^2 \\ &\leqslant \frac{|C+c|^2 - 4\mathrm{Re}(\overline{c}C)}{4|C+c|}\|y\|^2 \\ &= \frac{|C-c|^2}{4|C+c|}\|y\|^2.\end{aligned}$$

The last one is proved in [3]. We have given here alternate, simpler proof.

THEOREM 5. *If* (A2) *is satisfied, then it holds*

$$\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2 \leqslant \frac{1}{4}|C-c|^2\,\|y\|^4. \tag{13}$$

*Proof.* From Lemma 1 we have

$$\begin{aligned}
\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2 &\leqslant |C+c||\langle x,y\rangle|\|y\|^2 - \mathrm{Re}(\overline{c}C)\|y\|^4 - |\langle x,y\rangle|^2 \\
&= \frac{1}{4}|C-c|^2\|y\|^4 - \Big(\Big[\mathrm{Re}(\overline{c}C) + \frac{1}{4}|C-c|^2\Big]\|y\|^4 \\
&\qquad - |C+c||\langle x,y\rangle|\|y\|^2 + |\langle x,y\rangle|^2\Big) \\
&= \frac{1}{4}|C-c|^2\|y\|^4 - \Big(\frac{|C+c|^2}{4}\|y\|^4 - |C+c||\langle x,y\rangle|\|y\|^2 + |\langle x,y\rangle|^2\Big) \\
&= \frac{1}{4}|C-c|^2\|y\|^4 - \Big(\frac{|C+c|}{2}\|y\|^2 - |\langle x,y\rangle|\Big)^2 \\
&\leqslant \frac{1}{4}|C-c|^2\|y\|^4.
\end{aligned}$$

In proving our Theorems 1.–5. we used the fact that (A2) implies (1). Each of obtained inequalities can be improved using the same idea and equivalence of (A2) and (3) instead of (1). In this way, we obtain four additional and stronger inequalities.

THEOREM 6. *If* (A2) *is satisfied (and* (A1) *when it is necessary), then it holds*

$$\|x\|^2\|y\|^2 \leqslant \frac{\{\mathrm{Re}[(\overline{c}+\overline{C})\langle x,y\rangle]\}^2}{4\mathrm{Re}(\overline{c}\,C)}, \tag{6'}$$

$$\|x\|^2\|y\|^2 - \frac{\{\mathrm{Re}[(\overline{c}+\overline{C})\langle x,y\rangle]\}^2}{|c+C|^2} \leqslant \frac{|C-c|^2}{4\mathrm{Re}(\overline{c}C)}|\langle x,y\rangle|^2, \tag{11'}$$

$$\|x\|\|y\| - \frac{\mathrm{Re}[(\overline{c}+\overline{C})\langle x,y\rangle]}{|c+C|} \leqslant \frac{|C-c|^2}{4|C+c|}\|y\|^2, \tag{12'}$$

$$\|x\|^2\|y\|^2 - \frac{\{\mathrm{Re}[(\overline{c}+\overline{C})\langle x,y\rangle]\}^2}{|c+C|^2} \leqslant \frac{1}{4}|C-c|^2\,\|y\|^4. \tag{13'}$$

Note that these inequalities are improvements of (6), (11), (12) and (13), but might not be very convenient for applications.

## 2. Applications to Grüss' inequality

Inequalities proved in the first section have direct application in proving various forms of Grüss' inequality.

Let $H$ be an inner product space, $x, y \in H$, and $e$ unit vector in $H$. The starting point for Grüss' inequality is the following inequality

$$|\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle|^2 \leqslant (\|x\|^2 - |\langle x, e\rangle|^2)(\|y\|^2 - |\langle y, e\rangle|^2) \tag{14}$$

which follows directly from Cauchy-Bunyakowsky-Schwarz inequality, applied to vectors $x - \langle x, e\rangle e$ and $y - \langle y, e\rangle e$.

Let $c$ and $C$ be complex numbers and $\alpha$ nonnegative real number such that it holds

$$\text{(G2)} \qquad \operatorname{Re}\langle Cy - x, x - cy\rangle \geqslant \alpha\|y\|^2 \geqslant 0.$$

By inspection of the proofs of previous theorems it is easy to see that all results remain valid if $\operatorname{Re}(\overline{c}C)$ is replaced by $\alpha + \operatorname{Re}(\overline{c}C)$ and particularly if (A1) is replaced by

$$(\text{A1}^*) \qquad \alpha + \operatorname{Re}(\overline{c}\,C) > 0.$$

Therefore, we have the following:

*If* (A1*) *and* (G2) *are satisfied, then*

$$\|x\|^2\|y\|^2 - |\langle x, y\rangle|^2 \leqslant \left(|C + c| - 2[\alpha + \operatorname{Re}(\overline{c}C)]^{1/2}\right)|\langle x, y\rangle|\,\|y\|^2. \tag{15}$$

*If* (A1*) *and* (G2) *are satisfied, then*

$$\|x\|^2\|y\|^2 - |\langle x, y\rangle|^2 \leqslant \frac{|C - c|^2 - 4\alpha}{4[\alpha + \operatorname{Re}(\overline{c}C)]}|\langle x, y\rangle|^2. \tag{16}$$

*If* (G2) *is satisfied, then*

$$\|x\|^2\|y\|^2 - |\langle x, y\rangle|^2 \leqslant \left(\frac{1}{4}|C - c|^2 - \alpha\right)\|y\|^4. \tag{17}$$

Each of these inequalities gives two estimations for each of the two factors on the right-hand side of Grüss' inequality (14). Let us denote $\alpha = \alpha_{e,x}$ in the case of $y = e$ in (G2), and $\alpha = \alpha_{x,e}$ in the case of $y = e$ and $y$ changed to $x$ in (G2). Thus we can consider these four possible cases:

$$\operatorname{Re}\langle Ce - x, x - ce\rangle \geqslant \alpha_{e,x} \geqslant 0, \tag{18a}$$

$$\operatorname{Re}\langle Cx - e, e - cx\rangle \geqslant \alpha_{x,e}\|x\|^2 \geqslant 0, \tag{18b}$$

$$\operatorname{Re}\langle De - y, y - de\rangle \geqslant \alpha_{e,y} \geqslant 0, \tag{19a}$$

$$\operatorname{Re}\langle Dy - e, e - dy\rangle \geqslant \alpha_{y,e}\|y\|^2 \geqslant 0, \tag{19b}$$

The bounds for the factor $\|x\|^2 - |\langle x, e\rangle|^2$ folows from (15), (16) and (17). If (18a) holds (and (A1* ) when it is necessary) then:

$$\|x\|^2 - |\langle x, e\rangle|^2 \leqslant \Big(|C+c| - 2[\alpha_{e,x} + \mathrm{Re}(\bar{c}C)]^{1/2}\Big)|\langle x, e\rangle|, \tag{20a}$$

$$\|x\|^2 - |\langle x, e\rangle|^2 \leqslant \frac{|C-c|^2 - 4\alpha_{e,x}}{4[\alpha_{e,x} + \mathrm{Re}(\bar{c}C)]}|\langle x, e\rangle|^2, \tag{21a}$$

$$\|x\|^2 - |\langle x, e\rangle|^2 \leqslant \frac{1}{4}|C-c|^2 - \alpha_{e,x}, \tag{22a}$$

and if (18b) holds, then:

$$\|x\|^2 - |\langle x, e\rangle|^2 \leqslant \Big(|C+c| - 2[\alpha_{x,e} + \mathrm{Re}(\bar{c}C)]^{1/2}\Big)|\langle x, e\rangle|\|x\|^2, \tag{20b}$$

$$\|x\|^2 - |\langle x, e\rangle|^2 \leqslant \frac{|C-c|^2 - 4\alpha_{x,e}}{4[\alpha_{x,e} + \mathrm{Re}(\bar{c}C)]}|\langle x, e\rangle|^2, \tag{21b}$$

$$\|x\|^2 - |\langle x, e\rangle|^2 \leqslant \Big(\frac{1}{4}|C-c|^2 - \alpha_{x,e}\Big)\|x\|^4. \tag{22b}$$

Similar inequalities hold for the pair $y, e$, if (19a) or (19b) are satisfied.

This gives $6 \times 6$ possible estimates for the right-hand side in Grüss' inequality. Let us pick only three of them, with the same choice for each factor.

THEOREM 7. *Let* $\mathrm{Re}(\bar{c}C) + \alpha_{e,x} > 0$ *and* $\mathrm{Re}(\bar{d}D) + \alpha_{e,y} > 0$. *Suppose that* (18a) *and* (19a) *are satisfied. Then it holds*

$$\begin{aligned} |\langle x, y\rangle - \langle x, e\rangle\langle e, y\rangle| \leqslant{}& \Big(|C+c| - 2[\alpha_{e,x} + \mathrm{Re}(\bar{c}C)]^{1/2}\Big)^{1/2} \times \\ &\times \Big(|D+d| - 2[\alpha_{e,y} + \mathrm{Re}(\bar{d}D)]^{1/2}\Big)^{1/2}\big(|\langle x, e\rangle||\langle y, e\rangle|\big)^{1/2}. \end{aligned} \tag{23}$$

Inequality (16) is more symmetric with respect to both variables:

THEOREM 8. *Let* $\mathrm{Re}(\bar{c}C) + \alpha_x > 0$ *and* $\mathrm{Re}(\bar{d}D) + \alpha_y > 0$. *Suppose that for* $\alpha_x \geqslant 0$ *at least one of the following inequalities is satisfied*

$$\mathrm{Re}\langle Ce - x, x - ce\rangle \geqslant \alpha_x$$
$$\mathrm{Re}\langle Cx - e, e - cx\rangle \geqslant \alpha_x\|x\|^2,$$

*and similarly for* $\alpha_y$. *Then it holds*

$$|\langle x, y\rangle - \langle x, e\rangle\langle e, y\rangle| \leqslant \left(\frac{|C-c|^2 - 4\alpha_x}{4[\alpha_x + \mathrm{Re}(\bar{c}C)]} \cdot \frac{|D-d|^2 - 4\alpha_y}{4[\alpha_y + \mathrm{Re}(\bar{d}D)]}\right)^{1/2} |\langle x, e\rangle||\langle y, e\rangle|. \tag{24}$$

The next choice of right-hand sides gives a direct generalization of classical Grüss' inequality. This result is already known in similar form, see [2], Theorem 3 or [10].

THEOREM 9. *If* $(18a)$ *and* $(19a)$ *are satisfied, then it holds*

$$|\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle| \leqslant \sqrt{\frac{1}{4}|C-c|^2-\alpha_{e,x}}\sqrt{\frac{1}{4}|D-d|^2-\alpha_{e,y}}. \tag{25}$$

Let $e \in H$ be unit vector. It is not easy to give useful upper bound, greater than zero, for all $\alpha_{x,e}$. But, for some fixed $x \in H$, the choice of $\alpha_{e,x}$ given by

$$\mathrm{Re}\langle Ce - x, x - ce\rangle = \alpha_{e,x}$$

is the best one. For this choice $(22)$ gives the improvement of Grüss' inequality.

Let us give an example.

EXAMPLE. Take $x(t) = P_n(t)$, $n \geqslant 1$, the $n$-th Legendre's polynomial defined on $[-1,1]$. Let us take $e(t) \equiv 1$. The function $e$ has unit norm with respect to the scalar product with weight $\rho = \frac{1}{2}$, and $P_n$ is orthogonal to $e$. Let us take natural bounds $c = -1$, $C = 1$. Then $(18\mathrm{a})$ reads as

$$\mathrm{Re}\langle Ce - P_n, P_n - ce\rangle \geqslant \alpha,$$
$$\|e\|^2 - \|P_n\|^2 \geqslant \alpha.$$

Since $\|P_n\|^2 = \dfrac{2}{2n+1}$, the best choice for constant is $\alpha = \dfrac{2n-1}{2n+1}$. Hence, we obtain the following Grüss' inequality for arbitrary bounded integrable function $y$ defined on $[-1,1]$:

$$\left|\frac{1}{2}\int_{-1}^{1} y(t)P_n(t)dt - \frac{1}{2}\int_{-1}^{1} P_n(t)dt \cdot \frac{1}{2}\int_{-1}^{1} y(t)dt\right| \leqslant \sqrt{\frac{2}{2n+1}} \cdot \frac{1}{2} \cdot |M-m| \tag{26}$$

where $M$ and $m$ are respective upper and lower bounds for $y(t)$, $t \in [-1,1]$. The estimate $(26)$ is an improvement of the classical Gruss' inequality. This can be useful in analysis of Chebyshev functional.

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(Received January 20, 2004)

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