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NOTE ON GRÜSS TYPE INEQUALITY

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Dedicated to the memory
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(*communicated by J. Sándor*)

Abstract. In this note we prove, using Karamata's estimations of the Chebyshev quotient, Grüss type inequality which improves and generalize the inequality given by Dragomir-Khan.

1. Introduction

G. Grüss in [4] proved that for measurable functions $f,\ g : [a,b] \to \mathbf{R}$ the following inequality holds

$$\left| \frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2} \int_a^b f(x)dx \int_a^b g(x)dx \right| \leqslant \frac{1}{4}(M_1 - m_1)(M_2 - m_2), \tag{1}$$

provided that involved integrals exist and that $m_1 \leqslant f(x) \leqslant M_1$ and $m_2 \leqslant g(x) \leqslant M_2$ a.e. on $[a,b]$.

The analogous inequality holds for finite sequences (cf. [2]). The weighted versions of the above inequalities also hold (cf. [1]).

The following Grüss type inequality was proved in [3] :

THEOREM 1. *Let* $\mathbf{a} = (a_1, \dots, a_n)$ *and* $\mathbf{b} = (b_1, \dots, b_n)$ *be two sequences of positive real numbers with*

$$0 < a \leqslant a_i \leqslant A < \infty, \quad 0 < b \leqslant b_i \leqslant B < \infty, \quad i = 1, \dots, n. \tag{2}$$

Then one has the inequality

$$\left| \frac{1}{n}\sum_{i=1}^n a_i b_i - \frac{1}{n}\sum_{i=1}^n a_i \cdot \frac{1}{n}\sum_{i=1}^n b_i \right| \leqslant \frac{1}{4}\frac{(A-a)(B-b)}{\sqrt{aAbB}} \frac{1}{n}\sum_{i=1}^n a_i \cdot \frac{1}{n}\sum_{i=1}^n b_i. \tag{3}$$

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The constant $1/4$ *is best possible in (3) in the sense that it cannot be replaced by a smaller constant.*

The purpose of this note is to generalize and improve (in the sense that we obtain smaller constant) the inequality (3). This is done by using Karamata's estimations of the Chebyshev quotient (cf. [7]).

2. The main result

We give Karamata's result in terms of positive normalized functionals.

DEFINITION 1. Let I be a set and X be a real linear space of functions $f : I \to \mathbf{R}$ such that $1 \in X$. We say that $\phi : X \to \mathbf{R}$ is a positive normalized functional if it has the following properties:

1. $\phi(\alpha f + \beta g) = \alpha\phi(f) + \beta\phi(g)$, $\alpha, \beta \in \mathbf{R}$, $f, g \in X$
2. If $f \geqslant 0$, then $\phi(f) \geqslant 0$
3. $\phi(1) = 1$

If I is a measure space with measure μ such that $\mu(I) < \infty$, then $\phi(f) = \frac{1}{\mu(I)} \int_I f\, d\mu$ is an (generic) example of positive normalized functional on space of real integrable functions on I.

THEOREM 2. *Let* $\phi : X \to \mathbf{R}$ *be a positive normalized functional and let* f, $g \in X$ *be such that* $fg \in X$ *and* $0 < m_1 \leqslant f(x) \leqslant M_1$, $0 < m_2 \leqslant g(x) \leqslant M_2$ *for every* $x \in I$. *Then*

$$\frac{1}{K^2} \leqslant \frac{\phi(fg)}{\phi(f)\phi(g)} \leqslant K^2, \tag{4}$$

where $K = \dfrac{\sqrt{m_1 m_2} + \sqrt{M_1 M_2}}{\sqrt{m_1 M_2} + \sqrt{M_1 m_2}}$.

Proof. For readers convenience we'll give a sketch of the proof analogous to that in [7] (see also [5],[8]).

First, it is obvious that without loss of generality we can assume that $1 \leqslant f(x) \leqslant \gamma_1$ and $1 \leqslant g(x) \leqslant \gamma_2$ for every $x \in I$ and some γ_1, $\gamma_2 > 1$.

Step 1. The following inequalities hold:

$$\begin{aligned} \frac{\gamma_2[\gamma_1 - \phi(f)] + \gamma_1[\phi(f) - 1]}{\gamma_2[\gamma_1 - \phi(f)] + \phi(f) - 1} &\leqslant \frac{\phi(fg)}{\phi(g)} \\ &\leqslant \frac{\gamma_1 - \phi(f) + \gamma_1\gamma_2[\phi(f) - 1]}{\gamma_1 - \phi(f) + \gamma_2[\phi(f) - 1]} \end{aligned} \tag{5}$$

The left hand side inequality is obtained by consecutive applying of the functional ϕ (with respect to independent variables $x, t \in I$) on obvious inequality

$$[\gamma_1 - f(x)][f(t) - 1][\gamma_2 g(t) - g(x)] \geqslant 0.$$

The right hand side inequality is similarly obtained by using inequality

$$[\gamma_1 - f(x)][f(t) - 1][\gamma_2 g(x) - g(t)] \geqslant 0.$$

Step 2. Define functions $h, H : [1,\gamma_1] \to (0,\infty)$ by

$$H(t) = \frac{1}{t}\frac{\gamma_1 - t + \gamma_1\gamma_2(t-1)}{\gamma_1 - t + \gamma_2(t-1)}, \quad h(t) = 1/H(\gamma_1/t).$$

It is straightforward to check that for $t_1 = \dfrac{\sqrt{\gamma_1}(\sqrt{\gamma_1}+\sqrt{\gamma_2})}{\sqrt{\gamma_1\gamma_2}+1}$,

$$\max_{t\in[1,\gamma_1]} H(t) = H(t_1) = \left(\frac{1+\sqrt{\gamma_1\gamma_2}}{\sqrt{\gamma_1}+\sqrt{\gamma_2}}\right)^2 = K^2$$

and

$$\min_{t\in[1,\gamma_1]} h(t) = h(\gamma_1/t_1) = 1/H(t_1) = 1/K^2.$$

Using Step 1, we have $h(\phi(f)) \leqslant \dfrac{\phi(fg)}{\phi(f)\phi(g)} \leqslant H(\phi(f))$ and the claim follows. □

Our main remark is contained in the following corollary.

COROLLARY 1. *If the assumptions of Theorem 2 hold, then*

$$\begin{aligned}
&-\frac{(M_1-m_1)(M_2-m_2)}{\left(\sqrt{M_1m_2}+\sqrt{m_1M_2}\right)^2}\phi(fg)\\
&\qquad\leqslant -\frac{(M_1-m_1)(M_2-m_2)}{\left(\sqrt{m_1m_2}+\sqrt{M_1M_2}\right)^2}\phi(f)\phi(g)\\
&\qquad\leqslant \phi(fg)-\phi(f)\phi(g)\\
&\qquad\leqslant \frac{(M_1-m_1)(M_2-m_2)}{\left(\sqrt{m_1m_2}+\sqrt{M_1M_2}\right)^2}\phi(fg)\\
&\qquad\leqslant \frac{(M_1-m_1)(M_2-m_2)}{\left(\sqrt{M_1m_2}+\sqrt{m_1M_2}\right)^2}\phi(f)\phi(g)
\end{aligned} \tag{6}$$

Proof. The first and the last inequality in (6) are direct consequences of (4). Using lower bound in (4) we obtain

$$\left(\frac{1}{K^2}-1\right)\phi(f)\phi(g) \leqslant \phi(fg)-\phi(f)\phi(g)$$

which easily implies the second inequality in (6). Analogously, using upper bound in (4), we obtain the third inequality in (6). □

Notice that for obtaining Grüss type inequalities for finite sequences (as in Theorem 1) $f : I \to \mathbf{R}$, where $I = \{1,\dots,n\}$, one can define positive normalized functional ϕ by $\phi(f) = \left(\sum_{i=1}^n f(i)\omega_i\right) / \left(\sum_{i=1}^n \omega_i\right)$, where $\omega_i > 0$, $i = 1,\dots,n$.

Finally, we compare estimates obtained in Theorem 1 and and in Corollary 1 using our notations. Since inequality

$$\frac{(M_1-m_1)(M_2-m_2)}{\left(\sqrt{m_1M_2}+\sqrt{M_1m_2}\right)^2} \leqslant \frac{(M_1-m_1)(M_2-m_2)}{4\sqrt{m_1M_1m_2M_2}}$$

is obviously equivalent with inequality $0 \leqslant \left(\sqrt{M_1 m_2} - \sqrt{m_1 M_2}\right)^2$, it is evident that our upper estimate is better. The upper estimates are equal iff $M_1/m_1 = M_2/m_2$. Since inequality

$$-\frac{(M_1-m_1)(M_2-m_2)}{4\sqrt{m_1 M_1 m_2 M_2}} \leqslant -\frac{(M_1-m_1)(M_2-m_2)}{\left(\sqrt{m_1 m_2}+\sqrt{M_1 M_2}\right)^2}$$

is obviously equivalent with $0 \leqslant \left(\sqrt{m_1 m_2} - \sqrt{M_1 M_2}\right)^2$, it is evident that our lower estimate is also better. The lower estimates are equal iff $m_1 = M_1$ and $m_2 = M_2$.

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