Mathematical
Inequalities
& Applications
Volume 8, Number 2 (2005), 237–257

ON WEIGHTED EULER HARMONIC IDENTITIES WITH APPLICATIONS

A. AGLIĆ ALJINOVIĆ, LJ. DEDIĆ, M. MATIĆ AND J. PEČARIĆ

Dedicated to the memory of prof. Mladen Alić

(*communicated by P. S. Bullen*)

Abstract. A weighted Euler identities involving harmonic sequences of functions are established. Consequently various generalizations of Ostrowski inequality involving weighted integrals are obtained.

1. Introduction

Assume that $(P_k(t), k \geqslant 0)$ is a harmonic sequence of polynomials i.e. the sequence of polynomials satisfying

$$P_k'(t) = P_{k-1}(t),\ k \geqslant 1;\ \ P_0(t) = 1.$$

Define $P_k^*(t)$, $k \geqslant 0$, to be a periodic functions of period 1, related to $P_k(t)$, $k \geqslant 0$, as

$$P_k^*(t) = P_k(t),\ 0 \leqslant t < 1,$$
$$P_k^*(t+1) = P_k^*(t),\ t \in \mathbf{R}.$$

Thus, $P_0^*(t) = 1$, while for $k \geqslant 1$, $P_k^*(t)$ is continuous on $\mathbf{R}\backslash\mathbf{Z}$ and has a jump of

$$\alpha_k = P_k(0) - P_k(1)$$

at every integer t, whenever $\alpha_k \neq 0$. Note that $\alpha_1 = -1$, since $P_1(t) = t + c$, for some $c \in \mathbf{R}$. Also, note that from the definition it follows

$$P_k^{*\prime}(t) = P_{k-1}^*(t),\ \ k \geqslant 1,\ \ t \in \mathbf{R}\backslash\mathbf{Z}.$$

Let $f : [a,b] \to \mathbf{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[a,b]$ for some $n \geqslant 1$. In the recent paper [4] the following two identities have been proved:

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t + \widetilde{T}_n(x) + \tau_n(x) + \widetilde{R}_n^1(x) \tag{1.1}$$

Mathematics subject classification (2000): 26D15, 26D20.

Key words and phrases: Euler identities, harmonic sequences of functions, Ostrowski inequality.

and

$$f(x)=\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t+\widetilde{T}_{n-1}(x)+\tau_n(x)+\widetilde{R}_n^2(x), \tag{1.2}$$

where

$$\widetilde{T}_m(x)=\sum_{k=1}^{m}(b-a)^{k-1}P_k\left(\frac{x-a}{b-a}\right)\left[f^{(k-1)}(b)-f^{(k-1)}(a)\right],$$

for $1\leqslant m\leqslant n$, and

$$\tau_m(x)=\sum_{k=2}^{m}(b-a)^{k-1}\alpha_k f^{(k-1)}(x),$$

with convention $\tilde{T}_0(x)=0$, $\tau_1(x)=0$, while

$$\widetilde{R}_n^1(x)=-(b-a)^{n-1}\int_{[a,b]}P_n^*\left(\frac{x-t}{b-a}\right)\mathrm{d}f^{(n-1)}(t)$$

and

$$\widetilde{R}_n^2(x)=-(b-a)^{n-1}\int_{[a,b]}\left[P_n^*\left(\frac{x-t}{b-a}\right)-P_n\left(\frac{x-a}{b-a}\right)\right]\mathrm{d}f^{(n-1)}(t).$$

Here, as in the rest of the paper, we write $\int_{[a,b]}g(t)d\varphi(t)$ to denote the Riemann-Stieltjes integral with respect to a function $\varphi:[a,b]\to\mathbf{R}$ of bounded variation, and $\int_a^b g(t)\mathrm{d}t$ for the Riemann integral.

The formulae (1.1) and (1.2) hold for every $x\in[a,b]$. They have been used in [4] to prove some generalized Ostrowski inequalities. Further natural generalization of such results arises by replacing harmonic sequence of polynomials by a harmonic sequence of functions generated by some weight function. Some results of this type involving integration by parts formula are recently obtained by Dragomir [8].

The aim of this paper is to generalize formulae (1.1) and (1.2), by replacing the harmonic sequence of polynomials by a weighted harmonic sequence of functions. Using those generalized formulae we prove some further generalizations of Ostrowski inequality.

For some other weighted generalizations of Euler identity, Ostrowski type inequalities and it's discrete analogues the reader is referred to the papers [1], [2], [3].

2. Weighted Euler harmonic identities

For $a,b\in\mathbf{R}$, $a<b$, let $w:[a,b]\to[0,\infty)$ be a probability density function i.e. integrable function satisfying

$$\int_a^b w(t)\mathrm{d}t=1.$$

For $n\geqslant 1$ and $t\in[a,b]$ let

$$w_n(t)=\frac{1}{(n-1)!}\int_a^t(t-s)^{n-1}w(s)\mathrm{d}s. \tag{2.1}$$

Also, let

$$w_0(t) = w(t), \ t \in [a,b]. \tag{2.2}$$

It is well known that w_n is equal to the n-th indefinite integral of w, being equal to zero at a, i.e. $w_n^{(n)}(t) = w(t)$ and $w_n(a) = 0$, for every $n \geqslant 1$.

A sequence of functions $H_n : [a,b] \to \mathbf{R}, \ n \geqslant 0$, is called w-harmonic sequence of functions on $[a,b]$ if

$$H_n'(t) = H_{n-1}(t), \ n \geqslant 1; \ \ H_0(t) = w(t), \ t \in [a,b].$$

The sequence $(w_n(t), n \geqslant 0)$ is an example of w-harmonic sequence of functions on $[a,b]$.

LEMMA 1. *Let $(H_n(t), n \geqslant 0)$ be a w-harmonic sequence of functions on $[a,b]$. Then there exists a unique sequence $(Q_n(t), n \geqslant 0)$ of polynomials satisfying*

$$Q_n'(t) = Q_{n-1}(t), \quad \deg Q_n \leqslant n-1 \ \ (n \geqslant 1), \quad Q_0(t) = 0$$

such that

$$H_n(t) = Q_n(t) + w_n(t), \ n \geqslant 0.$$

Proof. The n-th derivative, for $n \geqslant 1$, of the function $H_n(t) - w_n(t)$ is equal to zero by definition. Therefore, there exists a polynomial $Q_n(t)$ of degree at most $n-1$ such that

$$H_n(t) - w_n(t) = Q_n(t),$$

which proves the existence. The uniqueness of $Q_n(t)$ is evident. □

REMARK 1. In the special case when $w(t) = \frac{1}{b-a}$, $t \in [a,b]$, the w-harmonic sequence of functions becomes the harmonic sequence of polynomials from Introduction, up to multiplicative constant $\frac{1}{b-a}$. In this case

$$w_n(t) = \frac{1}{b-a} \frac{(t-a)^n}{n!}, \ n \geqslant 0.$$

Therefore, every harmonic sequence of polynomials has the form

$$(b-a)Q_n(t) + \frac{(t-a)^n}{n!}, \ n \geqslant 0,$$

where $Q_n(t)$, $n \geqslant 0$ are as stated in Lemma 1.

Assume that $(H_n(t), n \geqslant 0)$ is a w-harmonic sequence of functions on $[a,b]$. Define $H_n^*(t)$, for $n \geqslant 0$, to be a periodic function of period 1, related to $H_n(t)$ as

$$H_n^*(t) = \frac{H_n(a+(b-a)t)}{(b-a)^n}, \ 0 \leqslant t < 1,$$
$$H_n^*(t+1) = H_n^*(t), \ t \in \mathbf{R}.$$

Thus, for $n \geqslant 1$, $H_n^*(t)$ is continuous on $\mathbf{R} \backslash \mathbf{Z}$ and has a jump of

$$\beta_n = \frac{H_n(a) - H_n(b)}{(b-a)^n}$$

at every $t \in \mathbf{Z}$, whenever $\beta_n \neq 0$. Note that

$$\beta_1 = -\frac{1}{b-a},$$

since

$$H_1(t) = c + w_1(t) = c + \int_a^t w(s)\mathrm{d}s,$$

for some $c \in \mathbf{R}$. Also, note that

$$H_n^{*\prime}(t) = H_{n-1}^*(t),\ n \geqslant 1,\ \ t \in \mathbf{R} \setminus \mathbf{Z}.$$

LEMMA 2. *For $x \in [a,b]$ and $n \geqslant 0$ define $\varphi_n(x;\cdot) : [a,b] \to \mathbf{R}$ as*

$$\varphi_n(x;t) = H_n^*\left(\frac{x-t}{b-a}\right),\ a \leqslant t \leqslant b.$$

Then for every continuous function $F : [a,b] \to \mathbf{R}$, and $n \geqslant 1$, we have

$$\int_{[a,b]} F(t) d\varphi_n(x;t) = -\frac{1}{b-a}\int_a^b F(t)\varphi_{n-1}(x;t)\mathrm{d}t - \beta_n F(x),$$

for $a \leqslant x < b$, and

$$\int_{[a,b]} F(t) d\varphi_n(b;t) = -\frac{1}{b-a}\int_a^b F(t)\varphi_{n-1}(b;t)\mathrm{d}t - \beta_n F(a).$$

Further, for every integrable function $F : [a,b] \to \mathbf{R}$,

$$\int_a^b F(t)\varphi_0(x;t)\mathrm{d}t = \int_a^b F(t) W_x(t)\mathrm{d}t,$$

where

$$W_x(t) = \begin{cases} w(a+x-t), & a \leqslant t \leqslant x \\ w(b+x-t), & x < t \leqslant b \end{cases}. \tag{2.3}$$

Proof. Let $n \geqslant 1$ and assume that $a < x < b$. The function $\varphi_n(x;\cdot)$ is differentiable on $[a,b] \setminus \{x\}$ and its derivative is equal to $\frac{-1}{b-a}\varphi_{n-1}(x;\cdot)$. Further, it has a jump of $\varphi_n(x;x+0) - \varphi_n(x;x-0) = -\beta_n$ at x, which gives the first formula in this case. For $x = a$ the function $\varphi_n(a;\cdot)$ is differentiable on (a,b) and its derivative is equal to $\frac{-1}{b-a}\varphi_{n-1}(a;\cdot)$. Further, it has jump of $\varphi_n(a;a+0) - \varphi_n(a;a) = -\beta_n$ at the point a, while $\varphi_n(a;b) - \varphi_n(a;b-0) = 0$, which gives the first formula for $x = a$. The second formula is a consequence of the first one and the fact that $\varphi_n(b;\cdot) = \varphi_n(a;\cdot)$.

The last assertion follows by simple observation that $\varphi_0(x;\cdot) = W_x(\cdot)$, for all $x \in [a,b\rangle$, while $\varphi_0(b;\cdot)$ and $W_b(\cdot)$ differ only at point $t = a$. □

Let $f : [a,b] \to \mathbf{R}$ be such that $f^{(n-1)}$ exists on $[a,b]$ for some $n \geqslant 1$. For every $x \in [a,b]$ and $1 \leqslant m \leqslant n$ we introduce the following notation

$$S_m(x) = \sum_{k=1}^m H_k(x)\left[f^{(k-1)}(b) - f^{(k-1)}(a)\right] + \sum_{k=2}^m [H_k(a) - H_k(b)] f^{(k-1)}(x), \tag{2.4}$$

with convention $S_1(x) = H_1(x)[f(b) - f(a)]$.

THEOREM 1. *Let* $(H_k, k \geqslant 0)$ *be a* w*-harmonic sequence of functions on* $[a,b]$ *and* $f : [a,b] \to \mathbf{R}$ *such that* $f^{(n-1)}$ *is a continuous function of bounded variation on* $[a,b]$ *for some* $n \geqslant 1$. *Then for every* $x \in [a,b]$

$$f(x) = \int_a^b f(t) W_x(t) \mathrm{d}t + S_n(x) + R_n^1(x), \tag{2.5}$$

where $W_x(t)$ *and* $S_n(x)$ *are defined by (2.3) and (2.4), respectively, while*

$$R_n^1(x) = -(b-a)^n \int_{[a,b]} H_n^* \left(\frac{x-t}{b-a} \right) \mathrm{d}f^{(n-1)}(t).$$

Proof. For $1 \leqslant k \leqslant n$ consider the integral

$$I_k(x) = (b-a)^k \int_{[a,b]} H_k^* \left(\frac{x-t}{b-a} \right) \mathrm{d}f^{(k-1)}(t).$$

Integration by parts yields

$$\begin{aligned} I_k(x) = {} & (b-a)^k H_k^* \left(\frac{x-t}{b-a} \right) f^{(k-1)}(t) \Big|_a^b \\ & - (b-a)^k \int_{[a,b]} f^{(k-1)}(t) \mathrm{d}H_k^* \left(\frac{x-t}{b-a} \right). \end{aligned} \tag{2.6}$$

First, assume that $a \leqslant x < b$. For every $k \geqslant 1$ we have

$$H_k^* \left(\frac{x-b}{b-a} \right) = H_k^* \left(\frac{x-a}{b-a} - 1 \right) = H_k^* \left(\frac{x-a}{b-a} \right) = \frac{H_k(x)}{(b-a)^k}.$$

Therefore, using the first formula from Lemma 2, from (2.6) we get

$$\begin{aligned} I_k(x) = {} & H_k(x) \left[f^{(k-1)}(b) - f^{(k-1)}(a) \right] + (b-a)^k \beta_k f^{(k-1)}(x) \\ & + (b-a)^{k-1} \int_a^b f^{(k-1)}(t) H_{k-1}^* \left(\frac{x-t}{b-a} \right) \mathrm{d}t. \end{aligned} \tag{2.7}$$

Since $\beta_1 = -\frac{1}{b-a}$, by the last formula from Lemma 2, for $k = 1$ (2.6) reduces to

$$\begin{aligned} I_1(x) &= H_1(x)[f(b) - f(a)] - f(x) + \int_a^b f(t) H_0^* \left(\frac{x-t}{b-a} \right) \mathrm{d}t \\ &= H_1(x)[f(b) - f(a)] - f(x) + \int_a^b f(t) W_x(t) \mathrm{d}t, \end{aligned} \tag{2.8}$$

where $W_x(t)$ is given by (2.3). For $k \geqslant 2$ we have

$$\begin{aligned} & (b-a)^{k-1} \int_a^b f^{(k-1)}(t) H_{k-1}^* \left(\frac{x-t}{b-a} \right) \mathrm{d}t \\ & \quad = (b-a)^{k-1} \int_{[a,b]} H_{k-1}^* \left(\frac{x-t}{b-a} \right) \mathrm{d}f^{(k-2)}(t) \\ & \quad = I_{k-1}(x) \end{aligned}$$

and (2.7) can be rewritten as

$$
\begin{aligned}
I_k(x) = & H_k(x)\left[f^{(k-1)}(b) - f^{(k-1)}(a)\right] \\
& + (b-a)^k \beta_k f^{(k-1)}(x) + I_{k-1}(x) \\
= & H_k(x)\left[f^{(k-1)}(b) - f^{(k-1)}(a)\right] \\
& + [H_k(a) - H_k(b)] f^{(k-1)}(x) + I_{k-1}(x),
\end{aligned} \tag{2.9}
$$

since

$$\beta_k = \frac{H_k(a) - H_k(b)}{(b-a)^k}.$$

From (2.8) and (2.9) it follows

$$
\begin{aligned}
I_n(x) = & \sum_{k=1}^{n} H_k(x)\left[f^{(k-1)}(b) - f^{(k-1)}(a)\right] \\
& + \sum_{k=2}^{n} [H_k(a) - H_k(b)] f^{(k-1)}(x) - f(x) + \int_a^b f(t) W_x(t) \mathrm{d}t,
\end{aligned}
$$

which proves our assertion in this case, since $I_n(x) = -R_n^1(x)$. Thus, (2.5) holds for $a \leqslant x < b$.

If $x = b$, then

$$H_k^*\left(\frac{b-b}{b-a}\right) = H_k^*(0) = \frac{H_k(a)}{(b-a)^k},\ H_k^*\left(\frac{b-a}{b-a}\right) = H_k^*(0) = \frac{H_k(a)}{(b-a)^k}.$$

Similarly as we did for $a \leqslant x < b$, using the above equalities and the second formula from Lemma 2, we get

$$
\begin{aligned}
I_k(b) = & H_k(a)\left[f^{(k-1)}(b) - f^{(k-1)}(a)\right] \\
& + (b-a)^k \beta_k f^{(k-1)}(a) + I_{k-1}(b),
\end{aligned}
$$

for $k \geqslant 2$, and

$$I_1(b) = H_1(a)\left[f(b) - f(a)\right] - f(a) + \int_a^b f(t) W_b(t) \mathrm{d}t.$$

Applying the above identities and

$$H_1(a) - H_1(b) = -1,$$

we get

$$
\begin{aligned}
I_n(b) = & \sum_{k=1}^{n} H_k(a)\left[f^{(k-1)}(b) - f^{(k-1)}(a)\right] \\
& + \sum_{k=2}^{n} [H_k(a) - H_k(b)] f^{(k-1)}(a) - f(a) + \int_a^b f(t) W_b(t) \mathrm{d}t
\end{aligned}
$$

$$= \sum_{k=1}^{n} H_k(b)\left[f^{(k-1)}(b) - f^{(k-1)}(a)\right]$$
$$+ \sum_{k=2}^{n} [H_k(a) - H_k(b)] f^{(k-1)}(b) - f(b) + \int_a^b f(t) W_b(t) \mathrm{d}t,$$

which proves (2.7) for $x = b$, because $I_n(b) = -R_n^1(b)$. □

THEOREM 2. *Let $(H_k, k \geqslant 0)$ be w-harmonic sequence of functions on $[a,b]$ and $f : [a,b] \to \mathbf{R}$ such that $f^{(n-1)}$ is a continuous function of bounded variation on $[a,b]$ for some $n \geqslant 1$. Then for every $x \in [a,b]$ and $n \geqslant 2$*

$$f(x) = \int_a^b f(t) W_x(t) \mathrm{d}t + S_{n-1}(x) + [H_n(a) - H_n(b)] f^{(n-1)}(x) + R_n^2(x), \quad (2.10)$$

while for $n = 1$

$$f(x) = \int_a^b f(t) W_x(t) \mathrm{d}t + R_1^2(x),$$

where $S_{n-1}(x)$ is defined by (2.4), and

$$R_n^2(x) = -(b-a)^n \int_{[a,b]} \left[H_n^*\left(\frac{x-t}{b-a}\right) - \frac{H_n(x)}{(b-a)^n}\right] \mathrm{d}f^{(n-1)}(t).$$

for $n \geqslant 1$.

Proof. First note that for $n \geqslant 2$

$$S_n(x) - S_{n-1}(x) = H_n(x)\left[f^{(n-1)}(b) - f^{(n-1)}(a)\right] + [H_n(a) - H_n(b)] f^{(n-1)}(x).$$

Thus

$$R_n^2(x) = R_n^1(x) + H_n(x) \int_{[a,b]} \mathrm{d}f^{(n-1)}(t)$$
$$= R_n^1(x) + H_n(x)\left[f^{(n-1)}(b) - f^{(n-1)}(a)\right]$$
$$= R_n^1(x) + S_n(x) - S_{n-1}(x) - [H_n(a) - H_n(b)] f^{(n-1)}(x),$$

for $n \geqslant 2$, and

$$R_1^2(x) = R_1^1(x) + H_1(x)[f(b) - f(a)].$$

Therefore, our assertion follows from formula (2.7). □

REMARK 2. In the case when $\varphi : [a,b] \to \mathbf{R}$ is such that φ' exists and is integrable on $[a,b]$, then the Riemann-Stieltjes integral $\int_{[a,b]} g(t) d\varphi(t)$ is equal to the Riemann integral $\int_a^b g(t)\varphi'(t)\mathrm{d}t$. Therefore, if $f : [a,b] \to \mathbf{R}$ is such that $f^{(n)}$ exists and is integrable on $[a,b]$, for some $n \geqslant 1$, then Theorems 1 and 2 hold with

$$R_n^1(x) = -(b-a)^n \int_{[a,b]} H_n^*\left(\frac{x-t}{b-a}\right) f^{(n)}(t) \mathrm{d}t,$$

and

$$R_n^2(x) = -(b-a)^n \int_{[a,b]} \left[H_n^* \left(\frac{x-t}{b-a} \right) - \frac{H_n(x)}{(b-a)^n} \right] f^{(n)}(t)\mathrm{d}t.$$

REMARK 3. In the special case when $w(t) = \frac{1}{b-a}$, $t \in [a,b]$, formulae (2.5) and (2.10) are generalizations of (1.1) and (1.2), respectively, since in this case

$$\int_a^b f(t) W_x(t)\mathrm{d}t = \frac{1}{b-a} \int_a^b f(t)\mathrm{d}t,$$

for every $x \in [a,b]$, and the sum $S_n(x) + R_n^1(x)$, calculated with respect to w-harmonic sequence of functions on $[a,b]$

$$H_n(t) = Q_n(t) + \frac{1}{b-a} \frac{(t-a)^n}{n!},\ n \geqslant 0,$$

where $Q_n(t)$, $n \geqslant 0$ are as stated in Lemma 2, becomes $\widetilde{T}_n(x) + \tau_n(x) + \widetilde{R}_n^1(x)$, calculated with respect to the harmonic sequence of polynomials

$$P_n(t) = \frac{Q_n(a+(b-a)t)}{(b-a)^{n-1}} + \frac{t^n}{n!},\ n \geqslant 0,$$

as in [4].

There is a difference in the definition of our $H_n^*(t)$ and $P_n^*(t)$ from Introduction, because a w-harmonic sequence of functions on $[a,b]$ is not defined on $[0,1]$.

REMARK 4. It is easy to see that for $x = b$ we get

$$S_n(b) = H_1(b)[f(b) - f(a)] + \sum_{k=2}^{n} \left[H_k(a) f^{(k-1)}(b) - H_k(b) f^{(k-1)}(a) \right],$$

assuming the sum on the right hand side to be zero when $n = 1$. So, applying Theorem 1 with $x = b$ we get the identity

$$f(b) = \int_a^b f(t) W_b(t)\,\mathrm{d}t + S_n(b) + R_n^1(b). \tag{2.11}$$

Let us denote

$$W(t) = W_b(t) = w(a+b-t),\ t \in [a,b].$$

Note that $W(t) \geqslant 0$, $t \in [a,b]$ and $\int_a^b W(t)\mathrm{d}t = 1$. Further we have

$$R_n^1(b) = -(b-a)^n \int_{[a,b]} H_n^* \left(\frac{b-t}{b-a} \right) \mathrm{d}f^{(n-1)}(t)$$
$$= -\int_{[a,b]} H_n(a+b-t)\,\mathrm{d}f^{(n-1)}(t).$$

Also, since $H_1(x) = c + \int_a^x w(s)\mathrm{d}s$, for some real constant c,we have $H_1(b) = c + 1$ and putting all this together we can rewrite the identity (2.11) in the form

$$
\begin{aligned}
&(c+1)f(a) - cf(b) \\
&= \int_a^b f(t)\, W(t)\,\mathrm{d}t + \sum_{k=2}^{n} \left[H_k(a) f^{(k-1)}(b) - H_k(b) f^{(k-1)}(a) \right] \\
&- \int_{[a,b]} H_n(a+b-t)\,\mathrm{d}f^{(n-1)}(t).
\end{aligned}
$$

We can regard this identity as generalized trapezoid identity since for $n = 1$ and $c = -\frac{1}{2}$ it reduces to

$$
\frac{1}{2}\left[f(a) + f(b) \right] = \int_a^b f(t)\, W(t)\,\mathrm{d}t - \int_{[a,b]} H_1(a+b-t)\,\mathrm{d}f(t),
$$

where $H_1(x) = -\frac{1}{2} + \int_a^x w(s)\mathrm{d}s$. Similarly, applying Theorem 1 with $x = \frac{a+b}{2}$ we get generalized midpoint identity

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right) &= \int_a^b f(t)\, \tilde{W}(t)\,\mathrm{d}t + \sum_{k=1}^{n} H_k\left(\frac{a+b}{2}\right) \left[f^{(k-1)}(b) - f^{(k-1)}(a) \right] \\
&+ \sum_{k=2}^{n} \left[H_k(a) - H_k(b) \right] f^{(k-1)}\left(\frac{a+b}{2}\right) + R_n^1\left(\frac{a+b}{2}\right), \qquad (2.12)
\end{aligned}
$$

where

$$
\tilde{W}(t) = W_{\frac{a+b}{2}}(t) = \begin{cases} w\left(a + \frac{a+b}{2} - t\right), & a \leqslant t \leqslant \frac{a+b}{2} \\ w\left(b + \frac{a+b}{2} - t\right), & \frac{a+b}{2} < t \leqslant b \end{cases}
$$

and

$$
\begin{aligned}
R_n^1\left(\frac{a+b}{2}\right) &= -\int_{\left[a, \frac{a+b}{2}\right]} H_n\left(a + \frac{a+b}{2} - t\right)\mathrm{d}f^{(n-1)}(t) \\
&\quad - \int_{\left\langle \frac{a+b}{2}, b\right]} H_n\left(b + \frac{a+b}{2} - t\right)\mathrm{d}f^{(n-1)}(t).
\end{aligned}
$$

For $n = 1$ we have $H_1(x) = c + \int_a^x w(s)\mathrm{d}s$ so that choice $c = -\int_a^{\frac{a+b}{2}} w(s)\mathrm{d}s$ yields $H_1(x) = \int_{\frac{a+b}{2}}^x w(s)\mathrm{d}s$ and (2.12) reduces to the simple midpoint identity

$$
f\left(\frac{a+b}{2}\right) = \int_a^b f(t)\, \tilde{W}(t)\,\mathrm{d}t + R_1^1\left(\frac{a+b}{2}\right).
$$

For some recent results on trapezoid and midpoint identities the reader is referred to the papers [6] and [7].

3. Generalizations of Ostrowski inequality

In this section we use the identities obtained in Theorem 1 and Theorem 2 to prove some inequalities which can be regarded as generalizations of Ostrowski inequality for various classes of functions. First we prove a number of inequalities which hold for a class of functions f whose derivatives $f^{(n-1)}$ are either L-Lipschitzian on $[a,b]$ or continuous and of bounded variation on $[a,b]$. After that, for the sake of completeness, we state and prove some analogous results holding for a class of functions f possessing derivatives $f^{(n)}$ in $L_p[a,b]$.

Throughout this section we use the same notations as in the previous section. So, $(H_k(t),\ k \geqslant 0)$ always denotes a w-harmonic sequence of functions on $[a,b]$ as defined at the beginning of Section 2, while the special w-harmonic sequence $(w_k(t),$ $k \geqslant 0)$ is defined by (2.1) and (2.2).

THEOREM 3. *Let $(H_k(t),\ k \geqslant 0)$ be a w-harmonic sequence of functions on $[a,b]$ and $f : [a,b] \to \mathbf{R}$ such that $f^{(n-1)}$ is an L-Lipschitzian function on $[a,b]$ for some $n \geqslant 1$. Then for $n \geqslant 2$*

$$\left|f(x) - \int_a^b f(t)W_x(t)\mathrm{d}t - S_{n-1}(x) - [H_n(a) - H_n(b)]f^{(n-1)}(x)\right| \leqslant L\int_a^b |H_n(t) - H_n(x)|\,\mathrm{d}t, \tag{3.1}$$

while for $n = 1$

$$\left|f(x) - \int_a^b f(t)W_x(t)\mathrm{d}t\right| \leqslant L\int_a^b |H_1(t) - H_1(x)|\,\mathrm{d}t,$$

for every $x \in [a,b]$.

Proof. If $\varphi : [a,b] \to \mathbf{R}$ is L-Lipschitzian on $[a,b]$, i.e.

$$|\varphi(x) - \varphi(y)| \leqslant L \cdot |x-y|,\ x,\ y \in [a,b],$$

then for any integrable function $g : [a,b] \to \mathbf{R}$

$$\left|\int_{[a,b]} g(t)\mathrm{d}\varphi(t)\right| \leqslant L\int_a^b |g(t)|\,\mathrm{d}t. \tag{3.2}$$

Using this estimate we get

$$\begin{aligned}
\left|R_n^2(x)\right| &= (b-a)^n\left|\int_{[a,b]}\left[H_n^*\left(\frac{x-t}{b-a}\right) - \frac{H_n(x)}{(b-a)^n}\right]\mathrm{d}f^{(n-1)}(t)\right| \\
&\leqslant (b-a)^n L\int_a^b \left|H_n^*\left(\frac{x-t}{b-a}\right) - \frac{H_n(x)}{(b-a)^n}\right|\mathrm{d}t \\
&= (b-a)^n L\int_a^x \left|H_n^*\left(\frac{x-t}{b-a}\right) - \frac{H_n(x)}{(b-a)^n}\right|\mathrm{d}t \\
&+ (b-a)^n L\int_x^b \left|H_n^*\left(\frac{x-t}{b-a}+1\right) - \frac{H_n(x)}{(b-a)^n}\right|\mathrm{d}t
\end{aligned}$$

$$= (b-a)^n L \int_a^x \left| \frac{H_n(a+x-t)}{(b-a)^n} - \frac{H_n(x)}{(b-a)^n} \right| \mathrm{d}t$$
$$+ (b-a)^n L \int_x^b \left| \frac{H_n(b+x-t)}{(b-a)^n} - \frac{H_n(x)}{(b-a)^n} \right| \mathrm{d}t$$
$$= L \left(- \int_x^a |H_n(u) - H_n(x)| \, \mathrm{d}u - \int_b^x |H_n(v) - H_n(x)| \, \mathrm{d}v \right)$$
$$= L \int_a^b |H_n(t) - H_n(x)| \, \mathrm{d}t.$$

Therefore, our assertion follows from Theorem 2. □

COROLLARY 1. *If f is L-Lipschitzian on $[a,b]$, then*

$$\left| f(x) - \int_a^b f(t) W_x(t) \mathrm{d}t \right| \leqslant L \left[(2x-a-b) w_1(x) - 2w_2(x) + w_2(b) \right],$$

for every $x \in [a,b]$.

Proof. Put $n=1$ in the theorem above and

$$\int_a^b |H_1(t) - H_1(x)| \, \mathrm{d}t = \int_a^x (w_1(x) - w_1(t)) \, \mathrm{d}t + \int_x^b (w_1(t) - w_1(x)) \, \mathrm{d}t$$
$$= (2x-a-b) w_1(x) - 2w_2(x) + w_2(b),$$

since

$$H_1(t) = c + w_1(t) = c + \int_a^t w(s) \mathrm{d}s,$$

for some real c. □

REMARK 5. In the special case when $w(t) = \frac{1}{b-a}$ the inequality of Corollary 1 reduces to Ostrowski inequality for L-Lipschitzian functions, since in this case

$$\int_a^b f(t) W_x(t) \mathrm{d}t = \frac{1}{b-a} \int_a^b f(t) \mathrm{d}t$$

and

$$(2x-a-b) w_1(x) - 2w_2(x) + w_2(b) = (b-a) \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right].$$

COROLLARY 2. *If f' is L-Lipschitzian on $[a,b]$, then*

$$\left| f(x) - \int_a^b f(t) W_x(t) \mathrm{d}t - [c + w_1(x)] [f(b) - f(a)] + [c(b-a) + w_2(b)] f'(x) \right|$$
$$\leqslant L \int_a^b |c(t-x) + w_2(t) - w_2(x)| \, \mathrm{d}t,$$

for every $x \in [a,b]$ and $c \in \mathbf{R}$.

Proof. Apply Theorem 3 with $n = 2$ and note that by Lemma 1 we have

$$H_1(t) = c + w_1(t), \;\; H_2(t) = c_0 + c(t-a) + w_2(t),$$

where c_0, and c are some real constants. □

REMARK 6. In the special case, when $c = -\frac{1}{2}$ and $w(t) = \frac{1}{b-a}$, we have

$$w_1(t) = \frac{t-a}{b-a}, \;\; w_2(t) = \frac{(t-a)^2}{2(b-a)}$$

and the inequality from Corollary 2 reduces to

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - \left(x - \frac{a+b}{2}\right)\frac{f(b)-f(a)}{b-a} \right| \leqslant \frac{(b-a)^2}{2}\left[\frac{8}{3}\delta^3(x) - \delta^2(x) + \frac{1}{12}\right]\cdot L,$$

where

$$\delta(x) = \frac{\left|x - \frac{a+b}{2}\right|}{b-a}.$$

This result was proved in [5],

REMARK 7. For $c \geqslant 0$, by a simple but long calculation we get

$$\int_a^b |c(t-x) + w_2(t) - w_2(x)|\,\mathrm{d}t = \frac{c}{2}\left[(x-a)^2 + (b-x)^2\right] + (2x-a-b)w_2(x) - 2w_3(x) + w_3(b).$$

COROLLARY 3. *Assume f satisfies the conditions of Theorem 3. For $n \geqslant 2$ and for every $x \in [a,b]$ we have*

$$\left| f(x) - \int_a^b f(t)W_x(t)\mathrm{d}t - \overline{S}_{n-1}(x) + w_n(b)f^{(n-1)}(x) \right| \leqslant L\left[(2x-a-b)w_n(x) - 2w_{n+1}(x) + w_{n+1}(b)\right],$$

where

$$\overline{S}_m(x) = \sum_{k=1}^{m} w_k(x)\left[f^{(k-1)}(b) - f^{(k-1)}(a)\right] - \sum_{k=2}^{m} w_k(b)f^{(k-1)}(x).$$

Proof. We apply Theorem 3 with w-harmonic sequence $H_k(t) = w_k(t)$, $k \geqslant 0$ defined at the beginning of previous section. In that case $S_{n-1}(x)$ becomes $\overline{S}_{n-1}(x)$, while

$$\int_a^b |w_n(t) - w_n(x)|\,\mathrm{d}t = (2x-a-b)w_n(x) - 2w_{n+1}(x) + w_{n+1}(b).$$

□

COROLLARY 4. *Assume f satisfies the conditions of Theorem 3. For $n \geqslant 2$ and for every $x \in [a,b]$ we have*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t - \widehat{S}_{n-1}(x) + \frac{(b-a)^{n-1}}{n!} f^{(n-1)}(x) \right| \leqslant \frac{L}{(b-a)n!}\left[(2x-a-b)(x-a)^n - \frac{2(x-a)^{n+1}}{n+1} + \frac{(b-a)^{n+1}}{n+1} \right],$$

where

$$\widehat{S}_m(x) = \sum_{k=1}^{m} \frac{(x-a)^k}{(b-a)k!}\left[f^{(k-1)}(b) - f^{(k-1)}(a) \right] - \sum_{k=2}^{m} \frac{(b-a)^{k-1}}{k!} f^{(k-1)}(x).$$

Proof. Apply Corollary 3 for the special case when $w(t) = \frac{1}{b-a}$ and, therefore

$$w_k(t) = \frac{1}{b-a}\frac{(t-a)^k}{k!},\ k \geqslant 0.$$

□

THEOREM 4. *Let $(H_k(t),\ k \geqslant 0)$ be a w-harmonic sequence of functions on $[a,b]$ and $f : [a,b] \to \mathbf{R}$ such that $f^{(n-1)}$ is an L-Lipschitzian function on $[a,b]$ for some $n \geqslant 1$. Then*

$$\left| f(x) - \int_a^b f(t)W_x(t)\mathrm{d}t - S_n(x) \right| \leqslant L\int_a^b |H_n(t)|\,\mathrm{d}t, \tag{3.3}$$

for every $x \in [a,b]$.

Proof. Using the estimate (3.2) and arguing similarly as in the proof of Theorem 3, we get

$$\begin{aligned} \left| R_n^1(x) \right| &= (b-a)^n \left| \int_{[a,b]} H_n^*\left(\frac{x-t}{b-a} \right) \mathrm{d}f^{(n-1)}(t) \right| \\ &\leqslant (b-a)^n L \int_a^b \left| H_n^*\left(\frac{x-t}{b-a} \right) \right| \mathrm{d}t \\ &= L\int_a^b |H_n(t)|\,\mathrm{d}t. \end{aligned}$$

Therefore, our assertion follows from (2.5). □

COROLLARY 5. *If f is L-Lipschitzian on $[a,b]$, then for every $x \in [a,b]$ and $c \in \mathbf{R}$ we have*

$$\left| f(x) - \int_a^b f(t)W_x(t)\mathrm{d}t - [c + w_1(x)]\,[f(b) - f(a)] \right| \leqslant L\int_a^b |c + w_1(t)|\mathrm{d}t.$$

Proof. Put $n=1$ in the theorem above. □

COROLLARY 6. *If f is L-Lipschitzian on $[a,b]$ then for every $x,y\in[a,b]$ we have*

$$\begin{aligned}&\left|f(x)-\int_a^b f(t)W_x(t)\mathrm{d}t-[w_1(x)-w_1(y)][f(b)-f(a)]\right|\\&\leqslant L\left[(2y-a-b)w_1(y)-2w_2(y)+w_2(b)\right].\end{aligned}$$

Proof. Put $c=-w_1(y)$ in Corollary 5. □

REMARK 8. For $y=x$ the above inequality reduces to

$$\left|f(x)-\int_a^b f(t)W_x(t)\mathrm{d}t\right|\leqslant L\left[(2x-a-b)w_1(x)-2w_2(x)+w_2(b)\right],$$

i.e. to the inequality from Corollary 1.

COROLLARY 7. *Assume f satisfies the conditions of Theorem 4. For $n\geqslant 1$ and for every $x\in[a,b]$ we have*

$$\left|f(x)-\int_a^b f(t)W_x(t)\mathrm{d}t-\overline{S}_n(x)\right|\leqslant Lw_{n+1}(b),$$

where $\overline{S}_n(x)$ is defined as in Corollary 3.

Proof. We apply Theorem 4 with the w-harmonic sequence $H_k(t)=w_k(t)$, $k\geqslant 0$. Then $S_n(x)$ becomes $\overline{S}_n(x)$ while

$$\int_a^b |w_n(t)|\,\mathrm{d}t=\int_a^b w_n(t)\mathrm{d}t=w_{n+1}(b).$$

□

COROLLARY 8. *Assume f satisfies the conditions of Theorem 4. For $n\geqslant 1$ and for every $x\in[a,b]$ we have*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t-\widehat{S}_n(x)\right|\leqslant L\frac{(b-a)^n}{(n+1)!},$$

where $\widehat{S}_n(x)$ is defined as in Corollary 4.

Proof. Apply Corollary 7 to the special case when $w(t)=\frac{1}{b-a}$. □

THEOREM 5. *Let $(H_k(t),\ k\geqslant 0)$ be a w-harmonic sequence of functions on $[a,b]$ and $f:[a,b]\to\mathbf{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation* *on* $[a,b]$ *for some* $n \geqslant 1$. *Then for* $n \geqslant 2$

$$\begin{aligned}&\left|f(x)-\int_a^b f(t)W_x(t)\mathrm{d}t-S_{n-1}(x)-[H_n(a)-H_n(b)]f^{(n-1)}(x)\right|\\&\leqslant \max_{t\in[a,b]}|H_n(t)-H_n(x)|\,V_a^b(f^{(n-1)})\end{aligned}$$

and for $n=1$

$$\left|f(x)-\int_a^b f(t)W_x(t)\mathrm{d}t\right|\leqslant \max_{t\in[a,b]}|H_1(t)-H_1(x)|\,V_a^b(f),$$

for every $x\in[a,b]$, *where* $V_a^b(f^{(n-1)})$ *is the total variation of* $f^{(n-1)}$ *on* $[a,b]$.

Proof. If $F:[a,b]\to\mathbf{R}$ is bounded and the Stieltjes integral

$$\int_{[a,b]}F(t)\mathrm{d}f^{(n-1)}(t)$$

exists, then

$$\left|\int_{[a,b]}F(t)\mathrm{d}f^{(n-1)}(t)\right|\leqslant \max_{t\in[a,b]}|F(t)|\cdot V_a^b(f^{(n-1)}).$$

Let us apply this estimation to the formula (2.10). We have

$$\begin{aligned}\left|R_n^2(x)\right|&=\left|-(b-a)^n\int_a^b\left[H_n^*\left(\frac{x-t}{b-a}\right)-\frac{H_n(x)}{(b-a)^n}\right]\mathrm{d}f^{(n-1)}(t)\right|\\&\leqslant (b-a)^n\max_{t\in[a,b]}\left|H_n^*\left(\frac{x-t}{b-a}\right)-\frac{H_n(x)}{(b-a)^n}\right|V_a^b(f^{(n-1)})\\&=\max_{t\in[a,b]}|H_n(t)-H_n(x)|\,V_a^b(f^{(n-1)}),\end{aligned}$$

which proves our assertion. □

COROLLARY 9. *If* f *is a continuous function of bounded variation on* $[a,b]$, *then*

$$\left|f(x)-\int_a^b f(t)W_x(t)\mathrm{d}t\right|\leqslant\frac{1}{2}\left[1+|1-2w_1(x)|\right]V_a^b(f).$$

Proof. Put $n=1$ in the theorem above and note that $w_1(b)=1$. Therefore

$$\begin{aligned}\max_{t\in[a,b]}|H_1(t)-H_1(x)|&=\max_{t\in[a,b]}|w_1(t)-w_1(x)|\\&=\max\{w_1(x),w_1(b)-w_1(x)\}\\&=\frac{1}{2}\left[1+|1-2w_1(x)|\right].\end{aligned}$$

□

COROLLARY 10. *If f is a continuous function of bounded variation on $[a,b]$, then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t\right| \leqslant \left[\frac{1}{2}+\frac{\left|x-\frac{a+b}{2}\right|}{b-a}\right] V_a^b(f).$$

Proof. Apply Corollary 9 with $w(t)=\frac{1}{b-a}$ and $w_1(t)=\frac{t-a}{b-a}$. □

COROLLARY 11. *If f' is a continuous function of bounded variation on $[a,b]$, then for every $x\in[a,b]$ and $c\in\mathbf{R}$ we have*

$$\left|f(x)-\int_a^b f(t)W_x(t)\mathrm{d}t-[c+w_1(x)][f(b)-f(a)]+[c(b-a)+w_2(b)]f'(x)\right|$$
$$\leqslant \max_{t\in[a,b]}|c(t-x)+w_2(t)-w_2(x)|\,V_a^b(f').$$

Proof. Put $n=2$ in Theorem 5. □

COROLLARY 12. *Assume f satisfies the conditions of Theorem 5. For $n\geqslant 2$ and for every $x\in[a,b]$ we have*

$$\left|f(x)-\int_a^b f(t)W_x(t)\mathrm{d}t-\overline{S}_{n-1}(x)+w_n(b)f^{(n-1)}(x)\right|$$
$$\leqslant \frac{1}{2}\left[w_n(b)+|w_n(b)-2w_n(x)|\right]V_a^b(f^{(n-1)}),$$

where $\overline{S}_{n-1}(x)$ is defined as in Corollary 3.

Proof. We apply Theorem 5 with the w-harmonic sequence $H_k(t)=w_k(t)$, $k\geqslant 0$. Then $S_{n-1}(x)$ becomes $\overline{S}_{n-1}(x)$ while

$$\max_{t\in[a,b]}|w_n(t)-w_n(x)|=\frac{1}{2}\left[w_n(b)+|w_n(b)-2w_n(x)|\right].$$

□

COROLLARY 13. *Assume f satisfies the conditions of Theorem 5. For $n\geqslant 2$ and for every $x\in[a,b]$ we have*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t-\widehat{S}_{n-1}(x)+\frac{(b-a)^{n-1}}{n!}f^{(n-1)}(x)\right|$$
$$\leqslant \frac{(b-a)^{n-1}}{2\cdot n!}\left[1+\left|1-2\left(\frac{x-a}{b-a}\right)^n\right|\right]V_a^b(f^{(n-1)}),$$

where $\widehat{S}_{n-1}(x)$ is defined as in Corollary 4.

Proof. Apply Corollary 12 in the special case when $w(t) = \frac{1}{b-a}$. □

THEOREM 6. *Let* $(H_k(t),\ k \geqslant 0)$ *be a* w*-harmonic sequence of functions on* $[a,b]$ *and* $f : [a,b] \to \mathbf{R}$ *be such that* $f^{(n-1)}$ *is a continuous function of bounded variation on* $[a,b]$ *for some* $n \geqslant 1$. *Then*

$$\left| f(x) - \int_a^b f(t) W_x(t) \mathrm{d}t - S_n(x) \right| \leqslant \max_{t \in [a,b]} |H_n(t)|\, V_a^b(f^{(n-1)}),$$

for every $x \in [a,b]$.

Proof. The result follows from (2.5) similarly as we proved Theorem 5. □

COROLLARY 14. *If* f *is a continuous function of bounded variation on* $[a,b]$, *then for every* $x \in [a,b]$ *and* $c \in \mathbf{R}$ *we have*

$$\left| f(x) - \int_a^b f(t) W_x(t) \mathrm{d}t - [c + w_1(x)]\,[f(b) - f(a)] \right| \leqslant \max\{|c|, |c+1|\}\, V_a^b(f).$$

Proof. Put $n = 1$ in the theorem above and note that $H_1(t) = c + w_1(t)$, $w_1(a) = 0$ and $w_1(b) = 1$. □

COROLLARY 15. *If* f *is a continuous function of bounded variation on* $[a,b]$, *then for every* $x, y \in [a,b]$ *we have*

$$\left| f(x) - \int_a^b f(t) W_x(t) \mathrm{d}t - [w_1(x) - w_1(y)]\,[f(b) - f(a)] \right|$$
$$\leqslant \frac{1}{2}\,[1 + |1 - 2w_1(y)|]\, V_a^b(f).$$

Proof. Put $c = -w_1(y)$ in Corollary 14. Then $|c| = w_1(y), |c+1| = 1 - w_1(y)$ and

$$\max\{w_1(y), 1 - w_1(y)\} = \frac{1}{2}\,[1 + |1 - 2w_1(y)|].$$

□

COROLLARY 16. *Assume* f *satisfies the conditions of Theorem 6. For* $n \geqslant 1$ *and for every* $x \in [a,b]$ *we have*

$$\left| f(x) - \int_a^b f(t) W_x(t) \mathrm{d}t - \overline{S}_n(x) \right| \leqslant w_n(b) V_a^b(f^{(n-1)}),$$

where $\overline{S}_n(x)$ *is defined as in Corollary 3.*

Proof. We apply Theorem 6 with the w-harmonic sequence $H_k(t) = w_k(t)$, $k \geqslant 0$. Then $S_n(x)$ becomes $\overline{S}_n(x)$ and

$$\max_{t\in[a,b]} |w_n(t)| = w_n(b).$$

□

COROLLARY 17. *Assume f satisfies the conditions of Theorem 6. For $n \geqslant 1$ and for every $x \in [a,b]$ we have*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t - \widehat{S}_n(x) \right| \leqslant \frac{(b-a)^{n-1}}{n!} V_a^b(f^{(n-1)}),$$

where $\widehat{S}_n(x)$ is defined as in Corollary 4.

Proof. Apply Corollary 16 with $w(t) = \frac{1}{b-a}$ and note that

$$w_n(b) = \frac{(b-a)^{n-1}}{n!}.$$

□

Now, we finish this section considering the case when $f : [a,b] \to \mathbf{R}$ is such that $f^{(n)}$ exists for some $n \geqslant 1$ and is integrable on $[a,b]$. In this case we can use the versions of the identities from Theorems 1 and 2 as noted in Remark 2.

THEOREM 7. *Let $(H_k(t),\ k \geqslant 0)$ be a w-harmonic sequence of functions on $[a,b]$ and $f : [a,b] \to \mathbf{R}$ be such that $f^{(n)}$ is integrable for some $n \geqslant 1$. Then for $n \geqslant 2$*

$$\left| f(x) - \int_a^b f(t) W_x(t)\mathrm{d}t - S_{n-1}(x) - [H_n(a) - H_n(b)] f^{(n-1)}(x) \right|$$
$$\leqslant \max_{t\in[a,b]} |H_n(t) - H_n(x)| \, \|f^{(n)}\|_1,$$

while for $n = 1$

$$\left| f(x) - \int_a^b f(t) W_x(t)\mathrm{d}t \right| \leqslant \frac{1}{2}\left[1 + |1 - 2w_1(x)|\right] \|f'\|_1,$$

for every $x \in [a,b]$.

Proof. Note that in this case

$$V_a^b(f^{(n-1)}) = \int_a^b \left| f^{(n)}(t) \right| \mathrm{d}t = \|f^{(n)}\|_1,$$

and apply Theorem 5. □

THEOREM 8. *Let* $(H_k(t),\ k \geqslant 0)$ *be a* w*-harmonic sequence of functions on* $[a,b]$ *and* $f : [a,b] \to \mathbf{R}$ *be such that* $f^{(n)}$ *is integrable for some* $n \geqslant 1$. *Then*

$$\left| f(x) - \int_a^b f(t) W_x(t) \mathrm{d}t - S_n(x) \right| \leqslant \max_{t \in [a,b]} |H_n(t)| \cdot \|f^{(n)}\|_1,$$

for every $x \in [a,b]$.

Proof. Apply Theorem 6 and note again that $V_a^b(f^{(n-1)}) = \|f^{(n)}\|_1$. □

THEOREM 9. *Let* $(H_k(t),\ k \geqslant 0)$ *be a* w*-harmonic sequence of functions on* $[a,b]$ *and* $f : [a,b] \to \mathbf{R}$ *be such that* $f^{(n)} \in L_\infty[a,b]$ *for some* $n \geqslant 1$. *Then for* $n \geqslant 2$

$$\left| f(x) - \int_a^b f(t) W_x(t) \mathrm{d}t - S_{n-1}(x) - [H_n(a) - H_n(b)] f^{(n-1)}(x) \right|$$
$$\leqslant \int_a^b |H_n(t) - H_n(x)| \, \mathrm{d}t \cdot \|f^{(n)}\|_\infty,$$

while for $n = 1$

$$\left| f(x) - \int_a^b f(t) W_x(t) \mathrm{d}t \right| \leqslant [(2x - a - b) w_1(x) - 2w_2(x) + w_2(b)] \, \|f'\|_\infty,$$

for every $x \in [a,b]$.

Proof. In this case $f^{(n-1)}$ is L-Lipschitzian with $L = \|f^{(n)}\|_\infty$. □

THEOREM 10. *Let* $(H_k(t),\ k \geqslant 0)$ *be a* w*-harmonic sequence of functions on* $[a,b]$ *and* $f : [a,b] \to \mathbf{R}$ *be such that* $f^{(n)} \in L_\infty[a,b]$ *for some* $n \geqslant 1$. *Then*

$$\left| f(x) - \int_a^b f(t) W_x(t) \mathrm{d}t - S_n(x) \right| \leqslant \int_a^b |H_n(t)| \, \mathrm{d}t \cdot \|f^{(n)}\|_\infty,$$

for every $x \in [a,b]$.

Proof. In this case $f^{(n-1)}$ is L-Lipschitzian with $L = \|f^{(n)}\|_\infty$. □

THEOREM 11. *Let* $(H_k(t),\ k \geqslant 0)$ *be a* w*-harmonic sequence of functions on* $[a,b]$ *and* $f : [a,b] \to \mathbf{R}$ *be such that* $f^{(n)} \in L_p[a,b]$ *for some* $n \geqslant 1$ *and* $1 < p < \infty$. *Then for* $n \geqslant 2$

$$\left| f(x) - \int_a^b f(t) W_x(t) \mathrm{d}t - S_{n-1}(x) - [H_n(a) - H_n(b)] f^{(n-1)}(x) \right|$$
$$\leqslant \|H_n - H_n(x)\|_q \|f^{(n)}\|_p,$$

while for $n = 1$

$$\left| f(x) - \int_a^b f(t) W_x(t) \mathrm{d}t \right| \leqslant \|H_1 - H_1(x)\|_q \|f'\|_p,$$

for every $x \in [a,b]$, *where* $1/p + 1/q = 1$.

Proof. By applying the Hölder inequality we have

$$\begin{aligned}
&\left| f(x) - \int_a^b f(t) W_x(t) \mathrm{d}t - S_{n-1}(x) - [H_n(a) - H_n(b)] f^{(n-1)}(x) \right| \\
&\leqslant (b-a)^n \int_a^b \left| H_n^*\left(\frac{x-t}{b-a}\right) - \frac{H_n(x)}{(b-a)^n} \right| \left| f^{(n)}(t) \right| \mathrm{d}t \\
&\leqslant (b-a)^n \left(\int_a^b \left| H_n^*\left(\frac{x-t}{b-a}\right) - \frac{H_n(x)}{(b-a)^n} \right|^q \mathrm{d}t \right)^{1/q} \| f^{(n)} \|_p \\
&= \left(\int_a^b |H_n(t) - H_n(x)|^q \mathrm{d}t \right)^{1/q} \| f^{(n)} \|_p ,
\end{aligned}$$

which proves our assertion. □

THEOREM 12. *Let* $(H_k(t),\ k \geqslant 0)$ *be a* w*-harmonic sequence of functions on* $[a,b]$ *and* $f : [a,b] \to \mathbf{R}$ *be such that* $f^{(n)} \in L_p[a,b]$ *for some* $n \geqslant 1$ *and* $1 < p < \infty$. *Then*

$$\left| f(x) - \int_a^b f(t) W_x(t) \mathrm{d}t - S_n(x) \right| \leqslant \|H_n\|_q \|f^{(n)}\|_p ,$$

for every $x \in [a,b]$, *where* $1/p + 1/q = 1$.

Proof. Similar to the proof of the theorem above. □

REFERENCES

[1] A. AGLIĆ ALJINOVIĆ, J. PEČARIĆ, *The weighted Euler identity*, Math. Inequal. Appl., **8**, 2 (2005), 207–221.

[2] A. AGLIĆ ALJINOVIĆ, J. PEČARIĆ, *A discrete Euler identity*, J. Inequal. in Pure and Appl. Math., **5**, 3 (2004), Article 58.

[3] A. AGLIĆ ALJINOVIĆ, J. PEČARIĆ, *Discrete weighted Montgomery identity and discrete Ostrowski type inequalities*, Computers and Mathematics with Applications, **48**, (2004), 731–745.

[4] LJ. DEDIĆ, M. MATIĆ, J. PEČARIĆ, AND A. VUKELIĆ, *On generalizations of Ostrowski inequality via Euler harmonic identities*, J. of Inequal. & Appl., **7**, 6 (2002), 787–805.

[5] LJ. DEDIĆ, M. MATIĆ AND J. PEČARIĆ, *On generalizations of Ostrowski inequality via some Euler-type identities*, Math. Inequal. & Appl., **3**, 3 (2000), 337–353.

[6] LJ. DEDIĆ, M. MATIĆ AND J. PEČARIĆ, *On Euler trapezoid formulae*, Appl. Math. Comp. **123**, (2001) 37–62.

[7] LJ. DEDIĆ, M. MATIĆ AND J. PEČARIĆ, *On Euler mid-point formulae*, ANZIAM Journal, **46**, (2005), 417–438.

[8] S. S. DRAGOMIR, *The generalised integration by parts formula for Appell sequences and related results*, Commun. Korean Math. Soc., **19**, 1 (2004), 75–92.

(Received October 15, 2003)

Lj. Dedić
Department of Mathematics
Faculty of Natural Sciences, Mathematics and Education
University of Split
Teslina 12
21000 Split
Croatia
e-mail: ljuban@pmfst.hr

M. Matić
Department of Mathematics
Faculty of Natural Sciences, Mathematics and Education
University of Split
Teslina 12
21000 Split
Croatia
e-mail: mmatic@pmfst.hr

J. Pečarić
Faculty of textile technology
University of Zagreb
Pierottijeva 6
10000 Zagreb
Croatia
e-mail: pecaric@element.hr

A. Aglić Aljinović
Department of Applied Mathematics
Faculty of Electrical Engineering and Computing
Unska 3
10 000 Zagreb
Croatia
e-mail: andrea@zpm.fer.hr