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ON THE BEST CONSTANT IN HILBERT'S INEQUALITY

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Dedicated to the memory
of prof. Mladen Alić

(*communicated by L. Pick*)

Abstract. The main objective of this paper is a study of some new generalizations of Hilbert's type inequalities. More precisely, we obtain, in some general cases, that the constants involved in the right-hand sides of such inequalities are the best possible.

1. Introduction

The Hardy-Hilbert's type inequalities are of some significant weight inequalities which play an important role in analysis and it's applications. So, at the beginning let us recall the famous Hilbert's theorems for double series: Let $\{a_m\}$ and $\{b_n\}$ be two non-negative sequences and $\frac{1}{p}+\frac{1}{q}=1,\ p>1$. Then the following inequalities hold

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n}\leqslant\frac{\pi}{\sin\frac{\pi}{p}}\left(\sum_{m=1}^{\infty}{a_m}^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}{b_n}^q\right)^{\frac{1}{q}} \tag{1}$$

and

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_m b_n}{m+n+1}\leqslant\frac{\pi}{\sin\frac{\pi}{p}}\left(\sum_{m=0}^{\infty}{a_m}^p\right)^{\frac{1}{p}}\left(\sum_{n=0}^{\infty}{b_n}^q\right)^{\frac{1}{q}}, \tag{2}$$

where the constant factor $\frac{\pi}{\sin\frac{\pi}{p}}$, contained in (1) and (2), is the best possible. Although classical, they are field of interest of numerous mathematicians and were generalized in many different ways. For more details see [14].

Very recently, Brnetić and Pečarić ([12], [13]) gave some further generalizations of Hardy-Hilbert's inequality. So we shall state their result that will take our attention. They considered special case $n=2$ ([13]), and obtained the following result in both equivalent forms

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THEOREM A. *If* $\lambda > 0$, $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, *then the following inequalities hold and are equivalent*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}dxdy < L\left(\int_0^\infty x^{1-\lambda+p(A_1-A_2)}f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty x^{1-\lambda+q(A_2-A_1)}g^q(x)dx\right)^{\frac{1}{q}}, \tag{3}$$

and

$$\int_0^\infty y^{(\lambda-1)(p-1)+p(A_1-A_2)}\left(\int_0^\infty \frac{f(x)}{(x+y)^\lambda}dx\right)^p dy < L^p\left(\int_0^\infty x^{1-\lambda+p(A_1-A_2)}f^p(x)dx\right) \tag{4}$$

where $L=(B(1-A_2p,\lambda-1+A_2p))^{\frac{1}{p}}\,(B(1-A_1q,\lambda-1+A_1q))^{\frac{1}{q}}$, $A_1\in\left(\frac{1-\lambda}{q},\frac{1}{q}\right)$, $A_2\in\left(\frac{1-\lambda}{p},\frac{1}{p}\right)$ *and* B *is a beta function. If* $0<p<1$ *then the reverse inequalities in (3) and (4) are valid for any* $A_1\in\left(\frac{1}{q},\frac{1-\lambda}{q}\right)$ *and* $A_2\in\left(\frac{1-\lambda}{p},\frac{1}{p}\right)$. *The inequality (4) holds also if* $p<0$.

In this paper we shall obtain some generalizations of Theorem A and also consider the constants involved in the right-hand sides of mentioned inequalities. The main purpose of this paper is to show that, in some cases, the constants are the best possible. Techniques that will be used in the proofs are mainly based on classical real analysis. Further, we shall also use, in discrete case, some general results on Hilbert's inequality from [19].

2. The best constants

In this section we shall obtain that the constants L and L^p involved in the right-hand sides of the inequalities (3) and (4) are the best possible for some choices of the parameters A_1 and A_2.

Let's suppose that the parameters A_1 and A_2 satisfy condition $pA_2+qA_1=2-\lambda$. We shall prove that for such choice of parameters A_1 and A_2, the constant L in the inequality (3) is the best possible.

THEOREM 1. *If the parameters* A_1 *and* A_2 *satisfy condition* $pA_2+qA_1=2-\lambda$, *then the constant* L *in Theorem A is the best possible.*

Proof. For this purpose, with $0<\varepsilon<1$, set $f(x)=x^{-qA_1}$ in $\langle\,\varepsilon,\frac{1}{\varepsilon}\rangle$, $f(x)=0$ elsewhere, and $g(y)=y^{-pA_2}$ in $\langle\,\varepsilon,\frac{1}{\varepsilon}\rangle$, $g(y)=0$ elsewhere. Then the left-hand side of the inequality (3) is

$$I=\int_\varepsilon^{\frac{1}{\varepsilon}}\int_\varepsilon^{\frac{1}{\varepsilon}}\frac{x^{-qA_1}y^{-pA_2}}{(x+y)^\lambda}dxdy=\int_\varepsilon^{\frac{1}{\varepsilon}}\frac{dx}{x}\int_{\frac{\varepsilon}{x}}^{\frac{1}{\varepsilon x}}\frac{u^{-pA_2}}{(1+u)^\lambda}du.$$

Further,

$$I = 2\ln\left(\frac{1}{\varepsilon}\right)L - R_1 - R_2,$$

where

$$R_1 = \int_{\varepsilon}^{\frac{1}{\varepsilon}} \frac{dx}{x}\int_0^{\frac{\varepsilon}{x}} \frac{u^{-pA_2}}{(1+u)^{\lambda}}du < \int_{\varepsilon}^{\frac{1}{\varepsilon}} \frac{dx}{x}\int_0^{\frac{\varepsilon}{x}} u^{-pA_2}du = \frac{1-\varepsilon^{\lambda+qA_1-pA_2}}{(1-pA_2)^2},$$

and

$$R_2 = \int_{\varepsilon}^{\frac{1}{\varepsilon}} \frac{dx}{x}\int_{\frac{1}{\varepsilon x}}^{\infty} \frac{u^{-pA_2}}{(1+u)^{\lambda}}du < \int_{\varepsilon}^{\frac{1}{\varepsilon}} \frac{dx}{x}\int_{\frac{1}{\varepsilon x}}^{\infty} u^{-pA_2-\lambda}du = \frac{1-\varepsilon^{\lambda-qA_1+pA_2}}{(1-qA_1)^2},$$

so we obtain the inequality

$$I > 2\ln\left(\frac{1}{\varepsilon}\right)L - \frac{1-\varepsilon^{\lambda+qA_1-pA_2}}{(1-pA_2)^2} - \frac{1-\varepsilon^{\lambda-qA_1+pA_2}}{(1-qA_1)^2}.$$

Now, let us suppose that there exist a smaller constant $0 < C < L$ such that the inequality (3) is valid. Then the right-hand side of the inequality (3) is equal to $2\ln\left(\frac{1}{\varepsilon}\right)C$. It follows that

$$-\frac{1-\varepsilon^{\lambda+qA_1-pA_2}}{(1-pA_2)^2} - \frac{1-\varepsilon^{\lambda-qA_1+pA_2}}{(1-qA_1)^2} < 2(C-L)2\ln\left(\frac{1}{\varepsilon}\right),$$

and we obtain contradiction by letting $\varepsilon \searrow 0$.
It remains to prove that L is also the best possible value in the reverse inequality. By using the same notation as before, we have

$$I < 2\ln\left(\frac{1}{\varepsilon}\right)L$$

Now, let us suppose that there exist a greater constant $D > L$ such that the reverse inequality in (3) is valid. Then the right-hand side of that inequality is equal to $2\ln\left(\frac{1}{\varepsilon}\right)D$. It follows that

$$2\ln\left(\frac{1}{\varepsilon}\right)L > 2\ln\left(\frac{1}{\varepsilon}\right)D,$$

what is a contradiction, since $D > L$. That completes the proof. $\square$

REMARK 1. It is easy to see that for the parameters A_1 and A_2 from the previous theorem, the constant L becomes $L = B(1-A_2p, \lambda-1+A_2p) = B(1-A_1q, \lambda-1+A_1q)$. Further, if we put $A_2 - A_1 = \alpha$, where $\frac{2-\lambda}{p} - 1 < \alpha < \frac{\lambda-2}{q} + 1$, then from the condition $pA_2+qA_1=2-\lambda$ we obtain $A_1=\dfrac{2-\lambda-p\alpha}{pq}$ and $A_2=\dfrac{2-\lambda+q\alpha}{pq}$ and the constant L is given by

$$L^*=B\left(\frac{\lambda-2+p}{p}+\alpha,\frac{\lambda-2+q}{q}-\alpha\right). \tag{5}$$

REMARK 2. If $pA_2+qA_1=2-\lambda$, then the constant involved in the right-hand side of the inequality (4) is also the best possible since the inequalities (3) and (4) are equivalent.

3. The main results

In this section we shall generalize Theorem A in the following way. Let $u:(0,\infty)\mapsto\mathbb{R}$ and $v:(0,\infty)\mapsto\mathbb{R}$ be non-negative differentiable functions such that the functions $xu(x)$ and $yv(y)$ are strictly increasing and $\lim\limits_{x\to\infty}xu(x)=\lim\limits_{y\to\infty}yv(y)=\infty$. Since $xu(x)$ and $yv(y)$ are strictly increasing, it follows that the functions $u(x)+xu'(x)$ and $v(y)+yv'(y)$ are non-negative.

Now, we generalize Theorem A by using the substitution $x=ru(r)$ and $y=sv(s)$. Hence, the constants involved in the right-hand sides of inequalities will be the best possible in some cases (see Theorem 2).

THEOREM 2. *Let* $\frac{1}{p}+\frac{1}{q}=1$, *with* $p>1$, *and* $f(x)$, $g(y)$ *be non-negative functions. Then the following inequalities hold and are equivalent*

$$\begin{aligned}&\int_0^\infty\int_0^\infty\frac{f(x)g(y)}{(xu(x)+yv(y))^\lambda}dxdy\\&\quad\leqslant L\left(\int_0^\infty(xu(x))^{1-\lambda+p(A_1-A_2)}\left(u(x)+xu'(x)\right)^{1-p}f^p(x)dx\right)^{\frac{1}{p}}\times\\&\qquad\times\left(\int_0^\infty(yv(y))^{1-\lambda+q(A_2-A_1)}\left(v(y)+yv'(y)\right)^{1-q}g^q(y)dy\right)^{\frac{1}{q}}\end{aligned} \tag{6}$$

and

$$\begin{aligned}&\int_0^\infty(yv(y))^{(\lambda-1)(p-1)+p(A_1-A_2)}\left(v(y)+yv'(y)\right)\left(\int_0^\infty\frac{f(x)}{(xu(x)+yv(y))^\lambda}\right)^p dy\\&\quad\leqslant L^p\int_0^\infty(xu(x))^{1-\lambda+p(A_1-A_2)}\left(u(x)+xu'(x)\right)^{1-p}f^p(x)dx,\end{aligned} \tag{7}$$

for any $A_1\in\left(\frac{1-\lambda}{q},\frac{1}{q}\right)$, $A_2\in\left(\frac{1-\lambda}{p},\frac{1}{p}\right)$, *where the constant* L *is defined in Theorem A. If* $0<p<1$ *then the reverse inequalities in (6) and (7) are valid for any* $A_1\in\left(\frac{1}{q},\frac{1-\lambda}{q}\right)$, $A_2\in\left(\frac{1-\lambda}{p},\frac{1}{p}\right)$. *The inequality (7) holds also if* $p<0$.

Further, if the parameters A_1 and A_2 satisfy condition $pA_2 + qA_1 = 2 - \lambda$ then the constants involved in the right-hand sides of the inequalities (6) and (7) and their's reverses are the best possible.

We also give the result in discrete case. If we use the general result from [19], we obtain

THEOREM 3. *Let $\frac{1}{p} + \frac{1}{q} = 1$, with $p > 1$, and $\{a_n\}$, $\{b_n\}$ be non-negative real sequences. Then the following inequalities hold and are equivalent*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m b_n}{(mu(m) + nv(n))^{\lambda}}$$
$$\leqslant L \left(\sum_{m=1}^{\infty} (mu(m))^{1-\lambda+p(A_1-A_2)} \left(u(m) + mu'(m)\right)^{1-p} a_m{}^p \right)^{\frac{1}{p}} \times$$
$$\times \left(\sum_{n=1}^{\infty} (nv(n))^{1-\lambda+q(A_2-A_1)} \left(v(n) + nv'(n)\right)^{1-q} b_n{}^q \right)^{\frac{1}{q}} \tag{8}$$

and

$$\sum_{n=1}^{\infty} (nv(n))^{(\lambda-1)(p-1)+p(A_1-A_2)} \left(v(n) + nv'(n)\right) \left(\sum_{m=1}^{\infty} \frac{a_m}{(mu(m) + nv(n))^{\lambda}} \right)^p$$
$$\leqslant L^p \sum_{m=1}^{\infty} (mu(m))^{1-\lambda+p(A_1-A_2)} \left(u(m) + mu'(m)\right)^{1-p} a_m{}^p, \tag{9}$$

for any $A_1 \in \left(\max\left\{ \frac{1-\lambda}{q}, 0 \right\}, \frac{1}{q} \right)$ and $A_2 \in \left(\max\left\{ \frac{1-\lambda}{p}, 0 \right\}, \frac{1}{p} \right)$, where the constant L is defined in Theorem 5. Further, if the parameters A_1 and A_2 satisfy condition $pA_2 + qA_1 = 2 - \lambda$ then the constants L and L^p are the best possible.

Proof. By using the general result from [19] we obtain the inequality

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m b_n}{(mu(m) + nv(n))^{\lambda}} \leqslant \left(\sum_{m=1}^{\infty} \Omega_p(m) a_m{}^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} \Omega_q(n) b_n{}^q \right)^{\frac{1}{q}},$$

where $\Omega_p(m) = \left(u(m)+mu'(m)\right)^{1-p} \sum_{n=1}^{\infty} \frac{v(n)+nv'(n)}{(mu(m)+nv(n))^{\lambda}} \times \frac{(mu(m))^{pA_1}}{(nv(n))^{pA_2}}$ and $\Omega_q(n) = \left(v(n)+nv'(n)\right)^{1-q} \sum_{m=1}^{\infty} \frac{u(m)+mu'(m)}{(mu(m)+nv(n))^{\lambda}} \times \frac{(nv(n))^{qA_2}}{(mu(m))^{qA_1}}$. Now, since the functions $mu(m)$ and $nv(n)$ are strictly increasing, one easily obtains that
$\Omega_p(m) \leqslant (mu(m))^{1-\lambda+p(A_1-A_2)} \left(u(m)+mu'(m)\right)^{1-p} B(1-pA_2, \lambda-1+pA_2)$, and
$\Omega_q(n) \leqslant (nv(n))^{1-\lambda+q(A_2-A_1)} \left(v(n)+nv'(n)\right)^{1-q} B(1-qA_1, \lambda-1+qA_1)$, hence we obtain (8).

It remains to prove that the constant L is the best possible if $pA_2 + qA_1 = 2 - \lambda$. Let $a_m = (mu(m))^{-qA_1 - \frac{\varepsilon}{p}} (u(m) + mu'(m))$ and $b_n = (nv(n))^{-pA_2 - \frac{\varepsilon}{q}} (v(n) + nv'(n))$. Since the function $xu(x)$ is strictly decreasing in $(0, \infty)$ we have

$$\begin{aligned}\frac{1}{\varepsilon} &= \int_1^\infty (xu(x))^{-1-\varepsilon} d(xu(x)) < \sum_{m=1}^\infty (mu(m))^{-1-\varepsilon}(u(m) + mu'(m)) \\ &= \sum_{m=1}^\infty (mu(m))^{1-\lambda+p(A_1-A_2)}(u(m) + mu'(m))^{1-p} a_m{}^p \\ &\quad < \varphi_1(1) + \int_1^\infty (xu(x))^{-1-\varepsilon} d(xu(x)) = \varphi_1(1) + \frac{1}{\varepsilon},\end{aligned}$$

where the function φ_1 is defined by $\varphi_1(x) = (xu(x))^{-1-\varepsilon}(u(x) + xu'(x))$. Hence we obtain $\sum_{m=1}^\infty (mu(m))^{1-\lambda+p(A_1-A_2)} \left(u(m) + mu'(m)\right)^{1-p} a_m{}^p = \frac{1}{\varepsilon} + O(1)$, and similarly $\sum_{n=1}^\infty (nv(n))^{1-\lambda+q(A_2-A_1)} \left(v(n) + nv'(n)\right)^{1-q} b_n{}^q = \frac{1}{\varepsilon} + O(1)$.

Now, let us suppose that there exist a smaller constant $0 < C < L$ such that the inequality (8) is valid. By putting a_m and b_n in (8) we obtain

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{(mu(m) + nv(n))^\lambda} < \frac{1}{\varepsilon}(C + o(1)). \tag{10}$$

On the other hand we have

$$\begin{aligned}&\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{(mu(m) + nv(n))^\lambda} \\ &\qquad > \int_1^\infty \int_1^\infty \frac{(xu(x))^{-qA_1 - \frac{\varepsilon}{p}} (yv(y))^{-pA_2 - \frac{\varepsilon}{q}}}{(xu(x) + yv(y))^\lambda} d(xu(x)) d(yv(y)).\end{aligned}$$

So, the right-hand side is equal to

$$\int_1^\infty \left(\int_{\frac{1}{xu(x)}}^\infty \frac{t^{-pA_2 - \frac{\varepsilon}{q}}}{(1+t)^\lambda} dt \right) (xu(x))^{-1-\varepsilon} d(xu(x)).$$

Now, since

$$\int_{\frac{1}{xu(x)}}^\infty \frac{t^{-pA_2 - \frac{\varepsilon}{q}}}{(1+t)^\lambda} dt > \int_0^\infty \frac{t^{-pA_2 - \frac{\varepsilon}{q}}}{(1+t)^\lambda} dt - \int_0^{\frac{1}{xu(x)}} t^{-pA_2 - \frac{\varepsilon}{q} - \lambda} dt,$$

after computing, we obtain

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(mu(m)+nv(n))^{\lambda}} > \frac{1}{\varepsilon}\left(L+o(1)\right). \tag{11}$$

Clearly, when ε is small enough, the inequality (10) is in contradiction with (11) and the proof is completed. □

As we already saw, if the parameters A_1 and A_2 satisfy condition $pA_2+qA_1=2-\lambda$, then the constants involved in the right-hand side of our inequalities are the best possible. Now, we observe the discrete case when $A_1=A_2=\dfrac{2-\lambda}{pq}$ and obtain some extensions of Hilbert's theorem for double series. Then $L=B\left(\dfrac{\lambda-2+p}{p},\dfrac{\lambda-2+q}{q}\right)$, and we define $B^*:=B\left(\dfrac{\lambda-2+p}{p},\dfrac{\lambda-2+q}{q}\right)$.

COROLLARY 1. *Let $\{a_n\}$ and $\{b_n\}$ be two non-negative sequences of real numbers and $2-\min\{p,q\}<\lambda\leqslant 2$. Then the following inequalities hold and are equivalent*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(mu(m)+nv(n))^{\lambda}} \leqslant B^*\left(\sum_{m=1}^{\infty}(mu(m))^{1-\lambda}\left(u(m)+mu'(m)\right)^{1-p}{a_m}^p\right)^{\frac{1}{p}}\times \times\left(\sum_{n=1}^{\infty}(nv(n))^{1-\lambda}\left(v(n)+nv'(n)\right)^{1-q}{b_n}^q\right)^{\frac{1}{q}}$$

and

$$\sum_{n=1}^{\infty}(nv(n))^{(\lambda-1)(p-1)}\left(v(n)+nv'(n)\right)\left(\sum_{m=1}^{\infty}\frac{a_m}{(mu(m)+nv(n))^{\lambda}}\right)^p \leqslant (B^*)^p\sum_{m=1}^{\infty}(mu(m))^{1-\lambda}\left(u(m)+mu'(m)\right)^{1-p}{a_m}^p$$

where the constants B^* and $(B^*)^p$ are the best possible.

REMARK 3. Note that for $\lambda=1$, B^* becomes $\dfrac{\pi}{\sin\frac{\pi}{p}}$ and we obtain generalization of the inequality (1).

In the Theorems 3 and 4 we didn't consider the constant factors of the functions $xu(x)$, $yv(y)$, $u(x)+xu'(x)$ and $v(y)+yv'(y)$. Now, let the constants factors of $xu(x)$, $yv(y)$, $u(x)+xu'(x)$ and $v(y)+yv'(y)$ are in turn A,B,C and D. Then they can be written in form

$$xu(x)=A\widetilde{u}(x),\ yv(y)=B\widetilde{v}(y),\ u(x)+xu'(x)=C\widetilde{u}'(x) \text{ and } v(y)+yv'(y)=D\widetilde{v}'(y).$$

We define

$$\mu = \left(\frac{A^{1-\lambda}}{D}\right)^{\frac{1}{p}} \left(\frac{B^{1-\lambda}}{C}\right)^{\frac{1}{q}} \left(\frac{A}{B}\right)^{A_1-A_2} \tag{12}$$

and obtain the following important result in the discrete case

THEOREM 4. *Let* $\frac{1}{p}+\frac{1}{q}=1$, *with* $p>1$, *and* $\{a_n\}$, $\{b_n\}$ *be non-negative real sequences. Then the following inequalities hold and are equivalent*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(mu(m)+nv(n))^{\lambda}} \leqslant \mu L\left(\sum_{m=1}^{\infty}(\widetilde{u}(m))^{1-\lambda+p(A_1-A_2)}\left(\widetilde{u}'(m)\right)^{1-p}{a_m}^{p}\right)^{\frac{1}{p}} \times \left(\sum_{n=1}^{\infty}(\widetilde{v}(n))^{1-\lambda+q(A_2-A_1)}\left(\widetilde{v}'(n)\right)^{1-q}{b_n}^{q}\right)^{\frac{1}{q}} \tag{13}$$

and

$$\sum_{n=1}^{\infty}(\widetilde{v}(n))^{(\lambda-1)(p-1)+p(A_1-A_2)}\left(\widetilde{v}'(n)\right)\left(\sum_{m=1}^{\infty}\frac{a_m}{(mu(m)+nv(n))^{\lambda}}\right)^{p} \leqslant \mu^p L^p\sum_{m=1}^{\infty}(\widetilde{u}(m))^{1-\lambda+p(A_1-A_2)}\left(\widetilde{u}'(m)\right)^{1-p}{a_m}^{p}, \tag{14}$$

for any $A_1\in\left(\max\left\{\frac{1-\lambda}{q},0\right\},\frac{1}{q}\right)$ *and* $A_2\in\left(\max\left\{\frac{1-\lambda}{p},0\right\},\frac{1}{p}\right)$.

Further, if the parameters A_1 *and* A_2 *satisfy condition* $pA_2+qA_1=2-\lambda$ *then the constants involved in the right-hand sides of the inequalities (13) and (14) are the best possible.*

REMARK 4. The integral analogue of Theorem 5 can be obtained in the same way, where the sums are replaced with the integrals and the non-negative real sequences with the non-negative real functions (see Theorem 3).

Now, we use Theorem 5 to obtain the generalization of the inequality (2), from the Introduction. For that purpose let's define two functions by

$$xu(x)=\begin{cases} a(x+\frac{c}{2a}) & x>0\\ \frac{c}{2} & x=0\end{cases} \quad \text{and} \quad yv(y)=\begin{cases} b(y+\frac{c}{2b}) & y>0\\ \frac{c}{2} & y=0\end{cases}, \tag{15}$$

where a, b and c are greater then zero. So if we put $A_1=A_2=\frac{2-\lambda}{pq}$, where $2-\min\{p,q\}<\lambda\leqslant 2$, the inequality (13) becomes

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_m b_n}{(am+bn+c)^{\lambda}}$$

$$\leqslant \mu^* B^* \left(\sum_{m=0}^{\infty}\left(m+\frac{c}{2a}\right)^{1-\lambda} {a_m}^p\right)^{\frac{1}{p}} \left(\sum_{n=0}^{\infty}\left(n+\frac{c}{2b}\right)^{1-\lambda} {b_n}^q\right)^{\frac{1}{q}}, \tag{16}$$

where $\mu^* = \left(a^{2-\lambda-p}\right)^{\frac{1}{p}}\left(b^{2-\lambda-q}\right)^{\frac{1}{q}}$. This is obviously a generalization of the inequality (2). We can also obtain more general inequalities without the constraint $A_1 = A_2$, but here they are omitted. It follows from (16) that the result of the paper [7] is yielded immediately. Actually, the various results in the papers [6]-[11] might be yielded from the inequalities (8) and (16).

4. Some applications in the discrete case

There are lots of applications of the inequalities from the previous section. In this section we shall enumerate only the discrete cases for which $u(x)$ and $v(y)$ are power function, logarithm function, inverse trigonometric function and the exponent function. We make such specifications in Theorems 4 and 5. We observe only the cases for which the parameters A_1 and A_2 satisfy constraint $pA_2 + qA_1 = 2-\lambda$. So, we put $\alpha = A_2 - A_1$, where

$$\max\left\{\frac{1-\lambda}{p},0\right\}-\frac{1}{q}<\alpha<\frac{1}{p}-\max\left\{\frac{1-\lambda}{q},0\right\}, \tag{17}$$

and the constant factor will be given by (5). In all the cases that follows the constant factors will be the best possible.

Power function

Let $u(x) = x^a$ and $v(y) = y^b$ where a and b are greater than -1. Then $xu(x) = x^{a+1}$ and $yv(y) = y^{b+1}$. In according to (12), it is easy to duce that $\mu = (b+1)^{-\frac{1}{p}}(a+1)^{-\frac{1}{q}}$. Hence, by Theorem 5, we have the following result

COROLLARY 2. *Let* $\frac{1}{p}+\frac{1}{q}=1$, *with* $p>1$, *and* α *be real parameter defined by (17). Then the following inequalities hold and are equivalent*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{\left(m^{a+1}+n^{b+1}\right)^{\lambda}}$$

$$\leqslant \mu L^* \left(\sum_{m=1}^{\infty} m^{(1-\lambda)(a+1)+a(1-p)-ap\alpha} {a_m}^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} n^{(1-\lambda)(b+1)+b(1-q)+bq\alpha} {b_n}^q\right)^{\frac{1}{q}}$$

and

$$\sum_{n=1}^{\infty} n^{(b+1)(\lambda-1)(p-1)+b-pb\alpha}\left(\sum_{m=1}^{\infty}\frac{a_m}{(m^{a+1}+n^{b+1})^{\lambda}}\right)^p \leqslant \mu^p (L^*)^p \sum_{m=1}^{\infty} m^{(1-\lambda)(a+1)+a(1-p)-ap\alpha} a_m{}^p,$$

where the constants involved in the right-hand sides of the inequalities are the best possible.

REMARK 5. In particular, if $\alpha = 0$, then $L^* = B^*$.

Logarithm function

Let $xu(x) = \ln(1+x)$ and $yv(y) = \ln(1+y)$. Then $(xu(x))' = \dfrac{1}{1+x}$ and $(yv(y))' = \dfrac{1}{1+y}$. It is known from (12) that $\mu = 1$ and according to Theorem 4 we have

COROLLARY 3. *Let* $\frac{1}{p}+\frac{1}{q}=1$, *with* $p>1$, *and* α *be real parameter defined by (17). Then the following inequalities hold and are equivalent*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(\ln(1+m)+\ln(1+n))^{\lambda}} \leqslant L^*\left(\sum_{m=1}^{\infty}(\ln(1+m))^{1-\lambda-p\alpha}(1+m)^{p-1}a_m{}^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}(\ln(1+n))^{1-\lambda+q\alpha}(1+n)^{q-1}b_n{}^q\right)^{\frac{1}{q}}$$

and

$$\sum_{n=1}^{\infty}(\ln(n+1))^{(\lambda-1)(p-1)-p\alpha}(1+n)^{-1}\left(\sum_{m=1}^{\infty}\frac{a_m}{(\ln(1+m)+\ln(1+n))^{\lambda}}\right)^p \leqslant (L^*)^p\sum_{m=1}^{\infty}(\ln(1+m))^{1-\lambda-p\alpha}(1+m)^{p-1}a_m{}^p,$$

where the constant factors L^* *and* $(L^*)^p$ *are the best possible.*

Inverse trigonometric function

We enumerate here only the cases for which $u(x)$ and $v(y)$ are inverse tangens function. Let $u(x) = \mathrm{arctg}x$ and $v(y) = \mathrm{arctg}y$. Define two functions by

$$\omega_p(x) = (x\mathrm{arctg}x)^{1-\lambda-p\alpha}\left(\mathrm{arctg}x + \frac{x}{1+x^2}\right)^{1-p}$$

and

$$\omega_q(y) = (y\mathrm{arctg}y)^{1-\lambda+q\alpha}\left(\mathrm{arctg}y + \frac{y}{1+y^2}\right)^{1-q}.$$

By Theorem 4 we have

COROLLARY 4. *Let* $\frac{1}{p}+\frac{1}{q}=1$, *with* $p>1$, *and* α *be real parameter defined by (17). Then the following inequalities hold and are equivalent*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m\mathrm{arctg}m+n\mathrm{arctg}n)^{\lambda}}\leqslant L^{*}\left(\sum_{m=1}^{\infty}\omega_p(m){a_m}^{p}\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}\omega_q(n){b_n}^{q}\right)^{\frac{1}{q}}$$

and

$$\sum_{n=1}^{\infty}(n\mathrm{arctg}n)^{(\lambda-1)(p-1)-p\alpha}\left(\mathrm{arctg}n+\frac{n}{1+n^2}\right)\left(\sum_{m=1}^{\infty}\frac{a_m}{(m\mathrm{arctg}m+n\mathrm{arctg}n)^{\lambda}}\right)^{p}\leqslant (L^{*})^{p}\sum_{m=1}^{\infty}\omega_p(m){a_m}^{p},$$

where the constant factors L^{*} *and* $(L^{*})^{p}$ *are the best possible.*

Exponent function

Let $u(x)=a^x$ and $v(y)=a^y$ where $a>1$. Then $xu(x)=xa^x$ and $yv(y)=ya^y$ and according to Theorem 4 we have the following result

COROLLARY 5. *Let* $\frac{1}{p}+\frac{1}{q}=1$, *with* $p>1$, *and* α *be real parameter defined by (17). Then the following inequalities hold and are equivalent*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(ma^m+na^n)^{\lambda}}\leqslant L^{*}\left(\sum_{m=1}^{\infty}(ma^m)^{1-\lambda-p\alpha}\left(a^m+ma^m\ln a\right)^{1-p}{a_m}^{p}\right)^{\frac{1}{p}}\times\left(\sum_{n=1}^{\infty}(na^n)^{1-\lambda+q\alpha}\left(a^n+na^n\ln a\right)^{1-p}{b_n}^{q}\right)^{\frac{1}{q}}$$

and

$$\sum_{n=1}^{\infty}(na^n)^{(\lambda-1)(p-1)-p\alpha}\left(a^n+na^n\ln a\right)\left(\sum_{m=1}^{\infty}\frac{a_m}{(ma^m+na^n)^{\lambda}}\right)^{p}\leqslant (L^{*})^{p}\sum_{m=1}^{\infty}(ma^m)^{1-\lambda-p\alpha}\left(a^m+ma^m\ln a\right)^{1-p}{a_m}^{p},$$

where the constant factors $L*$ *and* $(L^{*})^{p}$ *are the best possible.*

In a such way, a great deal of important inequalities might be established. Here they are omitted.

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