Mathematical
Inequalities
& Applications
Volume 8, Number 2 (2005), 305–316

HERMITE–HADAMARD'S INEQUALITIES FOR MULTIVARIATE g–CONVEX FUNCTIONS

M. KLARIČIĆ BAKULA, E. NEUMAN, J. PEČARIĆ AND VIDA ŠIMIĆ

Dedicated to the memory of prof. Mladen Alić

(*communicated by Charles E. M. Pearce*)

Abstract. Refinements of generalized Hermite-Hadamard's inequalities for multivariate g-convex functions are given. Since special instances of the g-convex functions include the r-convex and the logarithmically convex functions, those inequalities also give refinements of the Hermite-Hadamard's ineqalities for these families of functions.

1. Introduction and preliminaries

Among numerous inequalities obeyed by convex functions the ones discovered by C. Hermite and J. Hadamard are considered to be of great importance (see, e.g., [11, p. 137], [4]). These inequalities state that if $f : [a,b] \to \mathbb{R}$ is convex function, then

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(t)\,dt \leqslant \frac{f(a)+f(b)}{2}. \tag{1}$$

Many generalizations and refinements of this inequality have been obtained in recent years. The interested reader is referred to the monograph [4].

In order to present one of these results let us introduce more notation.

By

$$E_n = \left\{ \boldsymbol{u} = (u_0, \dots, u_n) : u_i \geqslant 0 \ (0 \leqslant i \leqslant n),\ \sum_{i=0}^{n} u_i = 1 \right\},\ n \in \mathbb{N}$$

we will denote Euclidean simplex. In what follows we will always choose $u_0 = 1 - (u_1 + \dots + u_n)$. By μ we will denote a probability measure on E_n. Natural weights w_i of the measure μ are defined by

$$w_i = \int_{E_n} u_i d\mu(\boldsymbol{u}),\ (i = 0, \dots, n). \tag{2}$$

Mathematics subject classification (2000): 26D15, 26B25.

Key words and phrases: Hermite-Hadamard's inequalities, g-convexity, r-convexity, log-convexity, functional Stolarsky means, logarithmic mean.

It is obvious that all weights w_i are nonnegative and that $\sum_{i=0}^{n} w_i = 1$.

Let U be an open subset of $\mathbb{R}^k$ $(k \in \mathbb{N})$ and let $\boldsymbol{x}^{(0)}, \boldsymbol{x}^{(1)}, ..., \boldsymbol{x}^{(n)} \in U$ $(n \geqslant k)$. Further, let X be an k by $n+1$ matrix whose columns are the vectors $\boldsymbol{x}^{(0)}, \boldsymbol{x}^{(1)}, ..., \boldsymbol{x}^{(n)}$. Let σ denote the convex hull of the columns of matrix X, i.e.,

$$\sigma = \left\{ \boldsymbol{y} \in \mathbb{R}^k : \boldsymbol{y} = \sum_{i=0}^{n} u_i \boldsymbol{x}^{(i)} = X\boldsymbol{u},\ \boldsymbol{u} \in E_n \right\}. \tag{3}$$

We will always assume that the columns of X span a proper simplex in $\mathbb{R}^k$, i.e., that $vol(\sigma) \neq 0$.

Weighted logarithmic mean $\mathscr{L}(\boldsymbol{x};\mu)$ of a positive $n+1-$tuple $\boldsymbol{x}$ is defined as [6]

$$\mathscr{L}(\boldsymbol{x};\mu) = \int_{E_n} \prod_{i=0}^{n} x_i^{u_i} d\mu(\boldsymbol{u}) = \int_{E_n} \exp(\boldsymbol{u} \cdot \ln(\boldsymbol{x}))\, d\mu(\boldsymbol{u}),$$

where $\boldsymbol{u} \cdot \ln(\boldsymbol{x})$ denotes the inner product of vectors $\boldsymbol{u}$ and $\ln(\boldsymbol{x}) = (\ln(x_0), ..., \ln(x_n)) \in \mathbb{R}^{n+1}$. It is well known that logarithmic mean interpolates the inequality for the arithmetic and geometric means, i.e.,

$$\prod_{i=0}^{n} x_i^{w_i} \leqslant \mathscr{L}(\boldsymbol{x};\mu) \leqslant \sum_{i=0}^{n} w_i x_i. \tag{4}$$

By $M_r(\boldsymbol{x},\boldsymbol{u})$ we will denote power mean of order r of a positive $n+1$-tuple $\boldsymbol{x}$ with weights $\boldsymbol{u}$, i.e.

$$M_r(\boldsymbol{x},\boldsymbol{u}) = \begin{cases} \left(\sum_{i=0}^{n} u_i x_i^r\right)^{\frac{1}{r}}, & r \neq 0 \\ \prod_{i=0}^{n} x_i^{u_i}, & r = 0 \end{cases} \tag{5}$$

It is well known that M_r is strictly increasing function of r.

The integral power mean of a positive real function F on E_n with a probability measure μ on E_n is defined as

$$\overline{M}_r(F;\mu) = \begin{cases} \left(\int_{E_n} (F(u))^r d\mu(u)\right)^{\frac{1}{r}}, & r \neq 0 \\ \exp\left[\int_{E_n} \ln(F(u))\, d\mu(u)\right], & r = 0 \end{cases} \tag{6}$$

In [12] J. Pečarić and V. Šimić have considered weighted Stolarsky-Tobey mean of several variables. For $\boldsymbol{x} = (x_0, ..., x_n) \in \mathbb{R}_+^{n+1}$ and $r, p \in \mathbb{R}$, the Stolarsky-Tobey mean is defined as

$$E_{r,p+r}(\boldsymbol{x};\mu)=\begin{cases}\left[\int\limits_{E_n}\left(\sum\limits_{i=0}^{n}u_i x_i^r\right)^{\frac{p}{r}}d\mu(\boldsymbol{u})\right]^{\frac{1}{p}}, & \text{if } rp\neq 0,\\ \exp\left(\int\limits_{E_n}\ln\left(\sum\limits_{i=0}^{n}u_i x_i^r\right)^{\frac{1}{r}}d\mu(\boldsymbol{u})\right), & \text{if } p=0, r\neq 0,\\ \left[\int\limits_{E_n}\left(\prod\limits_{i=0}^{n}x_i^{u_i}\right)^{p}d\mu(\boldsymbol{u})\right]^{\frac{1}{p}}, & \text{if } r=0, p\neq 0,\\ \exp\left(\int\limits_{E_n}\ln\left(\prod\limits_{i=0}^{n}x_i^{u_i}\right)d\mu(\boldsymbol{u})\right), & \text{if } p=r=0.\end{cases} \tag{7}$$

It is easy to see that this mean includes both the logarithmic mean and the power mean. We have

$$E_{0,1}(\boldsymbol{x};\mu)=\mathscr{L}(\boldsymbol{x};\mu) \text{ and } E_{r,2r}(\boldsymbol{x};\mu)=M_r(\boldsymbol{x},\boldsymbol{w}).$$

In [10] C. Pearce, J. Pečarić and V. Šimić considered a functional generalization of the Stolarsky-Tobey mean. For two strictly monotonic continuous functions h and g defined on a real interval I, a probability measure μ and an $n+1$-tuple $\boldsymbol{x}$ in I^{n+1}, the functional Stolarsky-Tobey mean is defined as

$$m_{h,g}(\boldsymbol{x};\mu)=h^{-1}\left\{\int_{E_n}h\left[g^{-1}\left(\sum_{i=0}^{n}u_i g(x_i)\right)\right]d\mu(\boldsymbol{u})\right\}. \tag{8}$$

It follows from (8) and (2) that

$$m_{g,g}(\boldsymbol{x};\mu)=g^{-1}\left(\sum_{i=0}^{n}w_i g(x_i)\right). \tag{9}$$

There are many important special cases of the above means, and among them is also Stolarsky-Tobey mean (7), which can be obtained from (8) and (9) by letting

$$h(x)=\begin{cases}x^p, & p\neq 0\\ \ln x, & p=0,\end{cases} \tag{1.10}$$

$$g(x)=\begin{cases}x^r, & r\neq 0\\ \ln x, & r=0,\end{cases} \tag{1.11}$$

$x\in(0,\infty)$. With h and g as defined above we have

$$m_{h,g}(x;\mu)=\begin{cases}E_{r,p+r}(x;\mu), & h\neq g\\ M_r(x;w), & h=g.\end{cases} \tag{12}$$

Logarithmically convex functions are important subfamily of the class of convex functions. We say that a function $f:I\to(0,\infty)$ is log-convex if

$$f[(1-\lambda)x+\lambda y]\leqslant[f(x)]^{1-\lambda}[f(y)]^{\lambda}$$

holds for all $x, y \in I$ and $0 \leqslant \lambda \leqslant 1$. Any log-convex function f also satisfies the inequality

$$f\,(\boldsymbol{x}\cdot\boldsymbol{u}) \leqslant \prod_{i=0}^{n} [f\,(x_i)]^{u_i}\,, \tag{13}$$

where $\boldsymbol{u} \in E_n$ and $\boldsymbol{x} \in I^{n+1}$. Logarithmically convex functions are of interest in the mathematical statistics [11], theory of special functions [1], and theory of means [7], to name a few areas.

The log-convex functions belong to a class of functions called the r-convex functions. A function $f : I \to (0,\infty)$ is said to be *r-convex* if the inequality

$$f\,[(1-\lambda)x+\lambda y] \leqslant \begin{cases} [(1-\lambda)[f\,(x)]^r+\lambda[f\,(y)]^r]^{\frac{1}{r}}, & r \neq 0 \\ [f\,(x)]^{1-\lambda}[f\,(y)]^{\lambda}, & r = 0 \end{cases}$$

holds for all $x, y \in I$ and $0 \leqslant \lambda \leqslant 1$.

Finally, we recall the definition of a more general class of functions called the g-convex functions. Let $f : I \to \mathbb{R}$ and let g be a strictly monotone continuous function defined on the range of f. The function f is said to be *g-convex* if the inequality

$$f\,[(1-\lambda)\,x+\lambda y] \leqslant g^{-1}\,[(1-\lambda)\,(g\circ f\,)\,(x)+\lambda\,(g\circ f\,)\,(y)]$$

holds for all $x, y \in I$ and $0 \leqslant \lambda \leqslant 1$. The function f is said to be g*-concave* if the reverse inequality holds. For $g(x)$ as defined in (1.11) the definition of g-convexity becomes the definition of r-convexity.

The functional Stolarsky-Tobey means and g-convexity are discussed in [8].

2. Refinements of the Hermite-Hadamard's inequalities for the multivariate log-convex functions

Some refinements of the Hermite-Hadamard's inequalities (1) have been obtained by S. Dragomir and B. Mond in [3]. They have proven that any log-convex function g satisfies the inequalities

$$\begin{aligned} g\left(\frac{a+b}{2}\right) &\leqslant \exp\left[\frac{1}{b-a}\int_a^b \ln\left[g\,(x)\right]dx\right] \\ &\leqslant \frac{1}{b-a}\int_a^b \sqrt{g\,(x)\,g\,(a+b-x)}dx \\ &\leqslant \frac{1}{b-a}\int_a^b g\,(x)\,dx \leqslant L\,[g\,(a)\,,g\,(b)] \\ &\leqslant \frac{g\,(a)+g\,(b)}{2}, \end{aligned} \tag{14}$$

where

$$\begin{aligned} L\,(p,q) &= \frac{p-q}{\ln p-\ln q},\ p \neq q \\ L\,(p,p) &= p, \end{aligned}$$

is the logarithmic mean of positive real numbers p and q.

Another refinement of the first inequality in (1) appears in [2, Theorem 1]. Let $A = \frac{a+b}{2}$. If the function g is log-convex and differentiable on $Int\,(I)$, then

$$\frac{1}{b-a}\int_a^b g(x)\,dx/g(A) \geqslant L\left(\exp\left[\frac{g'(A)}{g(A)}\left(\frac{b-a}{2}\right)\right], \exp\left[-\frac{g'(A)}{g(A)}\left(\frac{b-a}{2}\right)\right]\right) \geqslant 1. \tag{2.2}$$

In this section we shall give generalizations of the inequalities (14) and (2.2) for the log-convex functions of several variables.

In what follows we will always assume that X is a k by $n+1$ matrix whose columns are the vectors $\boldsymbol{x}^{(0)}, \boldsymbol{x}^{(1)}, ..., \boldsymbol{x}^{(n)} \in U$, $w_0, ..., w_n$ are the natural weights, and $\boldsymbol{u} \in E_n$.

In the proof of our main results, we will also make use of the following result ([5], Theorem 4.2):

THEOREM 1. *If $\phi : \sigma \to \mathbb{R}$ is a convex function, then*

$$\phi\left(\sum_{i=0}^{n} w_i \boldsymbol{x}^{(i)}\right) \leqslant \int_{E_n} \phi(X\boldsymbol{u})\,d\mu(\boldsymbol{u}) \leqslant \sum_{i=0}^{n} w_i \phi\left(\boldsymbol{x}^{(i)}\right). \tag{16}$$

Inequalities (16) are reversed if ϕ is a concave function.

THEOREM 2. *Let $f : \sigma \to (0,\infty)$ be a* log*-convex function. Then*

$$\begin{aligned} f\left(\sum_{i=0}^{n} w_i \boldsymbol{x}^{(i)}\right) &\leqslant \exp\left[\int_{E_n} \ln f\,(X\boldsymbol{u})\,d\mu(\boldsymbol{u})\right] \\ &\leqslant \int_{E_n} f\,(X\boldsymbol{u})\,d\mu(\boldsymbol{u}) \\ &\leqslant \mathscr{L}\left(f\left(\boldsymbol{x}^{(0)}\right), ..., f\left(\boldsymbol{x}^{(n)}\right); \mu\right) \leqslant \sum_{i=0}^{n} w_i f\left(\boldsymbol{x}^{(i)}\right). \end{aligned} \tag{17}$$

Proof. In order to prove the first inequality in (17) we utilize the first inequality in (16) with $h = \ln f$ to obtain

$$(\ln f)\left(\sum_{i=0}^{n} w_i \boldsymbol{x}^{(i)}\right) \leqslant \int_{E_n} (\ln f)(X\boldsymbol{u})\,d\mu(\boldsymbol{u}),$$

from which we get

$$f\left(\sum_{i=0}^{n} w_i \boldsymbol{x}^{(i)}\right) \leqslant \exp\left[\int_{E_n} (\ln f)(X\boldsymbol{u})\,d\mu(\boldsymbol{u})\right].$$

Since the function exp is convex, application of Jensen's inequality for integrals gives

$$\exp\left[\int_{E_n} (\ln f)(X\boldsymbol{u})\, d\mu(\boldsymbol{u})\right] \leqslant \int_{E_n} f(X\boldsymbol{u})\, d\mu(\boldsymbol{u}),$$

which is the second inequality in (17). The third inequality in (17) can be established using (13). We have

$$\begin{aligned}\int_{E_n} f(X\boldsymbol{u})\, d\mu(\boldsymbol{u}) &\leqslant \int_{E_n} \prod_{i=0}^{n} \left[f\left(\boldsymbol{x}^{(i)}\right)\right]^{u_i} d\mu(\boldsymbol{u}) \\ &= \mathscr{L}\left(f\left(\boldsymbol{x}^{(0)}\right), \dots, f\left(\boldsymbol{x}^{(n)}\right); \mu\right).\end{aligned}$$

The last inequality in (17) follows from the second inequality in (4). This completes the proof.

REMARK 1. The inequalities (14), without the third member, now follow from (17) by letting $n = 1$, $\boldsymbol{x}^{(0)} = a$ and $\boldsymbol{x}^{(1)} = b$ $(a \neq b)$.

Before we will state and prove the next result, let us introduce more notation. For $\boldsymbol{y} \in \sigma$ let $\boldsymbol{c} = \nabla \ln f(\boldsymbol{y})$ be the gradient of the function $\ln f$. Also, let

$$z_i = \left(\boldsymbol{x}^{(i)} - \boldsymbol{y}\right) \cdot \boldsymbol{c},\ i = 0, \dots, n,$$

let $\boldsymbol{z} = (z_0, \dots, z_n)$, and let $\exp(\boldsymbol{z}) = (\exp(z_0), \dots, \exp(z_n))$.

THEOREM 3. *Let $f : \sigma \to (0, \infty)$ be a $\log$-convex function. If f has continuous partial derivations of order one on $Int(\sigma)$, then*

$$\int_{E_n} f(X\boldsymbol{u})\, d\mu(\boldsymbol{u}) \geqslant f(\boldsymbol{y})\, \mathscr{L}(\exp(z); \mu) \tag{18}$$

holds for any $\boldsymbol{y} \in Int(\sigma)$. If $\boldsymbol{y} = \sum_{i=0}^{n} w_i \boldsymbol{x}^{(i)}$, then

$$\mathscr{L}(\exp(z); \mu) \geqslant 1. \tag{19}$$

Proof. Logarithmic convexity of function f implies the following inequality

$$\ln f(\boldsymbol{x}) - \ln f(\boldsymbol{y}) \geqslant (\boldsymbol{x} - \boldsymbol{y}) \cdot \nabla \ln f(\boldsymbol{y}),$$

which is valid for all $\boldsymbol{x}, \boldsymbol{y} \in Int(\sigma)$. Hence

$$f(\boldsymbol{x}) \geqslant f(\boldsymbol{y}) \exp[(\boldsymbol{x} - \boldsymbol{y}) \cdot \boldsymbol{c}].$$

Letting $\boldsymbol{x} = X\boldsymbol{u}$ above, and next integrating both sides against a probability measure μ, we obtain

$$\int_{E_n} f(X\boldsymbol{u})\, d\mu(\boldsymbol{u}) \geqslant f(\boldsymbol{y}) \int_{E_n} \exp[(X\boldsymbol{u} - \boldsymbol{y}) \cdot \boldsymbol{c}]\, d\mu(\boldsymbol{u}).$$

Since

$$\begin{aligned}(X\boldsymbol{u}-\boldsymbol{y})\cdot\boldsymbol{c} &= \left(\sum_{i=0}^{n}u_i\boldsymbol{x}^{(i)}-\sum_{i=0}^{n}u_i\boldsymbol{y}\right)\cdot\boldsymbol{c}\\ &= \left(\sum_{i=0}^{n}u_i\left(\boldsymbol{x}^{(i)}-\boldsymbol{y}\right)\right)\cdot\boldsymbol{c}\\ &= \sum_{i=0}^{n}u_iz_i=\boldsymbol{u}\cdot\boldsymbol{z},\end{aligned}$$

the last inequality together with the definition of logarithmic mean imply

$$\begin{aligned}\int_{E_n}f\,(X\boldsymbol{u})\,d\mu\,(\boldsymbol{u}) \geqslant f\,(\boldsymbol{y})\int_{E_n}\exp\,(\boldsymbol{u}\cdot\boldsymbol{z})\,d\mu\,(\boldsymbol{u})\\ =f\,(\boldsymbol{y})\,\mathscr{L}\,(\exp\,(\boldsymbol{z})\,;\mu)\,.\end{aligned}$$

For the proof of (19) we utilize the first inequality in (4) to obtain

$$\begin{aligned}\mathscr{L}\,(\exp\,(\boldsymbol{z})\,;\mu) \geqslant \exp\,(\boldsymbol{w}\cdot\boldsymbol{z}) &= \exp\left[\sum_{i=0}^{n}w_i\left(\boldsymbol{x}^{(i)}-\boldsymbol{y}\right)\cdot\boldsymbol{c}\right]\\ &= \exp\left[\left(\sum_{i=0}^{n}w_i\boldsymbol{x}^{(i)}-\boldsymbol{y}\right)\cdot\boldsymbol{c}\right]\\ &= \exp\,(0\cdot\boldsymbol{c})=1,\end{aligned}$$

where 0 stands for the origin in $\mathbb{R}^k$. The proof is complete.

COROLLARY 1. *Let $k=n$. Then under the assumptions of Theorem 3, the following inequality*

$$\frac{1}{|vol\,(\sigma)|}\int_{\sigma}f\,(\boldsymbol{x})\,d\boldsymbol{x}\geqslant f\,(\boldsymbol{y})\,(n!\,[z_0,...,z_n]\,e^t) \tag{20}$$

holds true for any $\boldsymbol{y}\in Int\,(\sigma)$, where $[z_0,...,z_n]\,e^t$ stands for the divided difference of order n of the function e^t, $\boldsymbol{x}=(x_1,\dots,x_n)$ and $d\boldsymbol{x}=dx_1\dots dx_n$.

Proof. We let $\mu\,(\boldsymbol{u})=n!$ (the Lebesgue measure on E_n) in (18). In that case we have

$$\mathscr{L}\,(\exp\,(\boldsymbol{z})\,;\mu)=n!\,[z_0,...,z_n]\,e^t \tag{21}$$

(see [6, (4.21)]). To complete the proof of (20) we let $\boldsymbol{x}=X\boldsymbol{u}$ ($\boldsymbol{u}\in E_n$) in the integral on the left side of (18). Then

$$\boldsymbol{x}=\boldsymbol{x}^{(0)}+\sum_{i=1}^{n}v_i\left(\boldsymbol{x}^{(i)}-\boldsymbol{x}^{(0)}\right),$$

where $v_i=u_i$ for $1\leqslant i\leqslant n$. Hence

$$\boldsymbol{x}-\boldsymbol{x}^{(0)}=A\boldsymbol{v}, \tag{22}$$

where $A = \left[\boldsymbol{x}^{(1)} - \boldsymbol{x}^{(0)}, \ldots, \boldsymbol{x}^{(n)} - \boldsymbol{x}^{(0)}\right]$ and $\boldsymbol{v} = (v_1, \ldots, v_n)$. Since the vectors $\boldsymbol{x}^{(i)}$ $(0 \leqslant i \leqslant n)$ span a proper simplex σ in $\mathbb{R}^n$, the matrix A is nonsingular. Thus $\boldsymbol{v} = A^{-1}\left(\boldsymbol{x} - \boldsymbol{x}^{(0)}\right)$ and the Jacobian of the transformation (22) is equal to $\det(A^{-1})$. It is clear that the first member of (18) is equal to

$$n!|\det(A^{-1})| \int_\sigma f(\boldsymbol{x})\, d\boldsymbol{x}.$$

Since $\operatorname{vol}(\sigma) = \frac{1}{n!}\det(A)$,

$$n! \int_{E_n} f(X\boldsymbol{u})\, d\boldsymbol{u} = \frac{1}{|\operatorname{vol}(\sigma)|} \int_\sigma f(\boldsymbol{x})\, d\boldsymbol{x}.$$

Combining this with (18) and (21) gives the assertion (20).

REMARK 2. Let us note that the inequalities (2.2) follow from (20), (21) and (19) by letting $n = 1$, $\boldsymbol{x}^{(0)} = a$, $\boldsymbol{x}^{(1)} = b$ $(a \neq b)$ and $\boldsymbol{y} = \frac{a+b}{2}$.

3. Refinements of the Hermite-Hadamard's inequalities for the multivariate g-convex functions

In this section we shall give a generalization of Theorem 2 to the case of g-convex functions. Since r-convex functions are a special case of g-convex functions, the obtained results are applicable to these functions.

We need the following result.

LEMMA 1. *Let $f : \sigma \to (0,\infty)$ be a g-convex function on σ. Then for any $\boldsymbol{x}^{(0)}, \ldots, \boldsymbol{x}^{(n)} \in U$ and $\boldsymbol{u} \in E_n$ we have*

$$f\left(\sum_{i=0}^{n} u_i \boldsymbol{x}^{(i)}\right) \leqslant g^{-1}\left(\sum_{i=0}^{n} u_i (g \circ f)\left(\boldsymbol{x}^{(i)}\right)\right)$$

with the inequality reversed if f is g-concave.

Proof. This follows immediately from the definition of g-convexity.

THEOREM 4. *Let $f : \sigma \to \mathbb{R}$ and let g and h be strictly monotone continuous functions defined on the range of f. If f is g-convex, $h \circ g^{-1}$ is convex (concave) and h is increasing (decreasing), then for any probability measure μ on E_n*

$$\begin{aligned} f\left(\int_{E_n} (X\boldsymbol{u}) d\mu(\boldsymbol{u})\right) &\leqslant g^{-1}\left\{\int_{E_n} (g \circ f)(X\boldsymbol{u}) d\mu(\boldsymbol{u})\right\} \\ &\leqslant h^{-1}\left\{\int_{E_n} (h \circ f)(X\boldsymbol{u}) d\mu(\boldsymbol{u})\right\} \\ &\leqslant m_{h,g}(\boldsymbol{y};\mu) \leqslant m_{h,h}(\boldsymbol{y};\mu), \end{aligned} \tag{23}$$

where $X = [\boldsymbol{x}^{(0)} \cdots \boldsymbol{x}^{(n)}]$ is a k by $n+1$ matrix ($n \geqslant k$), $\boldsymbol{u} \in E_n$, and $\boldsymbol{y} = \left(f\left(\boldsymbol{x}^{(0)}\right), \ldots, f\left(\boldsymbol{x}^{(n)}\right)\right)$. If f is g-concave, $h \circ g^{-1}$ is convex (concave) and h is decreasing (increasing), then the inequalities (23) are reversed.

Proof. We shall establish inequalities (23) when f is a g-convex function, h is increasing, and $h \circ g^{-1}$ is convex. Since the remaining cases can be proven using the argument presented below, their proofs are not included here. Suppose that the function g is increasing. It follows from Lemma 1 that the function $g \circ f$ is convex. Making use of the first inequality in (16), with ϕ replaced by $g \circ f$, we obtain

$$(g \circ f)\left(\int_{E_n} (X\boldsymbol{u})d\mu(u)\right) \leqslant \int_{E_n} (g \circ f)(X\boldsymbol{u})d\mu(\boldsymbol{u}).$$

Hence the first inequality in (23) follows. For the proof of the second inequality in (23) we use Jensen's inequality for integrals. Since $h \circ g^{-1}$ is convex, the latter give,

$$(h \circ g^{-1})\left[\int_{E_n} (g \circ f)(X\boldsymbol{u})d\mu(\boldsymbol{u})\right] \leqslant \int_{E_n} (h \circ f)(X\boldsymbol{u})d\mu(\boldsymbol{u}).$$

Since h is an increasing function, the assertion follows. In order to establish the third inequality in (23) we use the fact that h is increasing together with Lemma 1 and (8) to obtain

$$\begin{aligned} h^{-1}\left[\int_{E_n} (h \circ f)(X\boldsymbol{u})d\mu(\boldsymbol{u})\right] &= h^{-1}\left[\int_{E_n} h\left(f\left(X\boldsymbol{u}\right)\right) d\mu(\boldsymbol{u})\right] \\ \leqslant h^{-1}\left[\int_{E_n} h\left(g^{-1}\left(\sum_{i=0}^{n} u_i (g \circ f)\big(\boldsymbol{x}^{(i)}\big)\right)\right) d\mu(\boldsymbol{u})\right] &= m_{h,g}(\boldsymbol{y};\mu). \end{aligned}$$

The last inequality in (23) can be proven as follows. We use the fact that $h \circ g^{-1}$ is convex and apply Jensen's inequality for sums to obtain

$$(h \circ g^{-1})\left[\sum_{i=0}^{n} u_i (g \circ f)\big(\boldsymbol{x}^{(i)}\big)\right] \leqslant \sum_{i=0}^{n} u_i (h \circ f)\big(\boldsymbol{x}^{(i)}\big).$$

Integrating both sides of the last inequality against the probability measure μ and next using (2) we obtain

$$\begin{aligned} \int_{E_n} (h \circ g^{-1})\left[\sum_{i=0}^{n} u_i (g \circ f)\big(\boldsymbol{x}^{(i)}\big)\right] d\mu(\boldsymbol{u}) &\leqslant \int_{E_n} \sum_{i=0}^{n} u_i (h \circ f)(\boldsymbol{x}^{(i)}) d\mu(\boldsymbol{u}) \\ &= \sum_{i=0}^{n} w_i (h \circ f)\big(\boldsymbol{x}^{(i)}\big) = h\big(m_{h,h}(\boldsymbol{y};\mu)\big). \end{aligned}$$

Since h is an increasing function, the assertion follows. This completes the proof.

Before we will state and prove a corollary of Theorem 4, let us introduce more notation. Let $\boldsymbol{w} = (w_0, \ldots, w_n)$ denote a vector of natural weights defined in (2) and let $X\boldsymbol{w} = \sum_{i=0}^{n} w_i x^{(i)}$. For $\boldsymbol{u} \in E_n$, let $F(\boldsymbol{u}) = f(X\boldsymbol{u})$ and let $\boldsymbol{y}$ have the same meaning as in Theorem 4.

COROLLARY 2. *Let $f : \sigma \to (0,\infty)$ be an r-convex function and let p and r be real numbers. If $p \geqslant r$, then for any probability measure μ on E_n the following inequalities*

$$\begin{aligned} f(X\boldsymbol{w}) &\leqslant \overline{M}_r(F;\mu) \leqslant \overline{M}_p(F;\mu) \\ &\leqslant E_{r,p+r}(\boldsymbol{y};\mu) \leqslant M_p(\boldsymbol{y};\boldsymbol{w}) \end{aligned} \tag{24}$$

hold true. If f is r-concave and $p \leqslant r$, then the inequalities (24) are reversed.

Proof. Let the functions h and g be the same as in (1.10) and (1.11), respectively. If $p \neq 0$, then

$$(h \circ g^{-1})(x) = \begin{cases} x^{\frac{p}{r}}, & \text{if } r \neq 0 \\ e^{px}, & \text{if } r = 0. \end{cases} \tag{25}$$

When $p = 0$, then

$$(h \circ g^{-1})(x) = \begin{cases} \frac{1}{r}\ln x, & \text{if } r \neq 0 \\ x, & \text{if } r = 0. \end{cases} \tag{26}$$

We shall establish the inequalities (24) when the function f is r-convex and $p \geqslant r$. The case when f is r-concave and $p \leqslant r$ can be proven in a similar way. First, consider the case when $r > 0$. Taking into account that

$$\int_{E_n} (X\boldsymbol{u})d\mu(\boldsymbol{u}) = X\boldsymbol{w}$$

we obtain from (23), (8), and (7)

$$\begin{aligned} f(X\boldsymbol{w}) &\leqslant \left[\int_{E_n} f^r(X\boldsymbol{u})d\mu(\boldsymbol{u})\right]^{1/r} \leqslant \left[\int_{E_n} f^p(X\boldsymbol{u})d\mu(\boldsymbol{u})\right]^{1/p} \\ &\leqslant E_{r,p+r}(\boldsymbol{y};\mu) \leqslant M_p(\boldsymbol{y};\boldsymbol{w}). \end{aligned}$$

Application of (6) to the second and third members in the above chain of inequalities completes the proof of (24) in the case under discussion. Assume now that $r < 0$. It follows from (1.11) that the function g is decreasing. If $p < 0$, then the function h is decreasing on $(0,\infty)$ and $h \circ g^{-1}$ is concave. Inequalities (24) follow immediately from Theorem 4. Similarly, if $p > 0$, then the function h is increasing and $h \circ g^{-1}$ is convex. Again, we invoke Theorem 4 to obtain inequalities (24). Finally, if $p = 0$, then h is strictly increasing and $h \circ g^{-1}$ is convex. The latter statement follows easily from (26). Making use of Theorem 4 we obtain

$$\begin{aligned} f(X\boldsymbol{w}) &\leqslant \left[\int_{E_n} f^r(X\boldsymbol{u})d\mu(\boldsymbol{u})\right]^{1/r} \leqslant \exp\left[\int_{E_n} \ln f(X\boldsymbol{u})d\mu(\boldsymbol{u})\right] \\ &\leqslant E_{r,r}(\boldsymbol{y};\mu) \leqslant M_0(\boldsymbol{y};\boldsymbol{w}). \end{aligned}$$

Application of (6) to the second and third members completes the proof of (24) when $r < 0$ and $p = 0$. Consider now the case when $r = 0$. This implies that $p \geqslant 0$. If

$p > 0$, then both functions h and g are strictly increasing and $h \circ g^{-1}$ is convex (see (25)). In this case the inequalities (23) of Theorem 4 become

$$f(X\boldsymbol{w}) \leqslant \exp\left[\int_{E_n} \ln f(X\boldsymbol{u}) d\mu(\boldsymbol{u})\right] \leqslant \left[\int_{E_n} f^p(X\boldsymbol{u}) d\mu(\boldsymbol{u})\right]^{1/p}$$
$$\leqslant E_{0,p}(\boldsymbol{y};\mu) \leqslant M_p(\boldsymbol{y};\boldsymbol{w}).$$

To complete the proof it suffices to consider the case when $p = 0$. It follows from (1.10), (1.11), and (26) that the functions h and g are both strictly increasing and $h \circ g^{-1}$ is convex. Making use of Theorem 4 we obtain

$$f(X\boldsymbol{w}) \leqslant \exp\left[\int_{E_n} \ln f(X\boldsymbol{u}) d\mu(\boldsymbol{u})\right]$$
$$\leqslant E_{0,0}(\boldsymbol{y};\mu) = M_0(\boldsymbol{y};\boldsymbol{w}).$$

This completes the proof.

REMARK 3. If in Corollary 2 we let $r = 0$ and $p = 1$, then the inequalities (24) become the inequalities (17) of Theorem 2.

REFERENCES

[1] B. C. CARLSON, *Special Functions of Applied Mathematics*, Academic Press, New York, 1977.

[2] S. S. DRAGOMIR, *Refinements of the Hermite-Hadamard integral inequality for log-convex functions*, Austral. Math. Gazette **28**, (2001), 129–134.

[3] S. S. DRAGOMIR, B. MOND, *Integral inequalities of Hadamard's type for log-convex functions*, Demonstratio Math. **31**, 2 (1998), 354–364.

[4] S. S. DRAGOMIR, C. E. M. PEARCE, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000.
(Online: http://rgmia.vu.edu.au/monographs/hermite_hadamard.html).

[5] E. NEUMAN, *Inequalities involving multivariate convex functions II*, Proc. Amer. Math. Soc., **109**, (1990), 965–974.

[6] E. NEUMAN,*The weighted logarithmic mean*, J. Math. Anal. Appl., **188**, (1994), 885–900.

[7] E. NEUMAN, J. SÁNDOR, *Inequalities involving Stolarsky and Gini means*, Math. Pannonica, **14**, (2003), 29–44.

[8] E. NEUMAN, C. E. M. PEARCE, J. PEČARIĆ AND V. ŠIMIĆ, *The generalized Hadamard's inequality, g-convexity and functional Stolarsky means*, Bull. Austral. Math. Soc.**68**, (2003), 303–316.

[9] C. E. M. PEARCE, J. PEČARIĆ AND V. ŠIMIĆ, *Stolarsky means and Hadamard's inequality*, J. Math. Anal. Appl., **220**, (1998), 99–109.

[10] C. E. M. PEARCE, J. PEČARIĆ AND V. ŠIMIĆ, *Functional Stolarsky Means*, Mathematical Inequalities and Applications **2**, 4 (1999), 479–489.

[11] J. E. PEČARIĆ, F. PROSCHAN AND Y. L. TONG, *Convex Functions, Partial Orderings and Statistical Applications*, Academic Press, Boston, 1992.

[12] J. PEČARIĆ, V. ŠIMIĆ, *Stolarsky-Tobey mean in n variables*, Mathematical Inequalities and Applications **2**, 3 (1999), 325–341.

(Received October 7, 2004)

M. Klaričić Bakula
Department of Mathematics
Faculty of Natural Sciences, Mathematics and Education
University of Split
Teslina 12
21000 Split
Croatia
e-mail: milica@pmfst.hr

E. Neuman
Department of Mathematics
Southern Illinois University
Carbondale
IL 62901-4408
USA
e-mail: edneuman@math.siu.edu

J. Pečarić
Faculty of Textile Technology
University of Zagreb
Pierottijeva 6, 10000 Zagreb
Croatia
e-mail: pecaric@hazu.hr

Vida Šimić
Faculty of Textile Technology
University of Zagreb
Pierottijeva 6
11000 Zagreb
Croatia
e-mail: vidasim@ztk.ttf.hr