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GENERAL EULER–BOOLE'S AND DUAL EULER–BOOLE'S FORMULAE

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Dedicated to the memory of prof. Mladen Alić

(*communicated by D. Hinton*)

Abstract. The aim of this paper is to derive general Euler-Boole's and dual Euler-Boole's formulae. More precisely, we derive formulae of Boole type where the integral is approximated not only with the values of the function in certain points but also with values of its derivatives up to $(2n-1)$-th order in end points of the interval. Our method produces formulae of arbitrary degree of exactness. Dual Euler-Boole's formulae are derived by analogy with Simpson's and dual Simpson's rule, and Simpson's $3/8$ and Maclaurin's rule. Finally, by analogy with Bullen-Simpson's and Bullen-Simpson's $3/8$ inequality, general Bullen-Boole's inequality for a class of $(2k)$-convex functions is derived.

1. Introduction

Extended Euler formulae, obtained in [4], extend the well known formula for the expansion of an arbitrary function in Bernoulli polynomials (cf. [10]). Namely, for $f:[0,1]\to\mathbf{R}$ such that $f^{(n-1)}$ is continuous of bounded variation on $[0,1]$, for some $n\geqslant 1$, for every $x\in[0,1]$ we have

$$\int_0^1 f(t)\mathrm{d}t = f(x) - T_n(x) + \frac{1}{n!}\int_0^1 B_n^*(x-t)\,\mathrm{d}f^{(n-1)}(t) \tag{1.1}$$

$$\int_0^1 f(t)\mathrm{d}t = f(x) - T_{n-1}(x) + \frac{1}{n!}\int_0^1 \left[B_n^*(x-t) - B_n(x)\right]\mathrm{d}f^{(n-1)}(t) \tag{1.2}$$

where $T_0(x)=0$ and for $1\leqslant m\leqslant n$

$$T_m(x) = \sum_{k=1}^{m} \frac{B_k(x)}{k!}\left[f^{(k-1)}(1) - f^{(k-1)}(0)\right]. \tag{1.3}$$

Using these identities and their appropriate convex combinations one can produce quadrature formulae of arbitrary high degree of exactness. For example, for $f:[0,1]\to$

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$\mathbf{R}$ such that $f^{(n-1)}$ is continuous of bounded variation on $[0,1]$, for some $n \geqslant 1$, Euler-Boole's formulae were derived in [13]:

$$\begin{aligned}\int_0^1 f(t)\mathrm{d}t &= \frac{1}{90}[7f(0)+32f(1/4)+12f(1/2)+32f(3/4)+7f(1)] \\ &- \breve{T}_n(f)+\frac{1}{90n!}\int_0^1 \breve{G}_n(t)\mathrm{d}f^{(n-1)}(t) \qquad (1.4)\\ \int_0^1 f(t)\mathrm{d}t &= \frac{1}{90}[7f(0)+32f(1/4)+12f(1/2)+32f(3/4)+7f(1)] \\ &- \breve{T}_{n-1}(f)+\frac{1}{90n!}\int_0^1 [\breve{G}_n(t)-\breve{B}_n]\mathrm{d}f^{(n-1)}(t), \qquad (1.5)\end{aligned}$$

where $\breve{T}_0(f)=0$ and for $1 \leqslant m \leqslant n$

$$\breve{T}_m(f)=\sum_{k=1}^{m}\frac{\breve{B}_k}{k!}\left[f^{(k-1)}(1)-f^{(k-1)}(0)\right], \tag{1.6}$$

and, for $k \geqslant 1$ and $t \in \mathbf{R}$,

$$\begin{aligned}\breve{B}_k &= 7B_k(0)+32B_k(1/4)+12B_k(1/2)+32B_k(3/4)+7B_k(1)\\ \breve{G}_k(t) &= 14B_k^*(1-t)+32B_k^*(1/4-t)+12B_k^*(1/2-t)+32B_k^*(3/4-t).\end{aligned}$$

Applying the mean value theorem for integrals (cf. Theorem 2), we can easily rewrite the remainder in (1.5) and thus obtain classical Boole's formula (cf. [3]). Notice that $\breve{B}_{2k-1}=0$, for $k \geqslant 1$ but also $\breve{B}_2=\breve{B}_4=0$.

The aim of this paper is to derive general Euler-Boole's and dual Euler-Boole's formulae. More precisely, we'll derive formulae of Boole type such that $\tilde{B}_{2n-2} = \tilde{B}_{2n}=0$ ($\tilde{B}_k$ are defined by (2.3)), achieving an arbitrary degree of exactness. Dual Euler-Boole's formulae will be derived by analogy with Simpson's and dual Simpson's rule, and Simpson's 3/8 and Maclaurin's rule (cf. [3]).

For details on Bernoulli polynomials and Bernoulli numbers see [10] or [1].

2. General Euler-Boole's formulae

Let $f:[0,1]\to\mathbf{R}$ be such that $f^{(2n)}$ is continuous of bounded variation on $[0,1]$ for some $n \geqslant 1$. Put $x=0,\ 1/4,\ 1/2,\ 3/4,\ 1$ in (1.1), multiply by $\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4=\lambda_2,\ \lambda_5=\lambda_1$, respectively, with $2\lambda_1+2\lambda_2+\lambda_3=1$. We obtain:

$$\int_0^1 f(t)dt - D(0,1)+T_{2n}(f)=\frac{1}{(2n+1)!}\int_0^1 G_{2n+1}(t)df^{(2n)}(t), \tag{2.1}$$

where

$$D(0,1) = \lambda_1 \cdot [f(0)+f(1)] + \lambda_2 \cdot [f(1/4)+f(3/4)] + \lambda_3 \cdot f(1/2)$$

$$T_m(f) = \sum_{k=1}^{m} \frac{\tilde{B}_k}{k!}[f^{(k-1)}(1) - f^{(k-1)}(0)], \qquad 1 \leqslant m \leqslant 2n \tag{2.2}$$

$$\tilde{B}_k = \lambda_1 \cdot [B_k(0) + B_k(1)] + \lambda_2 \cdot [B_k(1/4) + B_k(3/4)] + \lambda_3 \cdot B_k(1/2), \tag{2.3}$$

$$G_k(t) = 2\lambda_1 \cdot B_k^*(1-t) + \lambda_2 \cdot [B_k^*(1/4-t) + B_k^*(3/4-t)] + \lambda_3 \cdot B_k^*(1/2-t). \tag{2.4}$$

for $k \geqslant 1$ and $t \in \mathbf{R}$.

Since $B_{2k-1}(1-t) = -B_{2k-1}(t)$ for $k \geqslant 1$, we have $\tilde{B}_{2k-1} = 0$. Thus, applying (1.2) instead of (1.1), we would get identity (2.1) again.

Analogously, assuming we have $f : [0,1] \to \mathbf{R}$ such that $f^{(2n-1)}$ is continuous of bounded variation on $[0,1]$ for some $n \geqslant 1$ from (1.1) we would get:

$$\int_0^1 f(t)dt - D(0,1) + T_{2n}(f) = \frac{1}{(2n)!}\int_0^1 G_{2n}(t)df^{(2n-1)}(t), \tag{2.5}$$

and if $f^{(2n+1)}$ was continuous of bounded variation on $[0,1]$, from (1.2) we would get:

$$\int_0^1 f(t)dt - D(0,1) + T_{2n}(f) = \frac{1}{(2n+2)!}\int_0^1 F_{2n+2}(t)df^{(2n+1)}(t), \tag{2.6}$$

where $F_k(t) = G_k(t) - \tilde{B}_k, \quad k \geqslant 1$. We shall call formulae (2.1), (2.5) and (2.6) general Euler-Boole's formulae.

Next, for $n \geqslant 2$, consider the following linear system:

$$2\lambda_1 + 2\lambda_2 + \lambda_3 = 1, \quad \tilde{B}_{2n-2} = 0, \quad \tilde{B}_{2n} = 0.$$

Using the fact that $B_{2k}(1-t) = B_{2k}(t)$, $B_{2k}(1/2) = -(1-2^{1-2k})B_{2k}$ and $B_{2k}(1/4) = 2^{-2k}B_{2k}(1/2)$, one can easily find the solution of this system:

$$\lambda_1 = \frac{16 - 10 \cdot 4^n + 4^{2n}}{8(4^n-1)(4^n-4)}, \quad \lambda_2 = \frac{4^{2n-1}}{(4^n-1)(4^n-4)}, \quad \lambda_3 = \frac{(4^n-10)\cdot 4^{n-1}}{(4^n-1)(4^n-4)}.$$

These are the coefficients we will work with. Interval $[0,1]$ is used for simplicity and involves no loss in generality.

What follows is a lemma that is a key step for all the results in this paper. To prove it, we will need an analogue of the Multiplication Theorem, stated for periodic functions B_n^*. The Multiplication Theorem for Bernoulli polynomials B_n states (cf. [1]):

$$B_n(mx) = m^{n-1}\sum_{k=0}^{m-1} B_n\left(x + \frac{k}{m}\right), \quad n \geqslant 0, \ \ m \geqslant 1 \tag{2.7}$$

That (2.7) is true for $B_n^*(x)$ and $x \in [0, 1/m)$ is obvious. For $x \in [j/m, (j+1)/m)$, $1 \leqslant j \leqslant m-1$:

$$B_n^*(mx) = B_n^*(m(x - j/m)) = m^{n-1}\sum_{k=0}^{m-1} B_n^*\left(x + \frac{k-j}{m}\right) = m^{n-1}\sum_{k=0}^{m-1} B_n^*\left(x + \frac{k}{m}\right),$$

so the statement is true again. Thus, we have

$$B_n^*(mx) = m^{n-1}\sum_{k=0}^{m-1} B_n^*\left(x+\frac{k}{m}\right), \quad n \geqslant 0, \ \ m \geqslant 1 \tag{2.8}$$

LEMMA 1. *For* $n \geqslant 2$, $G_{2n+1}(t)$ *has no zeros in the interval* $(0,1/2)$. *The sign of the function is determined by*

$$(-1)^n G_{2n+1}(t) > 0, \qquad 0 < t < 1/2.$$

Proof. Applying (2.8), we can rewrite $G_{2n+1}(t)$ as

$$G_{2n+1}(t) = \frac{-1}{4(4^n-1)(4^n-4)}[B_{2n+1}^*(4t) - 10B_{2n+1}^*(2t) + 16B_{2n+1}^*(t)]. \tag{2.9}$$

There cannot exist $t \in (1/4, 3/8)$ such that $G_{2n+1}(t) = 0$ because $B_{2n+1}^*(t)$, $-B_{2n+1}^*(2t)$ and $B_{2n+1}^*(4t)$ have the same sign on $(1/4, 3/8)$.

Let us assume there exists $t_1 \in (0,1/4]$ such that $G_{2n+1}(t_1) = 0$. Since $G_{2n+1}(0) = 0$, we conclude there must exist $t_2 \in (0,t_1)$ such that $G_{2n+1}'(t_2) = 0$. So, we must have

$$B_{2n}^*(4t_2) - 5B_{2n}^*(2t_2) + 4B_{2n}^*(t_2) = 0,$$

which is equivalent to

$$\frac{B_{2n}^*(4t_2) - B_{2n}^*(2t_2)}{B_{2n}^*(2t_2) - B_{2n}^*(t_2)} = 4$$

since for $z \in (0,1/2)$, $B_{2n}^*(2z) = B_{2n}^*(z)$ iff $z = 1/3$ and that cannot be the case. Define functions

$$f(x) = B_{2n}^*(2xt_2), \quad g(x) = B_{2n}^*(xt_2), \quad x \in [1,2].$$

Note that $g'(x) \neq 0$ for $x \in [1,2]$, since $0 < xt_2 < 1/2$. From Cauchy's mean value theorem we know there exists $x_1 \in (1,2)$ such that

$$\frac{B_{2n}^*(4t_2) - B_{2n}^*(2t_2)}{B_{2n}^*(2t_2) - B_{2n}^*(t_2)} = \frac{f'(x_1)}{g'(x_1)} = 4,$$

and from there

$$\frac{B_{2n-1}^*(2x_1t_2)}{B_{2n-1}^*(x_1t_2)} = 2, \quad \text{for some} \quad 0 < x_1t_2 < 1/2. \tag{2.10}$$

Next, define a function

$$h(t) = 2B_{2n-1}^*(t) - B_{2n-1}^*(2t).$$

From (2.10) it follows that $h(x_1t_2) = 0$. To obtain a contradiction, we will prove $h(t) \neq 0$ for $t \in (0,1/2)$. First, assume $t \in (0,1/4]$. Suppose there exists $t_3 \in (0,1/4]$ such that $h(t_3) = 0$. Since $h(0) = 0$, we conclude there must exist $t_4 \in (0,t_3)$ such that $h'(t_4) = 0$. But from there it would follow that $B_{2n-2}^*(t_4) = B_{2n-2}^*(2t_4)$ which cannot be the case. When $t \in (1/4, 1/2)$, $B^*_{2n-1}(t)$ and $-B^*_{2n-1}(2t)$ have the same sign, so our statement follows easily.

Finally, consider the case $t \in [3/8, 1/2)$. We have

$$B^*_{2n+1}(4t) - 10B^*_{2n+1}(2t) + 16B^*_{2n+1}(t) = k(t) - 8B^*_{2n+1}(2t) + 16B^*_{2n+1}(t),$$

where

$$k(t) = B^*_{2n+1}(4t) - 2B^*_{2n+1}(2t) = 2B^*_{2n+1}(1-2t) - B^*_{2n+1}[2(1-2t)].$$

It follows from the previous proof for the function $h(t)$, that $k(t)$ doesn't have zeros on $[3/8, 1/2)$. Furthermore, $k(t)$, $-B^*_{2n+1}(2t)$ and $B^*_{2n+1}(t)$ have the same sign on this interval. So, in conclusion, the function $G_{2n+1}(t)$ doesn't have zeros on $(0, 1/2)$.

It is clear now that $G_{2n+1}(t)$ doesn't change sign on $(0, 1/2)$. To determine the sign, it is enough to calculate the value of that function in any point from the interval $(0, 1/2)$, e.g. $t = 1/4$. □

The proof of the previous Lemma, compared to the proof of Lemma 2 in [13], is much more difficult, since we cannot reduce it to the case where we can explicitly calculate zeros of the function.

COROLLARY 1. *For $n \geqslant 2$, $(-1)^{n+1}F_{2n+2}(t)$ is strictly increasing on the interval $(0, 1/2)$ and strictly decreasing on the interval $(1/2, 1)$ and*

$$\max_{t\in[0,1]} |F_{2n+2}(t)| = \frac{2(4-4^{-n})}{(4^n-1)(4^n-4)}|B_{2n+2}|.$$

Also,

$$\int_0^1 |G_{2n+1}(t)|\,\mathrm{d}t = \int_0^1 |F_{2n+1}(t)|\,\mathrm{d}t = \frac{2(4-4^{-n})}{(n+1)(4^n-1)(4^n-4)}|B_{2n+2}|,$$
$$\int_0^1 |F_{2n+2}(t)|\,\mathrm{d}t = |\tilde{B}_{2n+2}| = \frac{45\cdot|B_{2n+2}|}{16(4^n-1)(4^n-4)}.$$

Proof. Using Lemma 1, from $F'_{2n+2}(t) = -(2n+2)G_{2n+1}(t)$ and $F_{2n+2}(t) = F_{2n+2}(1-t)$, we conclude that $(-1)^{n+1}F_{2n+2}(t)$ is strictly increasing on $(0, 1/2)$ and strictly decreasing on $(1/2, 1)$. Moreover, we have $F_{2n+2}(0) = F_{2n+2}(1) = 0$, so $\max_{t\in[0,1]} |F_{2n+2}(t)| = |F_{2n+2}(1/2)|$.

Further, using Lemma 1 again and the fact that $G'_{2n+2}(t) = F'_{2n+2}(t)$, we get

$$\int_0^1 |G_{2n+1}(t)|\,\mathrm{d}t \;=\; 2\left|\int_0^{1/2} G_{2n+1}(t)\mathrm{d}t\right| = \frac{1}{n+1}\left|F_{2n+2}\left(\frac{1}{2}\right)\right|.$$

Since $(-1)^{n+1}F_{2n+2}(t) > 0$ for each $t \in (0,1)$, we have

$$\int_0^1 |F_{2n+2}(t)|\,\mathrm{d}t = \left|\int_0^1 G_{2n+2}(t)\mathrm{d}t - \tilde{B}_{2n+2}\right| = |\tilde{B}_{2n+2}|.$$

□

THEOREM 1. *Assume* (p,q) *is a pair of conjugate exponents, that is* $1 \leqslant p,\ q \leqslant \infty,\ 1/p+1/q=1$. *If* $|f^{(2n)}|^p : [0,1] \to \mathbf{R}$ *is* R*-integrable for some* $n \geqslant 2$, *then we have*

$$\left| \int_0^1 f(t)\mathrm{d}t - D(0,1) + T_{2n}(f) \right| \leqslant K(2n,q) \cdot \|f^{(2n)}\|_p, \tag{2.11}$$

if $|f^{(2n+1)}|^p : [0,1] \to \mathbf{R}$ *is* R*-integrable for some* $n \geqslant 2$, *then we have*

$$\left| \int_0^1 f(t)\mathrm{d}t - D(0,1) + T_{2n}(f) \right| \leqslant K(2n+1,q) \cdot \|f^{(2n+1)}\|_p, \tag{2.12}$$

if $|f^{(2n+2)}|^p : [0,1] \to \mathbf{R}$ *is* R*-integrable for some* $n \geqslant 2$, *then we have*

$$\left| \int_0^1 f(t)\mathrm{d}t - D(0,1) + T_{2n}(f) \right| \leqslant K^*(2n+2,q) \cdot \|f^{(2n+2)}\|_p, \tag{2.13}$$

where

$$K(m,q) = \frac{1}{m!} \left[\int_0^1 |G_m(t)|^q \, \mathrm{d}t \right]^{\frac{1}{q}} \quad \textit{and} \quad K^*(m,q) = \frac{1}{m!} \left[\int_0^1 |F_m(t)|^q \, \mathrm{d}t \right]^{\frac{1}{q}}.$$

These inequalities are sharp for $1 < p \leqslant \infty$ *and best possible for* $p=1$.

Proof. Inequalities (2.11), (2.12) and (2.13) follow immediately after applying Hölder's inequality to the remainders in formulae (2.5), (2.1) and (2.6). To prove inequalities are sharp, put

$$\begin{aligned}
&f^{(m)}(t) = \operatorname{sgn} G_m(t) \cdot |G_m(t)|^{1/(p-1)} \quad \text{for} \quad 1<p<\infty \quad \text{and} \\
&f^{(m)}(t) = \operatorname{sgn} G_m(t) \quad \text{for} \quad p=\infty \quad \text{in} \quad (2.11) \quad \text{and} \quad (2.12), \\
&f^{(m)}(t) = \operatorname{sgn} F_m(t) \cdot |F_m(t)|^{1/(p-1)} \quad \text{for} \quad 1<p<\infty \quad \text{and} \\
&f^{(m)}(t) = \operatorname{sgn} F_m(t) \quad \text{for} \quad p=\infty \quad \text{in} \quad (2.13).
\end{aligned}$$

The proof that these inequalities are best possible for $p=1$ is the same as the proof of Theorem 2 in [14]. □

REMARK 1. For $p=\infty$, applying Corollary 1, (2.12) and (2.13) turn to:

$$\left| \int_0^1 f(t)\mathrm{d}t - D(0,1) + T_{2n}(f) \right| \leqslant \frac{|B_{2n+2}|}{(2n+2)!} \cdot \frac{4(4-4^{-n})}{(4^n-1)(4^n-4)} \cdot \|f^{(2n+1)}\|_\infty,$$

$$\left| \int_0^1 f(t)\mathrm{d}t - D(0,1) + T_{2n}(f) \right| \leqslant \frac{|B_{2n+2}|}{(2n+2)!} \cdot \frac{45}{16(4^n-1)(4^n-4)} \cdot \|f^{(2n+2)}\|_\infty.$$

For $p=1$, applying Corollary 1 again, (2.13) becomes:

$$\left| \int_0^1 f(t)\mathrm{d}t - D(0,1) + T_{2n}(f) \right| \leqslant \frac{|B_{2n+2}|}{(2n+2)!} \cdot \frac{2(4-4^{-n})}{(4^n-1)(4^n-4)} \cdot \|f^{(2n+2)}\|_1.$$

Using integration by parts and Lemma 1 from [4], for $p=2$ we obtain

$$\left|\int_0^1 f(t)\mathrm{d}t - D(0,1) + T_{2n}(f)\right|$$
$$\leqslant \frac{\|f^{(2n)}\|_2}{(4^n-1)(4^n-4)}\left[\frac{|B_{4n}|}{(4n)!}(2^{3-4n}-25\cdot 2^{1-2n}+42)\right]^{1/2},$$
$$\left|\int_0^1 f(t)\mathrm{d}t - D(0,1) + T_{2n}(f)\right|$$
$$\leqslant \frac{\|f^{(2n+1)}\|_2}{4(4^n-1)(4^n-4)}\left[\frac{B_{4n+2}}{(4n+2)!}\left(2^{3-4n}-85\cdot 2^{1-2n}+357\right)\right]^{1/2},$$
$$\left|\int_0^1 f(t)\mathrm{d}t - D(0,1) + T_{2n}(f)\right|$$
$$\leqslant \frac{\|f^{(2n+2)}\|_2}{16(2n+2)!(4^n-1)(4^n-4)}\left[2025B_{2n+2}^2\right.$$
$$\left.+\frac{[(2n+2)!]^2}{(4n+4)!}\left(2^{3-4n}-325\cdot 2^{1-2n}+4497\right)|B_{4n+4}|\right]^{1/2}$$

Next, denote:

$$R_{2n+2}(f) = \frac{1}{(2n+2)!}\int_0^1 F_{2n+2}(t)f^{(2n+2)}(t)\mathrm{d}t. \tag{2.14}$$

THEOREM 2. *If $f:[0,1]\to\mathbf{R}$ is such that $f^{(2n+2)}$ is continuous on $[0,1]$ for some $n\geqslant 2$, then there exists a point $\eta\in[0,1]$ such that*

$$R_{2n+2}(f) = -\frac{45\cdot B_{2n+2}}{16(2n+2)!(4^n-1)(4^n-4)}\cdot f^{(2n+2)}(\eta). \tag{2.15}$$

Proof. We can rewrite $R_{2n+2}(f)$ as

$$R_{2n+2}(f) = \frac{(-1)^{n+1}}{(2n+2)!}J_{2n+2},$$

where

$$J_{2n+2} = \int_0^1 (-1)^{n+1}F_{2n+2}(t)f^{(2n+2)}(t)\mathrm{d}t. \tag{2.16}$$

The claim follows from Corollary 1 and the mean value theorem for integrals. □

THEOREM 3. *If $f:[0,1]\to\mathbf{R}$ is such that $f^{(2n+2)}$ is a continuous function on $[0,1]$, for some $n\geqslant 2$, and does not change sign on $[0,1]$, then there exists a point $\theta\in[0,1]$ such that*

$$R_{2n+2}(f) = -\theta\frac{2(4-4^{-n})\cdot B_{2n+2}}{(2n+2)!(4^n-1)(4^n-4)}\left[f^{(2n+1)}(1)-f^{(2n+1)}(0)\right]. \tag{2.17}$$

Proof. Suppose $f^{(2n+2)}(t) \geqslant 0,\ 0 \leqslant t \leqslant 1$. If J_{2n+2} is given by (2.16), using Corollary 1, we obtain

$$0 \leqslant J_{2n+2} \leqslant (-1)^{n+1} F_{2n+2}(1/2) \cdot \int_0^1 f^{(2n+2)}(t)\mathrm{d}t.$$

which means that there must exist a point $\theta \in [0,1]$ such that

$$J_{2n+2} = \theta \cdot (-1)^{n+1} F_{2n+2}(1/2) \left[f^{(2n+1)}(1) - f^{(2n+1)}(0) \right].$$

When $f^{(2n+2)}(t) \leqslant 0,\ 0 \leqslant t \leqslant 1$, the statement follows similarly. □

REMARK 2. Using Theorem 2 and (2.6), for $n = 2$ we obtain

$$\begin{aligned} \int_0^1 f(t)\mathrm{d}t &- \frac{1}{90}\left[7f(0) + 32f(1/4) + 12f(1/2) + 32f(3/4) + 7f(1)\right] \\ &= -\frac{1}{1935360} f^{(6)}(\eta), \quad 0 < \eta < 1 \end{aligned}$$

which is the classical Boole's formula. For $n = 3$ we obtain

$$\begin{aligned} \int_0^1 f(t)\mathrm{d}t &- \frac{1}{1890}\left[217f(0) + 512f(1/4) + 432f(1/2) + 512f(3/4) + 217f(1)\right] \\ &+ \frac{1}{252}\left[f'(1) - f'(0)\right] = \frac{1}{1625702400} f^{(8)}(\eta), \quad 0 < \eta < 1 \end{aligned}$$

The second formula obviously has a higher degree of exactness. We call it corrected Boole's formula. Term "corrected" was first introduced in [16], where corrected Simpson's formula was derived. We shall call "corrected" every quadrature formula where in the approximation of the integral, values of the first derivative in the end points of the interval are involved as well. Notice that

$$\frac{1}{1625702400} \approx 6.15119 \cdot 10^{-10}.$$

3. General dual Euler-Boole's formulae

Boole's formula is a quadrature formula of closed type, and so are the general Euler-Boole's formulae. When the value of the function at the end point of the interval cannot be computed, formulae of closed type cannot be applied. For such functions, open formulae are much more effective. That is why quadrature formulae are usually considered in pairs: a closed and a corresponding open one, both with the same degree of exactness. For example, the well-known Simpson's rule

$$\int_0^1 f(t)\mathrm{d}t - \frac{1}{6}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right] = -\frac{1}{2880} f^{(4)}(\xi), \quad 0 < \xi < 1 \tag{3.1}$$

is sometimes studied in pair with the following formula, also known as dual Simpson's formula:

$$\int_0^1 f(t)\mathrm{d}t - \frac{1}{3}\left[2f\left(\frac{1}{4}\right) - f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right)\right] = \frac{7}{23040}f^{(4)}(\eta), \quad 0 < \eta < 1. \tag{3.2}$$

Another such pair of formulae is Simpson's 3/8 formula:

$$\int_0^1 f(t)dt - \frac{1}{8}\left[f(0) + 3f\left(\frac{1}{3}\right) + 3f\left(\frac{2}{3}\right) + f(1)\right] = -\frac{1}{6480}f^{(4)}(\zeta), \quad 0 < \zeta < 1 \tag{3.3}$$

and Maclaurin's formula, also known as dual 3/8 Simpson's formula:

$$\int_0^1 f(t)dt - \frac{1}{8}\left[3f\left(\frac{1}{6}\right) + 2f\left(\frac{1}{2}\right) + 3f\left(\frac{5}{6}\right)\right] = \frac{7}{51840}f^{(4)}(\vartheta), \quad 0 < \vartheta < 1. \tag{3.4}$$

Formulae (3.1)-(3.4) are valid for any function f for which $f^{(4)}$ is continuous and are exact for all polynomials of order $\leqslant 3$.

So, now the idea is to derive a formula of open type that will be dual to Boole's formula in this sense, or, more generally, open formulae dual to general Euler-Boole's formulae. We shall call those formulae general dual Euler-Boole's formulae.

Using a similar technique as in this paper, formulae (3.1)-(3.4) were considered and generalized in [5], [6], [8] and [7], respectively. One can easily check that in both of these cases we have

$$G_k^D(t) = 2^{1-k}G_k(2t) - G_k(t), \tag{3.5}$$

where G_k is obtained in case when a closed quadrature formula is considered and G_k^D in case of the corresponding dual quadrature formula. We will use this identity as a definition of dual formula, since from the function G_k we can deduce the quadrature formula. So, using (2.8) and (3.5), we obtain

$$\begin{aligned} G_k^D(t) &= \frac{1}{4(4^n-1)(4^n-4)}\left[4^{2n}B_k^*(1/8-t) - 10\cdot 4^n B_k^*(1/4-t)\right. \\ &+ 4^{2n}B_k^*(3/8-t) + 16B_k^*(1/2-t) + 4^{2n}B_k^*(5/8-t) \\ &- \left. 10\cdot 4^n B_k^*(3/4-t) + 4^{2n}B_k^*(7/8-t)\right], \end{aligned} \tag{3.6}$$

for $k \geqslant 1$ and $t \in \mathbf{R}$.

Similarly as in the previous section, take $f : [0,1] \to \mathbf{R}$ such that $f^{(2n)}$ is continuous of bounded variation on $[0,1]$ for some $n \geqslant 2$; put $x = 1/8,\ 1/4,\ 3/8,\ 1/2,$

$5/8,\ 3/4,\ 7/8$ in (1.1) and multiply by $4^{2n},\ -10\cdot 4^n,\ 4^{2n},\ 16,\ 4^{2n},\ -10\cdot 4^n,\ 4^{2n}$, respectively. Add those formulae and then divide by $4(4^n-1)(4^n-4)$. We obtain:

$$\int_0^1 f(t)dt - \tilde{D}(0,1) + T_{2n}^D(f) = \frac{1}{(2n+1)!}\int_0^1 G_{2n+1}^D(t)df^{(2n)}(t), \tag{3.7}$$

where

$$\begin{aligned}
\tilde{D}(0,1) &= \frac{1}{4(4^n-1)(4^n-4)}\left[4^{2n}f(1/8) - 10\cdot 4^n f(1/4) + 4^{2n}f(3/8)\right.\\
&+ \left. 16f(1/2) + 4^{2n}f(5/8) - 10\cdot 4^n f(3/4) + 4^{2n}f(7/8)\right]\\
T_m^D(f) &= \sum_{k=1}^{m}\frac{\tilde{B}_k^D}{k!}[f^{(k-1)}(1) - f^{(k-1)}(0)], \qquad 1\leqslant m\leqslant 2n \qquad (3.8)\\
\tilde{B}_k^D &= G_k^D(0), \quad k\geqslant 1. \qquad (3.9)
\end{aligned}$$

Once more, $\tilde{B}_{2k-1}^D = 0$ for $k\geqslant 1$, so the identity (3.7) is produced again if (1.2) is used instead of (1.1).

Assuming we have $f:[0,1]\to\mathbf{R}$ such that $f^{(2n-1)}$ is continuous of bounded variation on $[0,1]$ for some $n\geqslant 2$, from (1.1) we would get:

$$\int_0^1 f(t)dt - \tilde{D}(0,1) + T_{2n}^D(f) = \frac{1}{(2n)!}\int_0^1 G_{2n}^D(t)df^{(2n-1)}(t), \tag{3.10}$$

and if $f^{(2n+1)}$ was continuous of bounded variation on $[0,1]$, from (1.2) we would get:

$$\int_0^1 f(t)dt - \tilde{D}(0,1) + T_{2n}^D(f) = \frac{1}{(2n+2)!}\int_0^1 F_{2n+2}^D(t)df^{(2n+1)}(t), \tag{3.11}$$

where $F_k^D(t) = G_k^D(t) - \tilde{B}_k^D,\ \ k\geqslant 2$. Formulae (3.7), (3.10) and (3.11) are general dual Euler-Boole's formulae.

LEMMA 2. *For $n\geqslant 2$, $G_{2n+1}^D(t)$ has no zeros in the interval $(0,1/2)$. The sign of the function is determined by*

$$(-1)^{n-1}G_{2n+1}^D(t) > 0, \qquad 0<t<1/2.$$

Proof. We have $G_{2n+1}(1-t) = -G_{2n+1}(t)$, so from Lemma 1 it follows that $G_{2n+1}(2t)$ and $-G_{2n+1}(t)$ have the same sign on $(1/4,1/2)$ and from (3.5) we conclude $G_{2n+1}^D(t)$ cannot have any zeros there.

Next, we can rewrite $G_{2n+1}^D(t)$ as

$$\begin{aligned}
G_{2n+1}^D(t) &= \frac{-1}{4(4^n-1)(4^n-4)}\left[B_{2n+1}^*(4t-1/2)\right.\\
&- \left. 10B_{2n+1}^*(2t-1/2) + 16B_{2n+1}^*(t-1/2)\right]. \qquad (3.12)
\end{aligned}$$

Using this in the case when $t\in(0,1/4]$, the proof is completely analogous to the same part of the proof of Lemma 1. As for the sign of the function, again it is enough to calculate the value of the function in any point of the interval $(0,1/2)$, e.g. $t=1/4$. $\square$

Notice the analogy of the form of the dual function G_{2n+1}^D in (3.12) with the form of the function G_{2n+1} in (2.9). One can easily be obtained from the other having this connection in mind. Therefore, (3.12) can also be used as a definition of the dual function G_{2n+1}^D.

COROLLARY 2. *For $n \geqslant 2$, $(-1)^n F_{2n+2}^D(t)$ is strictly increasing on the interval $(0,1/2)$ and strictly decreasing on the interval $(1/2,1)$ and*

$$\max_{t\in[0,1]} \left|F_{2n+2}^D(t)\right| = \frac{2(4-4^{-n})}{(4^n-1)(4^n-4)}|B_{2n+2}|.$$

Also,

$$\int_0^1 \left|G_{2n+1}^D(t)\right| \mathrm{d}t = \int_0^1 \left|F_{2n+1}^D(t)\right| \mathrm{d}t = \frac{2(4-4^{-n})}{(n+1)(4^n-1)(4^n-4)}|B_{2n+2}|,$$
$$\int_0^1 \left|F_{2n+2}^D(t)\right| \mathrm{d}t = |\tilde{B}_{2n+2}^D| = \frac{45(1-2^{-2n-1})}{16(4^n-1)(4^n-4)}|B_{2n+2}|.$$

Proof. Analogous to the proof of Corollary 1. $\square$

THEOREM 4. *Assume (p,q) is a pair of conjugate exponents, that is $1 \leqslant p,\ q \leqslant \infty,\ 1/p+1/q=1$. If $|f^{(2n)}|^p : [0,1] \to \mathbf{R}$ is R-integrable for some $n \geqslant 2$, then we have*

$$\left|\int_0^1 f(t)\mathrm{d}t - \tilde{D}(0,1) + T_{2n}^D(f)\right| \leqslant K_D(2n,q)\cdot \|f^{(2n)}\|_p, \tag{3.13}$$

if $|f^{(2n+1)}|^p : [0,1] \to \mathbf{R}$ is R-integrable for some $n \geqslant 2$, then we have

$$\left|\int_0^1 f(t)\mathrm{d}t - \tilde{D}(0,1) + T_{2n}^D(f)\right| \leqslant K_D(2n+1,q)\cdot \|f^{(2n+1)}\|_p, \tag{3.14}$$

if $|f^{(2n+2)}|^p : [0,1] \to \mathbf{R}$ is R-integrable for some $n \geqslant 2$, then we have

$$\left|\int_0^1 f(t)\mathrm{d}t - \tilde{D}(0,1) + T_{2n}^D(f)\right| \leqslant K_D^*(2n+2,q)\cdot \|f^{(2n+2)}\|_p, \tag{3.15}$$

where

$$K_D(m,q) = \frac{1}{m!}\left[\int_0^1 \left|G_m^D(t)\right|^q \mathrm{d}t\right]^{\frac{1}{q}} \quad \textit{and} \quad K_D^*(m,q) = \frac{1}{m!}\left[\int_0^1 \left|F_m^D(t)\right|^q \mathrm{d}t\right]^{\frac{1}{q}}.$$

These inequalities are sharp for $1 < p \leqslant \infty$ and best possible for $p=1$.

Proof. Analogous to the proof of Theorem 1. □

REMARK 3. For $p=\infty$, applying Corollary 2, (3.14) and (3.15) turn to:

$$\left|\int_0^1 f(t)\mathrm{d}t - \tilde{D}(0,1) + T_{2n}^D(f)\right| \leqslant \frac{|B_{2n+2}|}{(2n+2)!}\cdot\frac{4(4-4^{-n})}{(4^n-1)(4^n-4)}\cdot\|f^{(2n+1)}\|_\infty,$$

$$\left|\int_0^1 f(t)\mathrm{d}t - \tilde{D}(0,1) + T_{2n}^D(f)\right| \leqslant \frac{|B_{2n+2}|}{(2n+2)!}\cdot\frac{45(1-2^{-2n-1})}{16(4^n-1)(4^n-4)}\cdot\|f^{(2n+2)}\|_\infty.$$

For $p=1$, applying Corollary 2 again, (3.15) becomes:

$$\left|\int_0^1 f(t)\mathrm{d}t - \tilde{D}(0,1) + T_{2n}^D(f)\right| \leqslant \frac{|B_{2n+2}|}{(2n+2)!}\cdot\frac{2(4-4^{-n})}{(4^n-1)(4^n-4)}\cdot\|f^{(2n+2)}\|_1.$$

Analogously as in the previous section, for $p=2$ we obtain

$$\begin{aligned}
&\left|\int_0^1 f(t)\mathrm{d}t - \tilde{D}(0,1) + T_{2n}^D(f)\right| \\
&\quad\leqslant \frac{\|f^{(2n)}\|_2}{(4^n-1)(4^n-4)}\left[\frac{|B_{4n}|}{(4n)!}(2^{4-8n}-25\cdot 2^{2-6n}-2^{3-4n}+25\cdot 2^{1-2n}+42)\right]^{1/2}, \\
&\left|\int_0^1 f(t)\mathrm{d}t - \tilde{D}(0,1) + T_{2n}^D(f)\right| \\
&\quad\leqslant \frac{\|f^{(2n+1)}\|_2}{4(4^n-1)(4^n-4)}\left[\frac{B_{4n+2}}{(4n+2)!}\left(2^{2-8n}-85\cdot 2^{-6n}-2^{3-4n}+85\cdot 2^{1-2n}+357\right)\right]^{1/2}, \\
&\left|\int_0^1 f(t)\mathrm{d}t - \tilde{D}(0,1) + T_{2n}^D(f)\right| \\
&\quad\leqslant \frac{\|f^{(2n+2)}\|_2}{16(2n+2)!(4^n-1)(4^n-4)}\Big[2025(1-2^{-2n-1})^2 B_{2n+2}^2 \\
&\quad+\frac{[(2n+2)!]^2}{(4n+4)!}\left(2^{-8n}-325\cdot 2^{-2-6n}-2^{3-4n}+325\cdot 2^{1-2n}+4497\right)|B_{4n+4}|\Big]^{1/2}.
\end{aligned}$$

Denote:

$$R_{2n+2}^D(f) = \frac{1}{(2n+2)!}\int_0^1 F_{2n+2}^D(t) f^{(2n+2)}(t)\mathrm{d}t. \tag{3.16}$$

THEOREM 5. *If $f:[0,1]\to\mathbf{R}$ is such that $f^{(2n+2)}$ is continuous on $[0,1]$ for some $n\geqslant 2$, then there exists a point $\eta\in[0,1]$ such that*

$$R_{2n+2}^D(f) = \frac{45(1-2^{-2n-1})\cdot B_{2n+2}}{16(2n+2)!(4^n-1)(4^n-4)}\cdot f^{(2n+2)}(\eta). \tag{3.17}$$

Proof. Analogous to the proof of Theorem 2. □

THEOREM 6. *If* $f : [0,1] \to \mathbf{R}$ *is such that* $f^{(2n+2)}$ *is a continuous function on* $[0,1]$*, for some* $n \geqslant 2$*, and does not change sign on* $[0,1]$*, then there exists a point* $\theta \in [0,1]$ *such that*

$$R_{2n+2}^{D}(f) = \theta \frac{2(4-4^{-n}) \cdot B_{2n+2}}{(2n+2)!(4^n-1)(4^n-4)} \left[f^{(2n+1)}(1) - f^{(2n+1)}(0) \right]. \tag{3.18}$$

Proof. Analogous to the proof of Theorem 3. □

REMARK 4. Using Theorem 5 and (3.11), for $n=2$ we obtain

$$\int_0^1 f(t)\mathrm{d}t - \frac{1}{45}\left[16\left(f(1/8)+f(3/8)+f(5/8)+f(7/8)\right) - 10\left(f(1/4)+f(3/4)\right) + f(1/2)\right] = \frac{31}{61931520} f^{(6)}(\eta), \quad 0<\eta<1$$

This is the dual formula for the classical Boole's formula. For $n=3$ we obtain

$$\int_0^1 f(t)\mathrm{d}t - \frac{1}{945}\left[256\left(f(1/8)+f(3/8)+f(5/8)+f(7/8)\right) - 40\left(f(1/4)+f(3/4)\right) + f(1/2)\right] - \frac{1}{504}\left[f'(1)-f'(0)\right] = -\frac{127}{208089907200} f^{(8)}(\eta), \quad 0<\eta<1$$

This formula is the dual formula for the corrected Boole's formula. Notice that

$$\frac{31}{61931520} \approx 5.00553 \cdot 10^{-7} \quad \text{and} \quad \frac{127}{208089907200} \approx 6.10313 \cdot 10^{-10}.$$

4. General Bullen-Boole's inequality

The following pair of inequalities is usually referred to in literature as Hadamard's inequalities:

$$f\left(\frac{1}{2}\right) \leqslant \int_0^1 f(t)\mathrm{d}t \leqslant \frac{f(0)+f(1)}{2}. \tag{4.1}$$

It holds for any convex function $f : [0,1] \to \mathbf{R}$. If f is concave, inequalities are reversed.

In [9], it was shown by a simple geometric argument that for a convex function f the following inequality is valid :

$$0 \leqslant \int_0^1 f(t)\mathrm{d}t - f\left(\frac{1}{2}\right) \leqslant \frac{f(0)+f(1)}{2} - \int_0^1 f(t)\mathrm{d}t. \tag{4.2}$$

An elementary analytic proof of (4.1) and (4.2), but stated on the interval $[-1,1]$, was given in [2]. Some other interesting results of similar type were given in that same paper. Namely, provided f is 4-convex, we have:

$$\begin{aligned} 0 &\leqslant \int_0^1 f(t)\mathrm{d}t - \frac{1}{3}\left[2f\left(\frac{1}{4}\right) - f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right)\right] \\ &\leqslant \frac{1}{6}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right] - \int_0^1 f(t)\mathrm{d}t \end{aligned} \tag{4.3}$$

This implies that dual Simpson's quadrature rule is more accurate than Simpson's quadrature rule. The inequality (4.3) is sometimes called Bullen-Simpson's inequality and was generalized for a class of $(2k)$-convex functions in [12]. Under same assumptions it was also proved that:

$$\begin{aligned} 0 &\leqslant \int_0^1 f(t)\mathrm{d}t - \frac{1}{8}\left[3f\left(\frac{1}{6}\right) + 2f\left(\frac{1}{2}\right) + 3f\left(\frac{5}{6}\right)\right] \\ &\leqslant \frac{1}{8}\left[f(0) + 3f\left(\frac{1}{3}\right) + 3f\left(\frac{2}{3}\right) + f(1)\right] - \int_0^1 f(t)\mathrm{d}t, \end{aligned} \tag{4.4}$$

which implies Maclaurin's quadrature rule is more accurate than Simpson's $3/8$ quadrature rule. The inequality (4.4) is sometimes called Bullen-Simpson's $3/8$ inequality and was generalized for a class of $(2k)$-convex functions in [11].

In this section we will derive an inequality of similar type, only this time starting from general Boole's formula and its dual formula. We call it general Bullen-Boole's inequality.

First, add (2.6) and (3.11) then divide by 2. We get:

$$\int_0^1 f(t)dt - \hat{D}(0,1) + \hat{T}_{2n}(f) = \hat{R}_{2n+2}(f), \tag{4.5}$$

where

$$\begin{aligned} \hat{D}(0,1) = \frac{1}{8(4^n-1)(4^n-4)} &\left[(2^{4n-1} - 5\cdot 4^n + 8)f(0) + 4^{2n}f\left(\frac{1}{8}\right)\right. \\ &+ 4^n(4^n-10)f\left(\frac{1}{4}\right) + 4^{2n}f\left(\frac{3}{8}\right) + (4^{2n} - 10\cdot 4^n + 16)f\left(\frac{1}{2}\right) \\ &\left.+ 4^{2n}f\left(\frac{5}{8}\right) + 4^n(4^n-10)f\left(\frac{3}{4}\right) + 4^{2n}f\left(\frac{7}{8}\right) + (2^{4n-1} - 5\cdot 4^n + 8)f(1)\right] \end{aligned}$$

$$\hat{T}_m(f) = \sum_{k=1}^{m} \frac{\hat{B}_k}{k!}[f^{(k-1)}(1) - f^{(k-1)}(0)], \qquad 1 \leqslant m \leqslant 2n$$

$$\begin{aligned} \hat{G}_k(t) = \frac{1}{8(4^n-1)(4^n-4)} &\left[(4^{2n} - 10\cdot 4^n + 16)B_k^*(1-t) + 4^{2n}B_k^*(1/8-t)\right. \\ &+ 4^n(4^n-10)B_k^*(1/4-t) + 4^{2n}B_k^*(3/8-t) + (4^{2n} - 10\cdot 4^n + 16)B_k^*(1/2-t) \\ &\left.+ 4^{2n}B_k^*(5/8-t) + 4^n(4^n-10)B_k^*(3/4-t) + 4^{2n}B_k^*(7/8-t)\right] \end{aligned}$$

$$\hat{B}_1 = 0, \quad \hat{B}_k = \hat{G}_k(0), \quad k \geqslant 2$$
$$\hat{F}_k(t) = \hat{G}_k(t) - \hat{B}_k, \quad k \geqslant 1$$
$$\hat{R}_{2n+2}(f) = \frac{1}{(2n+2)!} \int_0^1 \hat{F}_{2n+2}(t) df^{(2n+1)}(t)$$

The function $\hat{G}_k$ has the property $\hat{G}_k(t+1/2) = \hat{G}_k(t)$ so it is enough to study that function on the interval $(0, 1/4)$.

LEMMA 3. *For $n \geqslant 2$, $\hat{G}_{2n+1}(t)$ has no zeros in the interval $(0, 1/4)$. The sign of the function is determined by*

$$(-1)^n \hat{G}_{2n+1}(t) > 0, \qquad 0 < t < 1/4.$$

Proof. As (3.5) implies that $\hat{G}_{2n+1}(t) = 2^{-2n-1} G_{2n+1}(2t)$, the statement follows immediately from Lemma 1. □

THEOREM 7. *If $f : [0,1] \to \mathbf{R}$ is such that $f^{(2n+2)}$ is continuous on $[0,1]$ for some $n \geqslant 2$, then there exists a point $\eta \in [0,1]$ such that*

$$\hat{R}_{2n+2}(f) = -\frac{45 \cdot 4^{-n-3} \cdot B_{2n+2}}{(2n+2)!(4^n-1)(4^n-4)} \cdot f^{(2n+2)}(\eta). \tag{4.6}$$

Proof. Analogous to the proof of Theorem 2. □

Recall that a function $f : [a,b] \to \mathbf{R}$ is said to be n-convex, $n \geqslant 0$, on $[a,b]$ iff for all choices of $(n+1)$ distinct points in $[a,b]$ the nth order divided difference

$$[x_0, \dots, x_n]f \geqslant 0.$$

If this inequality is reversed, then f is said to be n-concave on $[a,b]$. Also, if $f^{(n)}$ exists, then f is n-convex iff $f^{(n)} \geqslant 0$. For further details on n-convex functions see [15].

THEOREM 8. *Let $f : [0,1] \to \mathbf{R}$ be such that $f^{(2n+2)}$ is continuous on $[0,1]$ for some $n \geqslant 2$. If f is a $(2n+2)-$convex function, then for even n we have*

$$0 \leqslant \int_0^1 f(t)\mathrm{d}t - \tilde{D}(0,1) + T_{2n}^D(f) \leqslant D(0,1) - T_{2n}(f) - \int_0^1 f(t)\mathrm{d}t. \tag{4.7}$$

For odd n inequalities are reversed.

Proof. Denote the middle part of (4.7) by LHS and the right-hand side by RHS. Then

$$LHS = R_{2n+2}^D(f) \quad \text{and} \quad RHS - LHS = -2\hat{R}_{2n+2}(f).$$

Now, applying (3.17) and (4.6), we conclude

$$LHS \geqslant 0, \quad RHS - LHS \geqslant 0, \quad \text{for even n}$$
$$LHS \leqslant 0, \quad RHS - LHS \leqslant 0, \quad \text{for odd n}$$

and thus the proof is complete. □

REMARK 5. For $n = 2$, (4.7) becomes

$$
\begin{aligned}
0 \leqslant \int_0^1 f(t)\mathrm{d}t - \frac{1}{45}\,&[16\,(f\,(1/8) + f\,(3/8) + f\,(5/8) + f\,(7/8)) \\
&- 10\,(f\,(1/4) + f\,(3/4)) + f\,(1/2)] \\
&\leqslant \frac{1}{90}\,[7f\,(0) + 32f\,(1/4) + 12f\,(1/2) + 32f\,(3/4) + 7f\,(1)] - \int_0^1 f(t)\mathrm{d}t
\end{aligned}
$$

which implies dual Boole's formula is more accurate than classical Boole's formula. For $n = 3$, (4.7) becomes

$$
\begin{aligned}
0 \leqslant\ & \frac{1}{945}\,[256\,(f\,(1/8) + f\,(3/8) + f\,(5/8) + f\,(7/8)) - 40\,(f\,(1/4) + f\,(3/4)) \\
&+ f\,(1/2)] + \frac{1}{504}[f'(1) - f'(0)] - \int_0^1 f(t)\mathrm{d}t \\
&\leqslant \int_0^1 f(t)\mathrm{d}t - \frac{1}{1890}\,[217f\,(0) + 512f\,(1/4) + 432f\,(1/2) + 512f\,(3/4) + 217f\,(1)] \\
&+ \frac{1}{252}[f'(1) - f'(0)].
\end{aligned}
$$

Therefore, dual corrected Boole's formula is more accurate than corrected Boole's formula.

For this new quadrature formula (4.5), similar results as those obtained in Section 2 for general Euler-Boole's and Section 3 for general dual Euler-Boole's formulae, can be derived analogously.

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