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DIFFERENCE DERIVED FROM WEIGHTED HÖLDER'S INEQUALITY

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Dedicated to the memory
of prof. Mladen Alić

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Abstract. We give the maximum of the difference

$$D_p(a,b;w) := \left(\sum_{k=1}^n w_k a_k^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^n w_k b_k^q\right)^{\frac{1}{q}} - \sum_{k=1}^n w_k a_k b_k$$

derived from a weighted Hölder's inequality for $p,q>1$, $p^{-1}+q^{-1}=1$ and for positive n-tuples $a:=(a_1,\dots,a_n)$, $b:=(b_1,\dots,b_n)$ and a weight $w:=(w_1,\dots,w_n)$ under certain conditions. The discussion in this note is simpler than our previous ones. It comes from the arrangement of a given weight and a linearization of $D_p(a,b;w)$ via Young's inequality. As a consequence, we give a, b and w which attain the maximum.

1. Introduction

Throughout this note, let $a:=(a_1,\dots,a_n)$ and $b:=(b_1,\dots,b_n)$ be n-tuples of positive numbers, and $w:=(w_1,\dots,w_n)$ be a weight. Suppose that

$$0<m_1 \leqslant a_k \leqslant M_1 \quad \text{and} \quad 0<m_2 \leqslant b_k \leqslant M_2 \quad (k=1,\dots,n).$$

Then we give the maximum of the difference

$$D_p(a,b;w) := \left(\sum_{k=1}^n w_k a_k^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^n w_k b_k^q\right)^{\frac{1}{q}} - \sum_{k=1}^n w_k a_k b_k$$

for $p,q>1$ with $p^{-1}+q^{-1}=1$, derived from a weighted Hölder's inequality, i.e., $D_p(a,b;w) \geqslant 0$.

In [1], an upper bound of $D_p(a,b;w)$ for $w=\left(\frac{1}{n},\dots,\frac{1}{n}\right)$ was given by using Ozeki's technique ([1], [2], [4], [5]). Moreover in [3] the estimation was shown to be the best possible in a reasonable sense by the minimax theorem which is implicitly discussed the maximum of $D_p(a,b;w)$.

In this note, we give the maximum of $D_p(a,b;w)$ by a simpler argument than previous ones in [1] and [3]. This simplification is based on a linearization of $D_p(a,b;w)$ via Young's inequality, see Lemma 1 in the next section.

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2. Maximum of the difference $D_p(a,b;w)$

Without loss of generality we may assume $M_1 = M_2 = 1$. Conveniently we write $m_1 = \alpha$ and $m_2 = \beta$ with $\alpha, \beta \in (0,1)$. Then $D_p(a,b;w)$ is considered as a function defined on the product of two n-dimensional cubes $[\alpha,1]^n$ and $[\beta,1]^n$. It follows from [3, Lemma 2.2] that $D_p(a,b;w)$ is convex in both a and b for a fixed weight w. Hence its maximum is attained at an extreme point of the definition domain $[\alpha,1]^n \times [\beta,1]^n$. So to compute the maximum, we only pay an attention to the extreme points (a,b), that is, a,b such that $a_i = \alpha, 1$ and $b_i = \beta, 1$ for $i,j \in I_n := \{1,2,\ldots,n\}$. Denote by J_a and J_b $(\subset I_n)$ the sets

$$J_a := \{i \in I_n; a_i = 1\} \quad \text{and} \quad J_b := \{i \in I_n; b_i = 1\}.$$

Moreover we define a *reformed weight* $\widetilde{w} := (\widetilde{w}_1, \widetilde{w}_2, \widetilde{w}_3, \widetilde{w}_4)$ for a given weight $w = (w_1, w_2, \ldots, w_n)$ by

$$\widetilde{w}_1 = \widetilde{w}_1(a,b) := \sum_{i \in J_a \cap J_b} w_i, \qquad \widetilde{w}_2 = \widetilde{w}_2(a,b) := \sum_{i \in J_a \cap J_b^c} w_i,$$
$$\widetilde{w}_3 = \widetilde{w}_3(a,b) := \sum_{i \in J_a^c \cap J_b} w_i \quad \text{and} \quad \widetilde{w}_4 = \widetilde{w}_4(a,b) := \sum_{i \in J_a^c \cap J_b^c} w_i.$$

We put $\widetilde{W}$ the set of all reformed weights $\widetilde{w}$ for a weight w. Then it follows from $w_i \geqslant 0$ ($i \in I_n$) and $\sum_{i \in I_n} w_i = 1$ that

$$\widetilde{w}_1, \widetilde{w}_2, \widetilde{w}_3, \widetilde{w}_4 \geqslant 0 \quad \text{and} \quad \widetilde{w}_1 + \widetilde{w}_2 + \widetilde{w}_3 + \widetilde{w}_4 = 1.$$

Using the reformed weight, we can rewrite $D_p(a,b;w)$ as follows:

$$D_p(a,b;w) = \{\widetilde{w}_1 + \widetilde{w}_2 + \alpha^p(\widetilde{w}_3 + \widetilde{w}_4)\}^{\frac{1}{p}} \{\widetilde{w}_1 + \widetilde{w}_3 + \beta^q(\widetilde{w}_2 + \widetilde{w}_4)\}^{\frac{1}{q}} - (\widetilde{w}_1 + \beta\widetilde{w}_2 + \alpha\widetilde{w}_3 + \alpha\beta\widetilde{w}_4). \tag{1}$$

Now we here introduce $\widetilde{D}(\widetilde{w},\tau)$ for $\tau > 0$, as a linearization of $D_p(a,b;w)$ in a sense

$$\widetilde{D}(\widetilde{w},\tau) := \frac{1}{p}\{\widetilde{w}_1 + \widetilde{w}_2 + \alpha^p(\widetilde{w}_3 + \widetilde{w}_4)\}\tau^p + \frac{1}{q}\{\widetilde{w}_1 + \widetilde{w}_3 + \beta^q(\widetilde{w}_2 + \widetilde{w}_4)\}\tau^{-q} - (\widetilde{w}_1 + \beta\widetilde{w}_2 + \alpha\widetilde{w}_3 + \alpha\beta\widetilde{w}_4). \tag{2}$$

Then, recalling Young's inequality that $h^{\frac{1}{p}}k^{\frac{1}{q}} \leqslant \frac{1}{p}h + \frac{1}{q}k$ for $h,k \geqslant 0$, we have $D_p(a,b;w) \leqslant \widetilde{D}(\widetilde{w},\tau)$ by (1) and (2). Further the following lemma ensures that D_p is given by using $\widetilde{D}(\widetilde{w},\tau)$ as a substitute for $D_p(a,b;w)$:

LEMMA 1. *The maximum D_p of $D_p(a,b;w)$ for a,b,w is represented as follows*:

$$D_p \left(:= \max_{a,b,w} D_p(a,b;w) \right) = \max_{\widetilde{w} \in \widetilde{W}} \min_{\tau > 0} \widetilde{D}(\widetilde{w},\tau) = \min_{\tau > 0} \max_{\widetilde{w} \in \widetilde{W}} \widetilde{D}(\widetilde{w},\tau).$$

Proof. The equality holds in $h^{\frac{1}{p}}k^{\frac{1}{q}} \leqslant \frac{1}{p}h + \frac{1}{q}k$ for all $h,k \geqslant 0$ if $h = k$. So we see that

$$D_p(a,b;w) = \min_{\tau>0} \widetilde{D}(\widetilde{w},\tau). \tag{3}$$

Indeed, it is attained at $\tau = \tau_{\widetilde{w}}$ such that

$$\{\widetilde{w}_1 + \widetilde{w}_2 + \alpha^p(\widetilde{w}_3 + \widetilde{w}_4)\}\tau_{\widetilde{w}}^p = \{\widetilde{w}_1 + \widetilde{w}_3 + \beta^q(\widetilde{w}_2 + \widetilde{w}_4)\}\tau_{\widetilde{w}}^{-q}. \tag{4}$$

So we have $\max_{a,b,w} D_p(a,b;w) = \max_{\widetilde{w}\in\widetilde{W}} \min_{\tau>0} \widetilde{D}(\widetilde{w},\tau)$.

Next we see that $\widetilde{w} \to \widetilde{D}(\widetilde{w},\tau)$ is linear (hence concave) on $\widetilde{W}$ and $\tau \to \widetilde{D}(\widetilde{w},\tau)$ is convex on $(0,\infty)$. So the identity $\max_{\widetilde{w}\in\widetilde{W}} \min_{\tau>0} \widetilde{D}(\widetilde{w},\tau) = \min_{\tau>0} \max_{\widetilde{w}\in\widetilde{W}} \widetilde{D}(\widetilde{w},\tau)$ holds by the mini-max theorem [6, pp.75-76], [3, Theorem S]. $\square$

In the below, we give some properties for $\widetilde{D}(\widetilde{w},\tau)$. Since $\widetilde{W}$ is a convex set with four extreme points $\widetilde{e}_1 = (1,0,0,0),\ldots,\widetilde{e}_4 = (0,0,0,1)$, we see that

$$\max_{\widetilde{w}\in\widetilde{W}} \widetilde{D}(\widetilde{w},\tau) = \max_{i\in\{1,2,3,4\}} \widetilde{D}(\widetilde{e}_i,\tau).$$

In addition, each $\widetilde{D}(\widetilde{e}_i,\tau)$ is calculated by (2) :

$$\begin{aligned} \widetilde{D}(\widetilde{e}_1,\tau) &= \frac{1}{p}\tau^p + \frac{1}{q}\tau^{-q} - 1, & \widetilde{D}(\widetilde{e}_2,\tau) &= \frac{1}{p}\tau^p + \frac{1}{q}\beta^q\tau^{-q} - \beta, \\ \widetilde{D}(\widetilde{e}_3,\tau) &= \frac{1}{p}\alpha^p\tau^p + \frac{1}{q}\tau^{-q} - \alpha \quad \text{and} & \widetilde{D}(\widetilde{e}_4,\tau) &= \frac{1}{p}\alpha^p\tau^p + \frac{1}{q}\beta^q\tau^{-q} - \alpha\beta. \end{aligned} \tag{5}$$

Now let us state some elementary facts about these functions for $\tau > 0$.

LEMMA 2.
(*i*) $\widetilde{D}(\widetilde{w},\tau) = \widetilde{D}(\widetilde{e}_1,\tau)\widetilde{w}_1 + \widetilde{D}(\widetilde{e}_2,\tau)\widetilde{w}_2 + \widetilde{D}(\widetilde{e}_3,\tau)\widetilde{w}_3 + \widetilde{D}(\widetilde{e}_4,\tau)\widetilde{w}_4$.
(*ii*) *Put* $\tau_1 = 1$, $\tau_2 = \beta^{\frac{1}{p}}$, $\tau_3 = \alpha^{-\frac{1}{q}}$ *and* $\tau_4 = \alpha^{-\frac{1}{q}}\beta^{\frac{1}{p}}$. *Then all functions* $\widetilde{D}(\widetilde{e}_k,\tau)$ ($k = 1,2,3,4$) *are strictly decreasing on* $0 < \tau \leqslant \tau_k$, *strictly increasing on* $\tau_k \leqslant \tau$ *and strictly convex on* $\tau > 0$. *Moreover we have* $\widetilde{D}(\widetilde{e}_k,\tau_k) = 0$ *and* $\lim_{\tau\to 0} \widetilde{D}(\widetilde{e}_k,\tau) = \lim_{\tau\to\infty} \widetilde{D}(\widetilde{e}_k,\tau) = \infty$.

There exist $\theta_{p,\alpha} \in (\alpha,1)$ and $\theta_{q,\beta} \in (\beta,1)$ such that

$$\frac{1-\alpha^p}{p(1-\alpha)} = \theta_{p,\alpha}^{p-1} \left(= \theta_{p,\alpha}^{\frac{p}{q}}\right) \quad \text{and} \quad \frac{1-\beta^q}{q(1-\beta)} = \theta_{q,\beta}^{q-1} \left(= \theta_{q,\beta}^{\frac{q}{p}}\right) \tag{6}$$

by applying the mean-value theorem to functions t^p and t^q respectively. Related to the notations $\theta_{p,\alpha}$ and $\theta_{q,\beta}$, we have some properties for functions $\widetilde{D}(\widetilde{e}_i,\tau)$ ($1 \leqslant i,j \leqslant 4$):

LEMMA 3. *Let* $\widetilde{D}_{i,j}(\tau) := \widetilde{D}(\widetilde{e}_i,\tau) - \widetilde{D}(\widetilde{e}_j,\tau)$ ($1 \leqslant i,j \leqslant 4$). *Then the following facts hold*:
(*i*) $\widetilde{D}_{1,2}(\tau)$ *is strictly decreasing and* $\widetilde{D}_{1,2}(\tau) = 0$ *has a* (*unique*) *solution* $\tau = \theta_{q,\beta}^{\frac{1}{p}} (\in (\beta^{\frac{1}{p}},1))$.

(*ii*) $\widetilde{D}_{1,3}(\tau)$ *is strictly increasing and* $\widetilde{D}_{1,3}(\tau)=0$ *has a solution* $\tau=\theta_{p,\alpha}^{-\frac{1}{q}}(\in(1,\alpha^{-\frac{1}{q}}))$.

(*iii*) $\widetilde{D}_{2,3}(\tau)$ *is strictly increasing and* $\widetilde{D}_{2,3}(\tau)=0$ *has a solution* $\tau=\tau_*(\in(\theta_{q,\beta}^{\frac{1}{p}},\theta_{p,\alpha}^{-\frac{1}{q}})\subset(\beta^{\frac{1}{p}},\alpha^{-\frac{1}{q}}))$.

(*iv*) $\widetilde{D}_{2,4}(\tau)$ *is strictly increasing and* $\widetilde{D}_{2,4}(\tau)=0$ *has a solution* $\tau=\beta^{\frac{1}{p}}\theta_{p,\alpha}^{-\frac{1}{q}}(\in(\beta^{\frac{1}{p}},\theta_{p,\alpha}^{-\frac{1}{q}}))$.

(*v*) $\widetilde{D}_{3,4}(\tau)$ *is strictly decreasing and* $\widetilde{D}_{3,4}(\tau)=0$ *has a solution* $\tau=\alpha^{-\frac{1}{q}}\theta_{q,\beta}^{\frac{1}{p}}(\in(\theta_{q,\beta}^{\frac{1}{p}},\alpha^{-\frac{1}{q}}))$.

Proof. We only prove (i) since (ii) – (v) are similarly shown. It follows from $\widetilde{D}_{1,2}(\tau)=\frac{1}{q}(1-\beta^q)\tau^{-q}-(1-\beta)$ that $\widetilde{D}'_{1,2}(\tau)=-(1-\beta^q)\tau^{-q-1}<0$. Moreover $\widetilde{D}_{1,2}(\tau)=0$ has a unique solution $\tau=\left\{\frac{1-\beta^q}{q(1-\beta)}\right\}^{\frac{1}{q}}=\theta_{q,\beta}^{\frac{1}{p}}$ and the solution is included in $(\beta^{\frac{1}{p}},1)$ by (6). □

Now we express the maximum D_p of $D_p(a,b;w)$ by using $\widetilde{D}(\widetilde{w},\tau)$:

LEMMA 4. *The maximum value* D_p *is given as follows*:

$$D_p=\widetilde{D}(\widetilde{e}_2,\tau_*)=\widetilde{D}(\widetilde{e}_3,\tau_*)=(1-t)\widetilde{D}(\widetilde{e}_2,\tau_*)+t\widetilde{D}(\widetilde{e}_3,\tau_*)=\widetilde{D}(\widetilde{w}_t,\tau_*) \qquad (7)$$

for all $t\in[0,1]$ *where* $\widetilde{w}_t:=(0,1-t,t,0)$, $\widetilde{e}_2=(0,1,0,0)$, $\widetilde{e}_3=(0,0,1,0)$ *and* τ_* *is defined by Lemma 3* (iii).

Proof. First of all we show that for each $\tau>0$

$$\max_{\widetilde{w}\in\widetilde{W}}\widetilde{D}(\widetilde{w},\tau)\left(=\max_{i\in\{1,2,3,4\}}\widetilde{D}(\widetilde{e}_i,\tau)\right)=\max\{\widetilde{D}(\widetilde{e}_2,\tau),\widetilde{D}(\widetilde{e}_3,\tau)\}, \qquad (8)$$

or

$$\widetilde{D}(\widetilde{e}_1,\tau)<\max\{\widetilde{D}(\widetilde{e}_2,\tau),\widetilde{D}(\widetilde{e}_3,\tau)\} \quad \text{and} \quad \widetilde{D}(\widetilde{e}_4,\tau)<\max\{\widetilde{D}(\widetilde{e}_2,\tau),\widetilde{D}(\widetilde{e}_3,\tau)\}. \qquad (9)$$

Note that $\widetilde{D}(\widetilde{e}_1,\tau)<\widetilde{D}(\widetilde{e}_2,\tau)$ for $\theta_{q,\beta}^{\frac{1}{p}}<\tau$ and $\widetilde{D}(\widetilde{e}_1,\tau)<\widetilde{D}(\widetilde{e}_3,\tau)$ for $0<\tau<\theta_{p,\alpha}^{-\frac{1}{q}}$ by Lemma 3 (i) and (ii), respectively. Hence since $\theta_{q,\beta}^{\frac{1}{p}}<\theta_{p,\alpha}^{-\frac{1}{q}}$, the first inequality of (9) holds for all $\tau>0$. Similarly it follows from Lemma 3 (iv) and (v) that the second inequality holds for all $\tau>0$ by $\beta^{\frac{1}{p}}\theta_{p,\alpha}^{-\frac{1}{q}}<\alpha^{-\frac{1}{q}}\theta_{q,\beta}^{\frac{1}{p}}$.

Moreover we see from (8), Lemmas 2 (ii) and 3 (iii) that $\max_{\widetilde{w}\in\widetilde{W}}\widetilde{D}(\widetilde{w},\tau)$ is a convex function of $\tau>0$ and its minimum attains at $\tau=\tau_*$, i.e.,

$$\min_{\tau>0}\max_{\widetilde{w}\in\widetilde{W}}\widetilde{D}(\widetilde{w},\tau)=\widetilde{D}(\widetilde{e}_2,\tau_*)=\widetilde{D}(\widetilde{e}_3,\tau_*)\left(=(1-t)\widetilde{D}(\widetilde{e}_2,\tau_*)+t\widetilde{D}(\widetilde{e}_3,\tau_*)=\widetilde{D}(\widetilde{w}_t,\tau_*)\right)$$

by Lemma 2 (i). The first inequality of (7) holds by Lemma 1. □

From the above lemmas we have the following theorem:

THEOREM 5. *Let* $\alpha,\beta \in (0,1)$ *and* $p,q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *be given, and* $\tau=\tau_*$ *be a unique solution of the equation* $\frac{1}{p}\tau^p+\frac{1}{q}\beta^q\tau^{-q}-\beta=\frac{1}{p}\alpha^p\tau^p+\frac{1}{q}\tau^{-q}-\alpha$ *which appears in Lemma 3* (iii). *Then*

$$\max_{a,b,w} D_p(a,b;w)=(\widetilde{w}_{2*}+\alpha^p\widetilde{w}_{3*})^{\frac{1}{p}}(\widetilde{w}_{3*}+\beta^q\widetilde{w}_{2*})^{\frac{1}{q}}-(\beta\widetilde{w}_{2*}+\alpha\widetilde{w}_{3*})$$
$$=\frac{1}{p}\tau_*^p+\frac{1}{q}\beta^q\tau_*^{-q}-\beta=\frac{1}{p}\alpha^p\tau_*^p+\frac{1}{q}\tau_*^{-q}-\alpha \tag{10}$$

where $a=(a_1,\dots,a_n)$ *and* $b=(b_1,\dots,b_n)$ *are* n*-tuples of positive numbers* $[\alpha,1]^n$ *and* $[\beta,1]^n$ *respectively,* $w=(w_1,\dots,w_n)$ *is a weight, and the constants* $\widetilde{w}_{2*}$ *and* $\widetilde{w}_{3*}$ *are defined by*

$$\widetilde{w}_{2*}:=\frac{1-\alpha^p\tau_*^{pq}}{1-\alpha^p\tau_*^{pq}+\tau_*^{pq}-\beta^q},\quad \widetilde{w}_{3*}:=\frac{\tau_*^{pq}-\beta^q}{1-\alpha^p\tau_*^{pq}+\tau_*^{pq}-\beta^q}\quad\in(0,1). \tag{11}$$

The maximum is attained at (a,b) *in the left side of* (10) *if and only if* $a_i=1,\alpha$, $b_i=1,\beta$ *and a reformed weight* $\widetilde{w}$ *generated by a given weight* w *is* $(0,\widetilde{w}_{2*},\widetilde{w}_{3*},0)$.

Proof. Since $D_p(a,b;w)$ is convex in both a and b, its maximum is attained at an extreme point (a,b) of the product $[\alpha,1]^n\times[\beta,1]^n$, that is, $a_i=\alpha,1$ and $b_i=\beta,1$ for $i,j\in\{1,2,\dots,n\}$. By Lemma 4 we see that the maximum D_p of $D_p(a,b;w)$ is given by $\widetilde{D}(\widetilde{w}_*,\tau_*)$ where $\widetilde{w}_*=(0,\widetilde{w}_{2*},\widetilde{w}_{3*},0)$ with $\widetilde{w}_{2*}+\widetilde{w}_{3*}=1$, $\widetilde{w}_{2*},\widetilde{w}_{3*}\geqslant 0$. Since this reformed weight $\widetilde{w}_*$ satisfies (4) for $\tau_{\widetilde{w}}=\tau_*$, we obtain (11). Here Lemma 3 (iii) implies that the constants $\widetilde{w}_{2*}$ and $\widetilde{w}_{3*}$ are included in $(0,1)$. Taking any weight $w=w_*$ which generates $\widetilde{w}_*$, we have $D_p(a,b;w_*)=\widetilde{D}(\widetilde{w}_*,\tau_*)$ by (3). Moreover by (1) we have $D_p(a,b,w_*)=(\widetilde{w}_{2*}+\alpha^p\widetilde{w}_{3*})^{\frac{1}{p}}(\widetilde{w}_{3*}+\beta^q\widetilde{w}_{2*})^{\frac{1}{q}}-(\beta\widetilde{w}_{2*}+\alpha\widetilde{w}_{3*})$. So we obtain the first equality of (10). Further since $D_p=D(\widetilde{e}_2,\tau_*)\,(=D(\widetilde{e}_3,\tau_*))$ by Lemma 4, the second equality of (10) holds. □

EXAMPLE 6. Let $\alpha=\beta=\frac{1}{2}$, $p=q=2$ and $n=4$ in Theorem 5. Then since the equality $D(\widetilde{e}_2,\tau_*)=D(\widetilde{e}_3,\tau_*)$ implies $\tau_*=1$, we have by (10)

$$\frac{1}{p}\tau_*^p+\frac{1}{q}\beta^q\tau_*^{-q}-\beta\left(=\frac{1}{p}\alpha^p\tau_*^p+\frac{1}{q}\tau_*^{-q}-\alpha\right)=\frac{1}{8}.$$

Moreover we have $\widetilde{w}_{2*}=\widetilde{w}_{3*}=\frac{1}{2}\in(0,1)$, so that

$$(\widetilde{w}_{2*}+\alpha^p\widetilde{w}_{3*})^{\frac{1}{p}}(\widetilde{w}_{3*}+\beta^q\widetilde{w}_{2*})^{\frac{1}{q}}-(\beta\widetilde{w}_{2*}+\alpha\widetilde{w}_{3*})=\frac{1}{8}.$$

On the other hand, the maximum of $D_p(a,b;w)$ is given by $(a_i,b_i)=(1,\beta)$ or $(\alpha,1)$ for $i\in I_4:=\{1,2,3,4\}$. We consider three cases $|J_a|=1$, 2 and 3 where $|J_a|$ expresses the cardinal number of J_a :

Let $|J_a| = 1$ (or $|J_b| = |I_4 \setminus J_a| = 3$). Then we may suppose that $J_a = \{i_1\}$ and $J_b = \{i_2, i_3, i_4\}$ for $i_1, i_2, i_3, i_4 \in I_4$. Hence we have $w_{i_1} = \frac{1}{2}$ and $w_{i_2}, w_{i_3}, w_{i_4} \in [0, \frac{1}{2}]$ with $w_{i_2} + w_{i_3} + w_{i_4} = \frac{1}{2}$, so that

$$
\begin{aligned}
&D_p(a,b;w) \\
&= \left\{ w_{i_1} \cdot 1^2 + (w_{i_2} + w_{i_3} + w_{i_4}) \cdot \left(\frac{1}{2}\right)^2 \right\}^{\frac{1}{2}} \left\{ w_{i_1} \cdot \left(\frac{1}{2}\right)^2 + (w_{i_2} + w_{i_3} + w_{i_4}) \cdot 1^2 \right\}^{\frac{1}{2}} \\
&\quad - \left(w_{i_1} \cdot 1 \cdot \frac{1}{2} + (w_{i_2} + w_{i_3} + w_{i_4}) \cdot \frac{1}{2} \cdot 1 \right) \\
&= \left\{ \frac{1}{2} \cdot 1^2 + \frac{1}{2} \cdot \left(\frac{1}{2}\right)^2 \right\}^{\frac{1}{2}} \left\{ \frac{1}{2} \cdot \left(\frac{1}{2}\right)^2 + \frac{1}{2} \cdot 1^2 \right\}^{\frac{1}{2}} - \left(\frac{1}{2} \cdot 1 \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot 1 \right) = \frac{1}{8}.
\end{aligned}
$$

Let $|J_a| = 2$ (or $|J_b| = |I_4 \setminus J_a| = 2$). Then we may suppose that $J_a = \{i_1, i_2\}$ and $J_b = \{i_3, i_4\}$. Hence we have $w_{i_1}, w_{i_2}, w_{i_3}, w_{i_4} \in [0, \frac{1}{2}]$ with $w_{i_1} + w_{i_2} = \frac{1}{2}$ and $w_{i_3} + w_{i_4} = \frac{1}{2}$, so that

$$
\begin{aligned}
&D_p(a,b;w) \\
&= \left\{ (w_{i_1} + w_{i_2}) \cdot 1^2 + (w_{i_3} + w_{i_4}) \cdot \left(\frac{1}{2}\right)^2 \right\}^{\frac{1}{2}} \left\{ (w_{i_1} + w_{i_2}) \cdot \left(\frac{1}{2}\right)^2 + (w_{i_3} + w_{i_4}) \cdot 1^2 \right\}^{\frac{1}{2}} \\
&\quad - \left\{ (w_{i_1} + w_{i_2}) \cdot 1 \cdot \frac{1}{2} + (w_{i_3} + w_{i_4}) \cdot \frac{1}{2} \cdot 1 \right\} = \frac{1}{8}.
\end{aligned}
$$

Let $|J_a| = 3$. Then similarly we have $D_p(a,b;w) = \frac{1}{8}$. Therefore we have (10).

3. Variational expression of the maximum D_p

We first recall an upper bound D_0 in our previous note [1, p.46]. For convenience, we put, for fixed $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $\alpha, \beta \in (0,1)$,

$$
K := \left\{ \frac{1-\alpha^p}{p(1-\alpha)} \right\}^{\frac{1}{p}} \left\{ \frac{1-\beta^q}{q(1-\beta)} \right\}^{\frac{1}{q}}.
$$

Moreover we need that the equation

$$
(1-\alpha)(1-K\tau^{\frac{1}{q}}) = (1-\beta)(1-K\tau^{-\frac{1}{p}}) \; (\equiv c) \tag{12}
$$

has a unique solution $\tau = \tau_0$, by which the upper bound D_0 of $D_p(a,b;w_0)$ for the weight $w_0 = (\frac{1}{n}, \ldots, \frac{1}{n})$ was given by

$$
D_0 := \frac{1-\alpha}{1-\alpha^p} + \frac{1-\beta}{1-\beta^q} - 1 - c\left(\frac{1}{1-\alpha^p} + \frac{1}{1-\beta^q} - 1 \right). \tag{13}
$$

In this section, we show the correspondence of the maximum D_p and D_0. The following lemma is technically essential:

LEMMA 7. *Let τ_0 be as in above. Then the unique solution τ_* in Lemma 3 (iii) is expressed as*

$$\tau_* = \left\{ \frac{p(1-\alpha)}{1-\alpha^p} \cdot \frac{1-\beta^q}{q(1-\beta)} \tau_0 \right\}^{\frac{1}{pq}}.$$

Proof. For convenience, we put

$$\tau_\sharp = \left\{ \frac{p(1-\alpha)}{1-\alpha^p} \cdot \frac{1-\beta^q}{q(1-\beta)} \tau_0 \right\}^{\frac{1}{pq}}.$$

Since $1 - \frac{c}{1-\alpha} = K\tau_0^{\frac{1}{q}}$ and $1 - \frac{c}{1-\beta} = K\tau_0^{-\frac{1}{p}}$ by (12), it follows from (13) that

$$D_0 = \frac{1-\alpha}{1-\alpha^p} + \frac{1-\beta}{1-\beta^q} - 1 - \frac{c}{1-\alpha^p} - \frac{c}{1-\beta^q} + c = \frac{1-\alpha}{1-\alpha^p} K\tau_0^{\frac{1}{q}} + \frac{1-\beta}{1-\beta^q} K\tau_0^{-\frac{1}{p}} - 1 + c.$$

Here, using $c = (1-\alpha)(1-K\tau_0^{\frac{1}{q}})$, we have

$$\begin{aligned}
D_0 &= \frac{1-\alpha}{1-\alpha^p} K\tau_0^{\frac{1}{q}} + \frac{1-\beta}{1-\beta^q} K\tau_0^{-\frac{1}{p}} - (1-\alpha)K\tau_0^{\frac{1}{q}} - \alpha \\
&= \alpha^p \frac{1-\alpha}{1-\alpha^p} K\tau_0^{\frac{1}{q}} + \frac{1-\beta}{1-\beta^q} K\tau_0^{-\frac{1}{p}} - \alpha \\
&= \frac{\alpha^p}{p} \left\{ \frac{1-\alpha^p}{p(1-\alpha)} \right\}^{-\frac{1}{q}} \left\{ \frac{1-\beta^q}{q(1-\beta)} \right\}^{\frac{1}{q}} \tau_0^{\frac{1}{q}} + \frac{1}{q} \left\{ \frac{1-\alpha^p}{p(1-\alpha)} \right\}^{\frac{1}{p}} \left\{ \frac{1-\beta^q}{q(1-\beta)} \right\}^{-\frac{1}{p}} \tau_0^{-\frac{1}{p}} - \alpha \\
&= \frac{1}{p} \alpha^p \tau_\sharp{}^p + \frac{1}{q} \tau_\sharp{}^{-q} - \alpha = \widetilde{D}(\widetilde{e}_3, \tau_\sharp) \quad (\text{by } (5)).
\end{aligned}$$

Similarly, using $c = (1-\beta)(1-K\tau_0^{-\frac{1}{p}})$, we obtain

$$D_0 = \frac{1}{p} \tau_\sharp{}^p + \frac{1}{q} \beta^q \tau_\sharp{}^{-q} - \beta = \widetilde{D}(\widetilde{e}_2, \tau_\sharp).$$

Namely we have $\widetilde{D}_{2,3}(\tau_\sharp) = \widetilde{D}(\widetilde{e}_2, \tau_\sharp) - \widetilde{D}(\widetilde{e}_3, \tau_\sharp) = 0$. Since the solution $\tau = \tau_*$ of $\widetilde{D}_{2,3}(\tau_\sharp) = 0$ is unique by Lemma 3 (iii), we have $\tau_* = \tau_\sharp$. □

THEOREM 8. *The maximum D_p $($of $D_p(a,b,;w)$ $)$ coincides with the upper bound D_0.*

Proof. Let $\tau_\sharp$ be as in the proof of Lemma 7. Then $D_0 = \widetilde{D}(\widetilde{e}_i, \tau_\sharp)$ for $i = 2,3$ and $\tau_\sharp = \tau_*$ are proved in Lemma 7. On the other hand we have $D_p = \widetilde{D}(\widetilde{e}_i, \tau_*)$ for $i = 2,3$ by Lemma 4 and so $D_p = D_0$. □

Incidentally, the proof of Theorem 8 says that Lemmas 4 and 7 play a role to combine Theorem 5 with the previous result.

Now to calculate D_p given by (7) directly, we have the following:

COROLLARY 9. *The maximum value D_p is a unique solution of the equation*

$$\frac{p^{\frac{1}{p}} q^{\frac{1}{q}}}{1-\alpha^p\beta^q} \{(1-\beta^q)t + \beta - \alpha\beta^q\}^{\frac{1}{p}} \{(1-\alpha^p)t + \alpha - \alpha^p\beta\}^{\frac{1}{q}} = 1 \quad (t > 0). \tag{14}$$

Proof. Let $\xi = \tau_*^p$ and $\eta = \tau_*^{-q}$. It follows from $\widetilde{D}(\widetilde{e}_2, \tau_*) = \widetilde{D}(\widetilde{e}_3, \tau_*) = D_p$ that

$$\frac{1}{p}\xi + \frac{1}{q}\beta^q\eta - \beta = \frac{1}{p}\alpha^p\xi + \frac{1}{q}\eta - \alpha = D_p.$$

So we have

$$\xi = \frac{p}{1-\alpha^p\beta^q}\{(1-\beta^q)D_p+\beta-\alpha\beta^q\} \quad \text{and} \quad \eta = \frac{q}{1-\alpha^p\beta^q}\{(1-\alpha^p)D_p+\alpha-\alpha^p\beta\}.$$

Since $\xi^{\frac{1}{p}}\eta^{\frac{1}{q}} = 1$, we see that D_p is a solution of (14). Next let $f(t)$ be the function defined by the left side of (14). Then $f(t)$ is strictly increasing and $\operatorname{ran} f(t) = (f(0), \infty)$. Moreover we have

$$\begin{aligned}(0 <)\ &\{p(\beta-\alpha\beta^q)\}^{\frac{1}{p}}\{q(\alpha-\alpha^p\beta)\}^{\frac{1}{q}} \\ &\leqslant \frac{1}{p}\cdot p(\beta-\alpha\beta^q) + \frac{1}{q}\cdot q(\alpha-\alpha^p\beta) \\ &= \beta(1-\alpha^p) + \alpha(1-\beta^q) \\ &\leqslant \left(\frac{1}{p}+\frac{1}{q}\beta^q\right)(1-\alpha^p) + \left(\frac{1}{q}+\frac{1}{p}\alpha^p\right)(1-\beta^q) \\ &= 1-\alpha^p\beta^q\end{aligned}$$

by Young's inequality, so that $f(0) = \frac{1}{1-\alpha^p\beta^q}\{p(\beta-\alpha\beta^q)\}^{\frac{1}{p}}\{q(\alpha-\alpha^p\beta)\}^{\frac{1}{q}} \in (0,1)$. Hence (14) has a unique solution $t = D_p$. $\square$

One of the authors introduced the following constant in [1, Theorem 3.2]:

$$C(u,v;\gamma) = \frac{1}{u}\left\{\frac{1-\gamma^v}{v(1-\gamma)}\right\}^{u-1} - \frac{\gamma-\gamma^v}{1-\gamma^v}$$

for $u > 1$, $v > 1$ with $\frac{1}{u}+\frac{1}{v} = 1$ and $0 < \gamma < 1$. Incidentally we consider the maximum of $D_p(a,b;w)$ in the following restriction for a and b:

$$1 \geqslant a_1 \geqslant \cdots \geqslant a_n \geqslant \alpha \quad \text{and} \quad 1 \geqslant b_1 \geqslant \cdots \geqslant b_n \geqslant \beta \quad (0 < \alpha, \beta < 1).$$

Then it is given by

$$\max_{a,b,w} D_p(a,b;w) = \max\{C(p,q;\beta), C(q,p;\alpha)\}.$$

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