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ON WEIGHTED DISCRETE HARDY'S INEQUALITY FOR NEGATIVE POWER NUMBERS

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Dedicated to the memory
of my supervisor
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(*communicated by L.- E. Persson*)

Abstract. In this paper we consider the weighted discrete Hardy's inequality for different real power numbers $0 \neq r < 1$ and obtain some new refinements of its finite sections. For $r < -1$ our results improve those previously given by Nguyen et al. in [19, 20]. Moreover, we prove that the constant factors involved in the right-hand sides of the obtained inequalities are the best possible, that is, they cannot be replaced with any smaller constant.

1. Introduction

Let $p > 1$ and $(a_n)_{n\in\mathbb{N}}$ be a sequence of non-negative real numbers, such that the series $\sum_{n=1}^{\infty} a_n$ converges. Then the well-known discrete Hardy's inequality

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} a_k^{1/p}\right)^p < \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} a_n \tag{1}$$

holds, unless $a_n = 0$ for all $n \in \mathbb{N}$. Moreover, the constant $[p/(p-1)]^p$ on its right-hand side is the best possible, that is, it cannot be replaced with any smaller constant.

The relation (1) was proved by G. H. Hardy in the paper [8]. In fact, it was known to him earlier (see [7]), but he had been unable to fix the best possible constant. In his estimate, he dealt with the constant $[p^2/(p-1)]^p$. Hardy's main aim in [8] was to find a new, more elementary proof of Hilbert's double series theorem and he showed that it follows from (1). Further remarks concerning the history and properties of Hardy's and Hilbert's inequalities can be found in [9, 15, 17].

Since published in 1925, the inequality (1) and its integral analogue (see [9]) have been discussed by several authors, who either reproved them using various techniques, or applied and generalized them in many different ways. The study of Hardy's inequality

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is also covered by a rich literature. Here, we just emphasize the monographs [14] and [21], related to this topic, and mention the references [3, 4, 5], [17, Chapter IV], and [19, 20], all of which to some extent have guided us in the research we present here.

One possible way of generalizing the discrete Hardy's inequality is to discuss the related power number p. Rewrite (1) in an equivalent form

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}a_k^r\right)^{1/r} < (1-r)^{-1/r}\sum_{n=1}^{\infty}a_n \tag{2}$$

obtained by substituting $p = 1/r$, where $0 < r < 1$. First, note that in the limiting case $r = 0$ of (2) we have the classical Carleman's inequality,

$$\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n}a_k\right)^{1/n} < e\sum_{n=1}^{\infty}a_n,$$

proved by T. Carleman in 1922, with the optimal constant factor e (see e.g. the recent review paper [10] for more details). Moreover, it is not hard to prove that in the case $r \geqslant 1$ the series on the left-hand side of (2) diverges for all non-trivial sequences $(a_n)_{n\in\mathbb{N}}$ of non-negative real numbers.

Recently, using the method of indeterminate coefficients, Nguyen and Nguyen, [19], concluded the analysis of parameters r by extending (2) to negative power numbers. They proved that

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}a_k^r\right)^{1/r} \leqslant \begin{cases} (1-r)^{-1/r}\sum_{n=1}^{\infty}a_n, & r\in[-1,1\rangle, r\neq 0,\\ \dfrac{r}{r-1}2^{1-1/r}\sum_{n=1}^{\infty}a_n, & r<-1 \end{cases} \tag{3}$$

holds for all real sequences $(a_n)_{n\in\mathbb{N}}$, such that $a_n > 0$, $n \in \mathbb{N}$, and $\sum_{n=1}^{\infty} a_n < \infty$. Especially, for $r = -1$ the relation (3) reads

$$\sum_{n=1}^{\infty}\frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}} \leqslant 2\sum_{n=1}^{\infty}a_n. \tag{4}$$

Unfortunately, the authors in [19, 20] did not determine whether the inequalities (3) and (4) are sharp or not. The weighted integral Hardy's inequalities with negative indices were discussed by Prokhorov, [22], in terms of necessary and sufficient conditions of Muckenhoupt type (or, more precisely, Stepanov–Persson type) for boundedness of Hardy's integral operator (for more information see [23]).

Another interesting way to generalize the discrete Hardy's inequality is either to obtain its finite sections, that is, to restrict the infinite series on its both-hand sides to a finite number of terms, to find its refinements by decreasing the weight coefficients in the series on its right-hand side, or to combine these two approaches.

For the case $0<r<1$, such result is given in [5], where it was shown that if both series in (2) are restricted to a finite number of terms, N, then the best possible constant for (2) can be replaced with a smaller one, dependent on N. Namely, the inequality

$$\sum_{n=1}^{N}\left(\frac{1}{n}\sum_{k=1}^{n}a_k^r\right)^{1/r} \leqslant N^{1-1/r}\left(\sum_{k=1}^{N}k^{-r}\right)^{1/r}\sum_{n=1}^{N}\left(1-\frac{\sum_{k=1}^{n-1}k^{-r}}{\sum_{k=1}^{N}k^{-r}}\right)a_n \qquad (5)$$

holds for all sequences $(a_n)_{n\in\mathbb{N}}$ of non-negative real numbers and all $N\in\mathbb{N}$. Equality in (5) holds if and only if $N>1$ and $a_1=\ldots=a_N=0$, or $N=1$.

On the other hand, the case $r=-1$ was considered in [20], where by using the same technique as in [19] Nguyen et al. proved the following refinement of the finite section of (4):

$$\begin{aligned}\sum_{n=1}^{N}\frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}} &\leqslant 2\sum_{n=1}^{N}a_n\sum_{k=n}^{N}\frac{2n^2}{k(k+1)^2}\\ &\leqslant 2\sum_{n=1}^{N}\left(1-\frac{1}{3n+1}-4n^2\sum_{k=n}^{N}\frac{1}{k(k+1)^2(3k+1)(3k+4)}\right)a_n\\ &\leqslant 2\sum_{n=1}^{N}\left(1-\frac{1}{3n+1}\right)a_n. \end{aligned} \qquad (6)$$

Finally, for the limiting case $r=0$ Kaijser, Persson, and Öberg, [11], proved the inequalities

$$\begin{aligned}\sum_{n=1}^{N}\left(\prod_{k=1}^{n}a_k\right)^{1/n} &+\sum_{n=1}^{N}\frac{1}{n(n+1)}\sum_{k=1}^{\lfloor n/2\rfloor}\left(\sqrt{x^*_{n-k+1}}-\sqrt{x^*_k}\right)^2\\ &\leqslant\sum_{n=1}^{N}\left(1-\frac{n}{N+1}\right)\left(1+\frac{1}{n}\right)^n a_n< e\sum_{n=1}^{N}\left(1-\frac{n}{N+1}\right)a_n \end{aligned} \qquad (7)$$

where $(x_n^*)_{n=1}^N$ denotes the non-increasing rearrangement of the sequence $(x_n)_{n=1}^N$, $x_n=n(1+1/n)^n a_n$, $n\in\mathbb{N}$.

Our aim in this paper is to state and prove some new generalizations of the relations (2) and (3) for the case of arbitrary positive weights and to discuss the obtained results with respect to the power number $r\in\mathbb{R}$, $r\neq 0$. First, for $r<1$, $r\neq 0$, we generalize (2) and (3) by putting weights and truncating the range of summation on their both-hand sides to a finite number of terms. In other words, we obtain finite sections of the strengthened weighted discrete Hardy's inequality. Further, for $r<-1$ we show that the constant on the right-hand side of the obtained relation is smaller than the corresponding one from [19, 20]. To conclude, we prove that our result is sharp, that is, the mentioned constant is the best possible for the obtained inequality.

CONVENTIONS. Throughout this paper we set $\sum_S=0$, if $S=\emptyset$, that is, empty sums are taken to be equal zero.

2. Weighted discrete Hardy's inequality and its finite sections

Before presenting our idea, we need to introduce some notation and, for the reader's convenience, to recall some definitions and results that will be used in our proofs. Suppose $\underline{a} = (a_n)_{n\in\mathbb{N}}$ is a given sequence of non-negative and $\underline{w} = (w_n)_{n\in\mathbb{N}}$ of positive real numbers. Let $t \in \mathbb{R}$ and $n \in \mathbb{N}$. By $M_n^{[t]}(\underline{a};\underline{w})$ we denote the n-th weighted power mean of order t, with the weights $\underline{w}$, of the sequence $\underline{a}$, that is,

$$M_n^{[t]}(\underline{a};\underline{w}) = \begin{cases} \left(\dfrac{1}{W_n}\displaystyle\sum_{k=1}^{n} w_k a_k^r\right)^{1/r}, & r>0, \text{ or } r<0,\ a_k>0,\ k=1,\dots,n, \\ \left(\displaystyle\prod_{k=1}^{n} a_k^{w_k}\right)^{1/W_n}, & r=0, \\ 0, & \text{otherwise}, \end{cases}$$

where $W_n = \sum_{k=1}^n w_k$. We also set $W_0 = 0$ and $M^{[t]}(\underline{a};\underline{w}) = (M_n^{[t]}(\underline{a};\underline{w}))_{n\in\mathbb{N}}$.

Observe that in our notation $[(\sum_{k=1}^n w_k a_k^r)/W_n]^{1/r} = 0$ holds if $r < 0$ and there exists $k \in \{1,\dots,n\}$ such that $a_k = 0$. Moreover, for $t = -1$ we have the weighted harmonic means of the sequence $\underline{a}$, so, instead of $M_n^{[-1]}(\underline{a};\underline{w})$, we shall write $H_n(\underline{a};\underline{w})$.

Further, if all weights in $\underline{w}$ are equal, their value $w = w_1 = w_2 = \dots$ does not affect the value of the related means, that is $M_n^{[t]}(\underline{a};\underline{w}) = M_n^{[t]}(\underline{a};\underline{1})$, where $\underline{1} = (1,1,\dots)$. In this case we can without loss of generality assume that $w = 1$ and denote $M_n^{[t]}(\underline{a}) = M_n^{[t]}(\underline{a};\underline{1})$ and $H_n(\underline{a}) = H_n(\underline{a};\underline{1})$.

More information concerning discrete means and their properties one can find in e.g. [9] and [18]. Here we just state the so-called weighted mixed (t,s)-means inequality,

$$M_N^{[s]}\left(M^{[t]}(\underline{a};\underline{w});\underline{w}\right) \leqslant M_N^{[t]}\left(M^{[s]}(\underline{a};\underline{w});\underline{w}\right) \tag{8}$$

where $N \in \mathbb{N}$ is arbitrary, $t,s \in \mathbb{R}$, $t < s$, and the weights $\underline{w}$ are such that the sequence $(W_n/w_n)_{n\in\mathbb{N}}$ is increasing. For positive sequences $\underline{a}$, there is equality in (8) if and only if $a_1 = \dots = a_N$.

The relation (8) was proved in 1999, independently by Kedlaya, [13], and Tarnavas and Tarnavas, [24], although some of its special cases (e.g. the non-weighted mixed-means inequality and the mixed arithmetic-geometric mean inequality) were known much earlier. A detailed historical overview related to (8) can be found in [4], while for different proofs and generalizations of this inequality we refer to [1, 2, 12, 13, 16, 24].

Now, we can state our basic result. The starting point of its proof is an application of the mixed-means inequality (8).

THEOREM 1. *Let $\underline{a}$ be a sequence of non-negative and $\underline{w}$ of positive real numbers, such that the sequence $(W_n/w_n)_{n\in\mathbb{N}}$ is increasing. If $0 \neq r < 1$, then the inequality*

$$\sum_{n=1}^{N} w_n \left(\frac{1}{W_n} \sum_{k=1}^{n} w_k a_k^r \right)^{1/r} \leqslant W_N^{1-1/r} \left(\sum_{k=1}^{N} w_k W_k^{-r} \right)^{1/r} \sum_{n=1}^{N} \left(1 - \frac{\sum_{k=1}^{n-1} w_k W_k^{-r}}{\sum_{k=1}^{N} w_k W_k^{-r}} \right) w_n a_n \tag{9}$$

holds for all $N \in \mathbb{N}$, with equality if and only if $N > 1$ and $a_1 = \ldots = a_N = 0$, or $N = 1$. If $r > 1$, then (9) holds with the reversed sign of inequality, while for $r = 1$ it becomes an equality regardless $\underline{a}$, $\underline{w}$, and N.

Proof. First, let $0 < r < 1$. Rewriting (8) for $t = 1$, $s = 1/r > 1$, and the sequence $\underline{a}^r = (a_n^r)_{n\in\mathbb{N}}$, we have

$$\left[\frac{1}{W_N} \sum_{n=1}^{N} w_n \left(\frac{1}{W_n} \sum_{k=1}^{n} w_k a_k^r \right)^{1/r} \right]^r \leqslant \frac{1}{W_N} \sum_{n=1}^{N} w_n \left(\frac{1}{W_n} \sum_{k=1}^{n} w_k a_k \right)^r,$$

that is,

$$\sum_{n=1}^{N} w_n \left(\frac{1}{W_n} \sum_{k=1}^{n} w_k a_k^r \right)^{1/r} \leqslant W_N^{1-1/r} \left[\sum_{n=1}^{N} w_n \left(\frac{1}{W_n} \sum_{k=1}^{n} w_k a_k \right)^r \right]^{1/r} = W_N^{1-1/r} \left[\sum_{n=1}^{N} w_n W_n^{-r} \left(\sum_{k=1}^{n} w_k a_k \right)^r \right]^{1/r}. \tag{10}$$

Denoting $S_N = \sum_{n=1}^{N} w_n W_n^{-r}$, the second line of (10) can be written as

$$\begin{aligned} & W_N^{1-1/r} S_N^{1/r} \left[\frac{1}{S_N} \sum_{n=1}^{N} w_n W_n^{-r} \left(\sum_{k=1}^{n} w_k a_k \right)^r \right]^{1/r} \\ & \leqslant W_N^{1-1/r} S_N^{1/r-1} \sum_{n=1}^{N} w_n W_n^{-r} \sum_{k=1}^{n} w_k a_k = W_N^{1-1/r} S_N^{1/r-1} \sum_{k=1}^{N} w_k a_k \sum_{n=k}^{N} w_n W_n^{-r} \\ & = W_N^{1-1/r} \left(\sum_{n=1}^{N} w_n W_n^{-r} \right)^{1/r} \sum_{k=1}^{N} \left(1 - \frac{\sum_{n=1}^{k-1} w_n W_n^{-r}}{\sum_{n=1}^{N} w_n W_n^{-r}} \right) w_k a_k \end{aligned}$$

so (9) holds for $0 < r < 1$. The last sequence of inequalities follows from Jensen's inequality for the convex function $x \mapsto x^{1/r}$. Note that for $N > 1$ equality in (9) holds if and only if there is equality in Jensen's and in the corresponding mixed-means inequality, that is, only if

$$w_1 a_1 = w_1 a_1 + w_2 a_2 = \ldots = w_1 a_1 + w_2 a_2 + \ldots + w_N a_N \text{ and } a_1 = \ldots = a_N.$$

Hence, $a_1 = \ldots = a_N = 0$. The case $N = 1$ is trivial since on the both-hand sides of (9) we have $w_1 a_1$. Moreover, for $r = 1$ and arbitrary $\underline{a}$, $\underline{w}$, and N, the both-hand sides of (9) are equal to $\sum_{n=1}^{N} w_n / W_n \sum_{k=1}^{n} w_k a_k$.

Now, consider $r < 0$. Applying the mixed $(1/r, 1)$-means inequality to the sequence $\underline{a}^r$, we obtain

$$\left[\frac{1}{W_N}\sum_{n=1}^{N} w_n \left(\frac{1}{W_n}\sum_{k=1}^{n} w_k a_k^r\right)^{1/r}\right]^r \geqslant \frac{1}{W_N}\sum_{n=1}^{N} w_n \left(\frac{1}{W_n}\sum_{k=1}^{n} w_k a_k\right)^r,$$

so (9) follows as in the previous case (the function $x \mapsto x^{1/r}$ is decreasing and convex). The case $r > 1$ is similar ($x \mapsto x^{1/r}$ is increasing and concave).

REMARK 1. Observe that the inequality (5) is only a special case of the relation (9), obtained when $0 < r < 1$ and all the weights in $\underline{w}$ are equal (that is, in the non-weighted case). Therefore, Theorem 1 may be seen as a weighted generalization of [5, Theorem 1].

REMARK 2. Unfortunately, we do not know whether the constant involved in the right-hand side of (9), $W_N^{1-1/r}(\sum_{k=1}^{N} w_k W_k^{-r})^{1/r}$, is the best possible constant factor α_N for the relation

$$\sum_{n=1}^{N} w_n \left(\frac{1}{W_n}\sum_{k=1}^{n} w_k a_k^r\right)^{1/r} < \alpha_N \sum_{n=1}^{N}\left(1 - \frac{\sum_{k=1}^{n-1} w_k W_k^{-r}}{\sum_{k=1}^{N} w_k W_k^{-r}}\right) w_n a_n$$

in the case $0 \neq r < 1$, or not. However, Theorem 1 provides an explicit upper bound for α_N, dependent on N. The best possible constant γ_N for the finite section of the non-weighted discrete Hardy's inequality,

$$\sum_{n=1}^{N}\left(\frac{1}{2}\sum_{k=1}^{n} a_k^{1/2}\right)^2 < \gamma_N \sum_{n=1}^{N} a_n,$$

was investigated by H. S. Wilf, [25], but only for $r = 1/2$. He obtained the asymptotic behavior of γ_N as $N \to \infty$ by proving that

$$\gamma_N = 4 - \frac{16\pi^2}{(\ln N)^2} + O\left(\frac{\ln\ln N}{(\ln N)^3}\right).$$

For further details, see [17, Chapter IV].

The following results show the relation between Theorem 1 and finite sections of the weighted discrete Hardy's inequality for the power numbers $0 \neq r < 1$.

THEOREM 2. *Suppose $\underline{a}$ and $\underline{w}$ are as in Theorem 1 and $0 \neq r < 1$. Then the inequalities*

$$\begin{aligned}
&\sum_{n=1}^{N} w_n \left(\frac{1}{W_n}\sum_{k=1}^{n} w_k a_k^r\right)^{1/r} \\
&\quad \leqslant W_N^{1-1/r}\left(\sum_{k=1}^{N} w_k W_k^{-r}\right)^{1/r} \sum_{n=1}^{N}\left(1 - \frac{\sum_{k=1}^{n-1} w_k W_k^{-r}}{\sum_{k=1}^{N} w_k W_k^{-r}}\right) w_n a_n \\
&\quad \leqslant (1-r)^{-1/r}\sum_{n=1}^{N}\left(1 - \frac{\sum_{k=1}^{n-1} w_k W_k^{-r}}{\sum_{k=1}^{N} w_k W_k^{-r}}\right) w_n a_n \leqslant (1-r)^{-1/r}\sum_{n=1}^{N} w_n a_n \qquad (11)
\end{aligned}$$

hold for all $N \in \mathbb{N}$, *with equality if and only if* $a_1 = \ldots = a_N = 0$.

Proof. The first inequality in (11) is literally the relation (9) from Theorem 1, while the last one is obvious. To prove the second inequality, note that the estimate

$$W_N^{1-1/r}\left(\sum_{k=1}^{N} w_k W_k^{-r}\right)^{1/r} \leqslant W_N^{1-1/r}\left(\int_0^{W_N} x^{-r}dx\right)^{1/r} = (1-r)^{-1/r}$$

holds since $\sum_{k=1}^{N} w_k W_k^{-r}$ is the lower Darboux sum for $\int_0^{W_N} x^{-r}dx$ in the case when $0 < r < 1$, and the upper Darboux sum for the same integral when $r < 0$.

REMARK 3. Observe that the non-weighted case of (11) reads

$$\sum_{n=1}^{N}\left(\frac{1}{n}\sum_{k=1}^{n} a_k^r\right)^{1/r} \leqslant N^{1-1/r}\left(\sum_{k=1}^{N} k^{-r}\right)^{1/r}\sum_{n=1}^{N}\left(1-\frac{\sum_{k=1}^{n-1}k^{-r}}{\sum_{k=1}^{N}k^{-r}}\right)a_n$$
$$\leqslant (1-r)^{-1/r}\sum_{n=1}^{N}\left(1-\frac{\sum_{k=1}^{n-1}k^{-r}}{\sum_{k=1}^{N}k^{-r}}\right)a_n \leqslant (1-r)^{-1/r}\sum_{n=1}^{N} a_n \qquad (12)$$

so Theorem 2 may be regarded as a weighted generalization of [5, Remark 1]. Moreover, even in the non-weighted case we have a new generalization of the finite section of the strengthened discrete Hardy's inequality since the set of the power numbers r is expanded to include all negative real numbers.

REMARK 4. Especially, for $r = -1$ the relation (12) becomes

$$\sum_{n=1}^{N} H_n(\underline{a}) \leqslant 2\sum_{n=1}^{N}\frac{N}{N+1}\left[1-\frac{(n-1)n}{N(N+1)}\right]a_n \leqslant 2\sum_{n=1}^{N}\left[1-\frac{(n-1)n}{N(N+1)}\right]a_n \qquad (13)$$

so we obtained a new refinement of the finite section of (4). Note that the sequence of the weight factors involved in the right-hand side of (13) is decreasing. Since the corresponding sequence from (6) is increasing, the mentioned relations are incomparable.

REMARK 5. The limiting case $r = 0$ of (11) was considered previously, in [6], where a weighted generalization of (7) was obtained.

The following step is to replace finite sums in Theorem 2 with infinite series, that is, to take the limit of (11) as $N \to \infty$ to obtain a strengthened weighted discrete Hardy's inequality.

THEOREM 3. *Suppose* $0 \neq r < 1$ *and the sequences* $\underline{a}$ *and* $\underline{w}$ *from Theorem* 2 *are such that the series* $\sum_{n=1}^{\infty} w_n a_n$ *converges. Then the inequalities*

$$\sum_{n=1}^{\infty} w_n\left(\frac{1}{W_n}\sum_{k=1}^{n} w_k a_k^r\right)^{1/r} < (1-r)^{-1/r}\sum_{n=1}^{\infty}\left(1-\frac{\sum_{k=1}^{n-1}w_k W_k^{-r}}{\sum_{k=1}^{\infty}w_k W_k^{-r}}\right)w_n a_n$$
$$< (1-r)^{-1/r}\sum_{n=1}^{\infty} w_n a_n \qquad (14)$$

hold, unless $a_n = 0$ *for all* $n \in \mathbb{N}$.

Proof. Directly from Theorem 2 by taking $\lim_{N\to\infty}$ of the relation (11). Observe that $0 < \sum_{k=1}^{\infty} w_k W_k^{-r} \leqslant \infty$.

REMARK 6. The inequality

$$\sum_{n=1}^{\infty} w_n \left(\frac{1}{W_n} \sum_{k=1}^{n} w_k a_k^r \right)^{1/r} < (1-r)^{-1/r} \sum_{n=1}^{\infty} w_n a_n \tag{15}$$

is known as the weighted discrete Hardy's inequality (see [9]). Note that the first relation in (14) is a refinement, and (11) is a finite section of this relation for $r < 1$, $r \neq 0$.

REMARK 7. The identity

$$\sum_{n=1}^{N} \left(1 - \frac{\sum_{k=1}^{n-1} w_k W_k^{-r}}{\sum_{k=1}^{N} w_k W_k^{-r}} \right) w_n a_n = \frac{1}{\sum_{k=1}^{N} w_k W_k^{-r}} \sum_{k=1}^{N} w_k W_k^{-r} \sum_{n=1}^{k} w_n a_n$$

means that the usual partial sum $\sum_{n=1}^{N} w_n a_n$ of the series on the right-hand side of the weighted Hardy's inequality (15) has been in (14) replaced with the corresponding, but strictly smaller "weighted Césaro sum", that is, the partial sums of the original sequence $(w_n a_n)_{n\in\mathbb{N}}$ have been arithmetically averaged with the weights $(w_n W_n^{-r})_{n\in\mathbb{N}}$.

REMARK 8. Note that the finite sections and refinements of the weighted Hardy's inequality were obtained only for the case $0 \neq r < 1$. Now, we consider $r \geqslant 1$. Let the sequence $\underline{w}$ from Theorem 3 be such that the series $\sum_{n=1}^{\infty} w_n/W_n$ diverges. It is not hard to see that the series on the left-hand side of (14) is divergent, unless $a_n = 0$ for all $n \in \mathbb{N}$. To prove this, assume that $m \in \mathbb{N}$ is such that $a_1 = \ldots = a_{m-1} = 0$ and $a_m > 0$ (if $m = 1$, then $a_1 > 0$). Then the monotonicity property of the means yields

$$\begin{aligned} \sum_{n=1}^{\infty} w_n \left(\frac{1}{W_n} \sum_{k=1}^{n} w_k a_k^r \right)^{1/r} &\geqslant \sum_{n=1}^{\infty} w_n \left(\frac{1}{W_n} \sum_{k=1}^{n} w_k a_k \right) \\ &= \sum_{n=m}^{\infty} \frac{w_n}{W_n} \sum_{k=m}^{n} w_k a_k \geqslant w_m a_m \sum_{n=m}^{\infty} \frac{w_n}{W_n} = \infty \end{aligned}$$

so the proof is completed.

To conclude this section, we compare our results to Nguyens relation (3). In particular, in the non-weighted setting (14) reads

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} a_k^r \right)^{1/r} &< (1-r)^{-1/r} \sum_{n=1}^{\infty} \left(1 - \frac{\sum_{k=1}^{n-1} k^{-r}}{\sum_{k=1}^{\infty} k^{-r}} \right) a_n \\ &< (1-r)^{-1/r} \sum_{n=1}^{\infty} a_n \end{aligned} \tag{16}$$

so we obtained a weighted generalization of (3). Observe that for $-1 \leqslant r < 1$, $r \neq 0$, the relations (3) and (16) have identical constants involved in their right-hand sides, while for $r < -1$ these constants are different. To determine which one of them is smaller, we consider the function $f : [1, \infty\rangle \to \mathbb{R}$,

$$f(x) = \ln x + \left(1 + \frac{1}{x}\right) \ln \frac{2}{1+x}.$$

Since $f(1) = 0$ and

$$f'(x) = \frac{1}{x^2} \ln \frac{1+x}{2} > 0, \quad x > 1,$$

the function f is strictly increasing. Hence,

$$\ln x + \left(1 + \frac{1}{x}\right) \ln \frac{2}{1+x} > 0, \quad x > 1,$$

so straightforward computations yield

$$\frac{x}{1+x} 2^{1+\frac{1}{x}} > (1+x)^{\frac{1}{x}}, \quad x > 1.$$

Substituting $x = -r$, for $r < -1$ we have

$$\frac{r}{r-1} 2^{\frac{r-1}{r}} > (1-r)^{-\frac{1}{r}}.$$

Thus, our constant is strictly smaller than the constant from (3).

3. The best possible constants

Finally, we show that the result from Theorem 3 is sharp, that is, the constant $(1-r)^{-1/r}$ is the best possible for (14). To prove this, we make use of the following technical lemma.

LEMMA 1. *Let $(b_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence of non-negative real numbers and $(c_n)_{n\in\mathbb{N}}$ a sequence of positive real numbers, such that $\lim_{n\to\infty} b_n = 1$ and $\sum_{n=1}^{\infty} c_n = \infty$. Then for each $\varepsilon \in \langle 0, 1-b_1\rangle$ there exists $N_0 \in \mathbb{N}$, such that the inequality*

$$\sum_{n=1}^{N} b_n c_n > (1-\varepsilon) \sum_{n=1}^{N} c_n$$

holds for all $N \in \mathbb{N}$, $N \geqslant N_0$.

Proof. Let $\varepsilon \in \langle 0, 1-b_1\rangle$ be arbitrary. Since $b_1 < 1-\varepsilon$ and $b_n \nearrow 1$, there exists a number $n_0 = \max\{n \in \mathbb{N} : b_n \leqslant 1-\varepsilon\}$. More precisely, we have

$$1 - b_1 - \varepsilon > \ldots > 1 - b_{n_0} - \varepsilon \geqslant 0, \;\; 1 - \varepsilon < b_{n_0+1} < \ldots < 1, \;\; \sum_{n=1}^{n_0} (1 - b_n - \varepsilon) c_n > 0.$$

Moreover, if $\delta = (1 + \varepsilon - b_{n_0+1})/2$, then $\varepsilon - \delta > 0$ and

$$1 - \delta < b_{n_0+1} < \ldots < 1. \tag{17}$$

Finally, since the series $\sum_{n=n_0+1}^{\infty} c_n$ is divergent, there exists $N_0 \in \mathbb{N}$ such that

$$\sum_{n=n_0+1}^{N} c_n > \frac{1}{\varepsilon-\delta}\sum_{n=1}^{n_0}(1-b_n-\varepsilon)c_n, \quad N \geqslant N_0,$$

or, equivalently,

$$(1-\varepsilon)\sum_{n=1}^{N} c_n < \sum_{n=1}^{n_0} b_n c_n + \sum_{n=n_0+1}^{N}(1-\delta)c_n, \quad N \geqslant N_0.$$

Using (17), the right-hand side of the previous inequality is further less than

$$\sum_{n=1}^{n_0} b_n c_n + \sum_{n=n_0+1}^{N} b_n c_n = \sum_{n=1}^{N} b_n c_n,$$

so the proof is completed.

Now, we can discuss the best possible constants for (14). Consider the case $0<r<1$ first.

THEOREM 4. *Let $0<r<1$, $\underline{a}$ be a sequence of non-negative real numbers, and $\underline{w}$ a decreasing sequence of positive real numbers. If the sequence $(w_n/W_n)_{n\in\mathbb{N}}$ is decreasing, the series $\sum_{n=1}^{\infty} w_n/W_n$ diverges, and $\lim_{n\to\infty} W_n=\infty$, then the constant $(1-r)^{-1/r}$ is the best possible for* (14).

Proof. Suppose $N \in \mathbb{N}$ is fixed and the sequence $\underline{a}$ is defined by

$$a_n = \begin{cases} \dfrac{1}{W_n}, & n \leqslant N, \\ 0, & n > N. \end{cases} \tag{18}$$

Then on the right-hand side of (14) we have

$$\begin{aligned} R &= (1-r)^{-1/r}\sum_{n=1}^{\infty}\left(1-\frac{\sum_{k=1}^{n-1} w_k W_k^{-r}}{\sum_{k=1}^{\infty} w_k W_k^{-r}}\right) w_n a_n \\ &< (1-r)^{-1/r}\sum_{n=1}^{\infty} w_n a_n = (1-r)^{-1/r}\sum_{n=1}^{N}\frac{w_n}{W_n} \end{aligned} \tag{19}$$

while the left-hand side of (14) becomes

$$L=\sum_{n=1}^{\infty} w_n M_n^{[r]}(\underline{a};\underline{w}) \geqslant \sum_{n=1}^{N} w_n M_n^{[r]}(\underline{a};\underline{w})$$

$$\begin{aligned} &= \sum_{n=1}^{N} w_n \left(\frac{1}{W_n} \sum_{k=1}^{n} w_k W_k^{-r} \right)^{1/r} \\ &> \sum_{n=1}^{N} w_n \left[\frac{1}{W_n} (1-r)^{-1} \left(W_n^{1-r} - W_1^{1-r} \right) \right]^{1/r} \\ &= (1-r)^{-1/r} \sum_{n=1}^{N} \frac{w_n}{W_n} \left[1 - \left(\frac{W_1}{W_n} \right)^{1-r} \right]^{1/r}. \end{aligned} \tag{20}$$

The second inequality in (20) was obtained from

$$\begin{aligned} \sum_{k=1}^{n} w_k W_k^{-r} &\geqslant \sum_{k=1}^{n} w_{k+1} W_k^{-r} \geqslant \int_{W_1}^{W_{n+1}} x^{-r} dx \\ &> \int_{W_1}^{W_n} x^{-r} dx = (1-r)^{-1} \left(W_n^{1-r} - W_1^{1-r} \right). \end{aligned}$$

Denote

$$b_n = \left[1 - \left(\frac{W_1}{W_n} \right)^{1-r} \right]^{1/r} \text{ and } c_n = \frac{w_n}{W_n}, \; n \in \mathbb{N}.$$

Since $b_1 = 0$ and $b_n \nearrow 1$, the sequences $(b_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ fulfill the conditions of Lemma 1. Hence, for each $\varepsilon \in \langle 0, 1 \rangle$ there exists $N_0 \in \mathbb{N}$ such that

$$\sum_{n=1}^{N} \frac{w_n}{W_n} \left[1 - \left(\frac{W_1}{W_n} \right)^{1-r} \right]^{1/r} > (1-\varepsilon) \sum_{n=1}^{N} \frac{w_n}{W_n} \tag{21}$$

holds for all $N \in \mathbb{N}$, $N \geqslant N_0$. Finally, combining (14), (19), (20), and (21) we obtain

$$\begin{aligned} (1-r)^{-1/r} \sum_{n=1}^{N} \frac{w_n}{W_n} &= (1-r)^{-1/r} \sum_{n=1}^{\infty} w_n a_n \\ &> R > L > (1-\varepsilon)(1-r)^{-1/r} \sum_{n=1}^{N} \frac{w_n}{W_n} \end{aligned} \tag{22}$$

so $(1-r)^{-1/r}$ cannot be replaced with any smaller constant.

REMARK 9. Theorem 5 obviously covers the non-weighted case $\underline{w} = \underline{1}$. Another such weights $\underline{w}$ are $w_n = 1/n$, $n \in \mathbb{N}$.

In the case $r < 0$, the conditions on the weights $\underline{w}$ are slightly different from those in Theorem 4.

THEOREM 5. *Suppose $r < 0$, $\underline{a}$ is a sequence of non-negative real numbers, and $\underline{w}$ an increasing sequence of positive real numbers. If the series $\sum_{n=1}^{\infty} w_n/W_n$ is divergent, the sequences $(W_n/w_n)_{n\in\mathbb{N}}$ and $(W_n/W_{n+1})_{n\in\mathbb{N}}$ are increasing, and $\lim_{n\to\infty} W_n/W_{n+1} = 1$, then the constant $(1-r)^{-1/r}$ is the best possible for* (14).

Proof. For an arbitrary $N \in \mathbb{N}$ let the sequence $\underline{a}$ be defined by (18). Then $M_n^{[r]}(\underline{a};\underline{w}) = 0$, $n > N$, so the left-hand side of (14) reads

$$\begin{aligned} L &= \sum_{n=1}^{\infty} w_n M_n^{[r]}(\underline{a};\underline{w}) = \sum_{n=1}^{N} w_n M_n^{[r]}(\underline{a};\underline{w}) = \sum_{n=1}^{N} w_n \left(\frac{1}{W_n} \sum_{k=1}^{n} w_k W_k^{-r} \right)^{1/r} \\ &> \sum_{n=1}^{N} w_n \left[\frac{1}{W_n} (1-r)^{-1} W_{n+1}^{1-r} \right]^{1/r} = (1-r)^{-1/r} \sum_{n=1}^{N} \frac{w_n}{W_n} \left(\frac{W_n}{W_{n+1}} \right)^{1-1/r}. \end{aligned} \tag{23}$$

Note that the inequality in (23) follows from

$$\sum_{k=1}^{n} w_k W_k^{-r} \leqslant \sum_{k=1}^{n} w_{k+1} W_k^{-r} \leqslant \int_{W_1}^{W_{n+1}} x^{-r} dx < \int_{0}^{W_{n+1}} x^{-r} dx = (1-r)^{-1} W_{n+1}^{1-r}.$$

Applying Lemma 1 to $b_n = (W_n/W_{n+1})^{1-1/r}$ and $c_n = w_n/W_n$, we have that for each $\varepsilon \in \langle 0, w_1/W_2 \rangle$ there exists $N_0 \in \mathbb{N}$, such that

$$\sum_{n=1}^{N} \frac{w_n}{W_n} \left(\frac{W_n}{W_{n+1}} \right)^{1-1/r} > (1-\varepsilon) \sum_{n=1}^{N} \frac{w_n}{W_n}, \quad N \geqslant N_0. \tag{24}$$

Therefore, (14), (19), (23) and (24) again imply (22), so $(1-r)^{-1/r}$ is the best possible constant for (14).

REMARK 10. Observe that Theorem 5 covers the non-weighted case $\underline{w} = \underline{1}$. Another example of such weights is $w_n = n$, $n \in \mathbb{N}$, since $w_n/W_n = 2/(n+1) \searrow 0$, $W_n/W_{n+1} = n/(n+2) \nearrow 1$ and the series $\sum_{n=1}^{\infty} 2/(n+1)$ diverges.

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