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SUFFICIENT CONDITIONS FOR CONVEXITY IN A CLASS OF FUNCTIONS ARISING IN TELECOMMUNICATIONS

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Dedicated with affection to the memory of our colleague, Mladen Alić

(*communicated by N. Elezović*)

Abstract. Convexity is a key property for a class of functions arising in telecommunications. We derive sufficient conditions for this to hold.

1. Introduction and Background

A central and long-standing problem in the modelling and dimensioning of telecommunications networks was to find the worst-possible time congestion in a $GI/M/N/N$ loss network with given mean arrival rate and mean holding time. In this system there are individual arrivals (calls) and interarrival times are identically and independently distributed with distribution function $F(\cdot)$, say, with fixed mean m. The service facility consists of N homogeneous servers (channels) in parallel, each with exponential services (holding times) occurring at rate μ. An arrival finding all channels occupied is lost. The time congestion is the equilibrium proportion of the time that all the channels are occupied. The classical problem is as follows: given m, μ, N, what is the corresponding maximal value of the time congestion?

This nonlinear optimisation problem has been addressed inter alia by Coyle [1], who solved the problem for the case $N = 1$ and conjectured the solution for general N. A difficulty with the methodology was in showing that a local maximum obtained is also a global maximum, and Coyle was unable to do this for $N > 1$. The problem was finally resolved by Peake and Pearce [3], who gave an analytical treatment holding for general N. This treatment showed incidentally that Coyle's conjecture is false for $N > 1$. A detailed bibliography of relevant work is also provided in [3].

The argument in [3] was based on a stochastic analysis of Takács [6, Chapter 4]. It follows from his analysis that if the inter–arrival time distribution function F of the $G/M/N/N$ system is non–lattice with mean $m < \infty$, then there is a well–defined

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steady–state limiting value P_N^* for the continuous time congestion. This is given by

$$P_N^* = \frac{1}{N\mu m} \cdot \frac{\frac{1-\phi(N\mu)}{\phi(N\mu)}}{1 + \sum_{j=1}^{N} \binom{N}{j} \prod_{i=1}^{j} \frac{1-\phi(i\mu)}{\phi(i\mu)}} ,$$

where

$$\phi(s) := \int_0^\infty e^{-st} dF(t) \quad (Re.s \geqslant 0)$$

is the Laplace–Stieltjes transform of F.

Peake and Pearce [3] used a tight double–sided inequality of Eckberg [2] for the Laplace–Stieltjes transform to reduce the variational maximisation problem to a technical question involving the polynomial function

$$\alpha(z) := 1 + \sum_{n=1}^{N} \binom{N}{n} \prod_{j=1}^{n} (z^j - 1). \tag{1}$$

Their preliminary reduction converted the problem to showing, for $N > 1$ and $z > 1$, that $\alpha(z)/(z^N - 1)$ is strictly convex in z. This was established in [3] using somewhat delicate and lengthy analysis. It is worth noting that $(z^N - 1)/\alpha(z)$ is not concave for $z > 1$.

A generalisation of this technical question is of interest in connection with some equilibrium–point problems in electromagnetism [4]. Let a_i, b_i ($1 \leqslant i \leqslant N$) and M be positive numbers and define

$$\alpha(z) = 1 + \sum_{n=1}^{N} \prod_{j=1}^{n} \left[\frac{b_j}{a_j} (z^{a_j} - 1) \right] . \tag{2}$$

The corresponding task for solving the equilibrium–point problems is to find sufficient conditions (on a_i, b_i, M) for the strict convexity in z of $\alpha(z)/(z^M - 1)$ for $z > 1$. Physical intuition suggests that convexity might hold under very general conditions. However counterexamples are given in [4], so that finding sufficient conditions is a non–vacuous task.

This problem is also of interest in fractional programming [5], since for α given by (1), $\alpha(z)/(z^N - 1)$ can be viewed as the quotient of two convex functions. The library of nontrivial such quotients which are themselves convex is quite small.

In this paper we present a considerably streamlined approach to giving sufficient conditions for the convexity of $\alpha(z)/(z^M - 1)$ for α defined by (2). This is based on a matrix formulation. The problem is a nice example of the power of inequality–based techniques.

In Section 2 we present some preliminaries and in Section 3 two key lemmata. We give our main result in Section 4.

2. Preliminaries

Let a_i, b_i ($1 \leqslant i \leqslant N$) and M be positive numbers and put

$$s_0 = 0, \quad s_i = \sum_{j=1}^{i} a_j \quad (1 \leqslant i \leqslant N).$$

For $z \geqslant 1$ we define

$$p_0(z) := 1, \quad p_i(z) := z^{a_i} - 1 \quad (1 \leqslant i \leqslant N)$$

and for $z > 1$

$$f_i := a_i/p_i \quad (1 \leqslant i \leqslant N),$$

so

$$g_i := -z df_i/dz = f_i^2 + a_i \quad \text{for} \quad 1 \leqslant i \leqslant N \quad \text{and} \quad z > 1. \tag{3}$$

It is convenient to encapsulate these quantities in matrix–vector form. Put

$$\begin{aligned}
&B := \left(1,\ \frac{b_1}{a_1},\ \frac{b_1 b_2}{a_1 a_2},\ \ldots,\ \frac{b_1 b_2 \ldots b_N}{a_1 a_2 \ldots a_N}\right),\\
&P := (1,\ p_1,\ p_1 p_2,\ \ldots,\ p_1 p_2 \cdots p_N)^T,\\
&T := \operatorname{diag}\,(0,\ 0,\ f_1,\ f_1 + f_2,\ \ldots,\ f_1 + \cdots + f_{N-1}).
\end{aligned}$$

By (3) we have

$$\begin{aligned}
U &:= -\, zdT/dz\\
&= \operatorname{diag}\,(0,\ 0,\ g_1,\ g_1 + g_2,\ \ldots,\ g_1 + \cdots + g_{N-1}).
\end{aligned}$$

We put

$$\alpha(z) := BP,\ \beta(z) := z\frac{d\alpha}{dz},\ \gamma(z) := z^2 \frac{d^2\alpha}{dz^2}, \tag{4}$$

$$G = \begin{bmatrix} s_0 & 0 & 0 & \ldots & 0 & 0\\ a_1 & s_1 & 0 & \ldots & 0 & 0\\ 0 & a_2 & s_2 & \ldots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \ldots & a_N & s_N \end{bmatrix}.$$

For $1 \leqslant m \leqslant N$ we have

$$\begin{aligned}
z\frac{d}{dz}(p_1 \ldots p_m) &= p_1 \ldots p_m z \frac{d}{dz}\ \ln(p_1 \ldots p_m)\\
&= p_1 \ldots p_m z \frac{d}{dz}\ \sum_{i=1}^{m} \ln p_i\\
&= p_1 \ldots p_m \sum_{i=1}^{m} (a_i + f_i)\\
&= p_1 \ldots p_m [s_m + f_1 + \ldots + f_m].
\end{aligned}$$

For $m > 1$ this gives

$$z\frac{d}{dz}(p_1 \dots p_m) = p_1 \dots p_m[s_m + f_1 + \dots + f_{m-1}] + p_1 \dots p_{m-1}a_m,$$

whilst for $m = 1$ we have

$$z\frac{dp_1}{dz} = p_1 s_1 + p_0 a_1.$$

Combining these rsults provides

$$z\frac{dP}{dz} = (G+T)P \tag{5}$$

and so

$$\beta(z) = B(G+T)P. \tag{6}$$

If

$$H := \text{diag}\ (b_1 + s_0,\ b_2 + s_1,\ \dots,\ b_{N-1} + s_N),$$

then we have that

$$BG = BH \tag{7}$$

and so

$$\beta(z) = B(H+T)P. \tag{8}$$

We shall make use also of the auxiliary matrices

$$K := \text{diag}\ (b_1(a_1 + b_2 - b_1),\ b_2(a_2 + b_3 - b_2),\ \dots,\ b_{N+1}(a_{N+1} + b_{N+2} - B_{N+1})),$$
$$L := \text{diag}\ (b_1 b_2,\ b_2 b_3,\ \dots,\ b_{N+1} b_{N+2}),$$

where $a_{N+1} = b_{N+1} = b_{N+2} = 0$. It is readily verified that

$$B(G^2 - H^2) = BK \quad \text{and} \quad B(TG - GT)P = BLP. \tag{9}$$

We have from (4) and (6) that

$$\begin{aligned} \beta + \gamma &= z\frac{d}{dz}\left(z\frac{d\alpha}{dz}\right) \\ &= z\frac{d\beta}{dz} \\ &= z\frac{d}{dz}[B(G+T)P] \\ &= B[(G+T)^2 - U]P, \end{aligned} \tag{10}$$

by (5).

Define

$$\delta(z) := B(H+T)^2 P \tag{11}$$

and put

$$\Omega_i = (H+T)_{i,i}, \quad (B)_i = B_i, \quad (P)_i = P_i \quad (0 \leqslant i \leqslant N).$$

Then by (4), (8) and (11) we have

$$\alpha = \sum_{i=0}^{N} B_i P_i, \quad \beta = \sum_{i=0}^{N} B_i \Omega_i P_i, \quad \delta = \sum_{i=0}^{N} B_i \Omega_i^2 P_i.$$

Since $B_i, P_i \geqslant 0$ for $0 \leqslant i \leqslant N$ and $z \geqslant 1$, we deduce from Jensen's inequality that

$$\alpha\delta - \beta^2 \geqslant 0. \tag{12}$$

3. Lemmata

Define

$$C := 2\text{diag}\,(0,\ 0,\ s_1,\ s_2/2,\ s_3/3,\ \ldots,\ s_{N-1}/(N-1))$$

and let D be the diagonal matrix given by $D_{i,i} = 0$ $(2 < i \leqslant N)$ and

$$D_{0,0} = b_1 b_2$$
$$D_{1,1} = \begin{cases} 0, & \textit{if} \quad a_2 \leqslant a_1 \\ b_2(a_2 - a_1)/2, & \textit{if} \quad a_2 > a_1 \end{cases}$$

and

$$D_{2,2} = \begin{cases} 0 & \text{if } a_2 \leqslant 2a_1 \\ (a_2 - a_1)(a_2 - 2a_1)/6 & \text{if } a_2 > 2a_1\,. \end{cases}$$

LEMMA 1. *For $z > 1$ we have*

$$B[2U - T^2 - CT - D]P \leqslant 0.$$

Proof. It suffices to show that

$$2U_m - T_m^2 - C_m T_m \leqslant 0 \quad \text{for} \quad m > 2 \tag{13}$$

and

$$B_2(2U_2 - T_2^2 - C_2 U_2)P_2 \leqslant BDP. \tag{14}$$

For the former, note first that the functions

$$\frac{x}{z^x - 1} \quad \text{and} \quad x + \frac{x}{z^x - 1}$$

are respectively strictly decreasing and strictly increasing for $x > 0$. Since $f_i = a_i/(z^{a_i} - 1)$, we thus have for $z > 1$ that $f_i > f_j$ and $f_i + a_i - (f_j + a_j)$ are either both zero or are of opposite sign. Hence

$$(f_i - f_j)(f_i + a_i - f_j - a_j) \leqslant 0$$

and so

$$(f_i - f_j)^2 \leqslant (f_i - f_j)(a_j - a_i).$$

Summation over i and j from 1 to $m-1$ provides

$$(m-1)\sum_{i=1}^{m-1} f_i^2 - \left(\sum_{i=1}^{m-1} f_i\right)^2 \leqslant \left(\sum_{i=1}^{m-1} a_i\right)\sum_{j=1}^{m-1} f_j - (m-1)\sum_{i=1}^{m-1} a_i f_i$$

and so

$$(m-1)\left[\sum_{i=1}^{m-1}\left(f_i^2 + a_i f_i\right)\right] \leqslant \left(\sum_{i=1}^{m-1} f_i\right)^2 + \left(\sum_{i=1}^{m-1} a_i\right)\sum_{j=1}^{m-1} f_j.$$

The left- and righthand sides of the last relation are respectively $(m-1)U_m$ and $T_m^2 + s_{m-1}T_m$, whence (13) follows by the definition of the matrix C.

For (14), consider the difference

$$X = 2U_2 - T_2^2 - C_2T_2 = T_2^2 = a_1^2/(z_1^a - 1)^2.$$

Pre– and post–multiplying respectively by B_2 and P_2 yields

$$B_2XP_2 = b_1b_2\frac{a_1}{a_2}\frac{z^{a_2}-1}{z^{a_1}-1} = b_1b_2\frac{f_1}{f_2}.$$

We distinguish three cases. First, if $a_2 \leqslant a_1$, then

$$B_2XP_2 \leqslant b_1b_2 = B_0D_{0,0}P_0 \leqslant BDP,$$

as required.

Secondly, suppose $a_1 < a_2 < 2a_1$. All the derivatives of $F(x) := (z^x - 1)/x$ are positive, so for $c_1, \ldots, c_k$ all different and $k > 1$, the finite differences

$$F[c_1, c_2, \ldots, c_k] := \sum_{i=1}^{k} \frac{F(c_i)}{\prod_{j\neq i}(c_i - c_j)}$$

are likewise all positive. In particular, for a, b, c distinct

$$F[a,b,c] = \frac{F(a)}{(a-b)(a-c)} + \frac{F(b)}{(b-a)(b-c)} + \frac{F(c)}{(c-a)(c-b)} > 0.$$

If we expand the inequality $F[a_1, a_2, 2a_1] > 0$, and recall that $f_1 = 1/F(a_1)$ and $f_2 = 1/F(a_2)$, we find that

$$\begin{aligned} B_2XP_2 &= b_1b_2\frac{f_1}{f_2} < b_1b_2 + B_1P_1b_2(a_2 - a_1)/2 \\ &= B_0D_{0,0}P_0 + B_1D_{1,1}P_1 \\ &< BDP. \end{aligned}$$

Finally, suppose that $a_2 \geqslant 2a_1$. Under equality the required result follows in the limit from the second case, so without loss of generality we may suppose that $a_2 > 2a_1$. The finite difference result

$$F[a_1, a_2, a_1 + a_2, 2a_1 + a_2] + 2F[a_1, 2a_1, a_2, a_2 + 2a_1] > 0$$

rearranges to provide

$$\begin{aligned} b_1 b_2 \frac{f_1}{f_2} &\leqslant b_1 b_2 + B_1 P_1 \frac{b_2(a_2 - a_1)}{2} + B_2 P_2 \frac{(a_2 - a_1)(a_2 - 2a_1)}{6} \\ &= BDP, \end{aligned}$$

completing the proof. □

LEMMA 2. *Suppose*

$$\begin{aligned} V &:= 2K + H^2 - D + 2L, \\ R &:= 2H - C. \end{aligned}$$

Then for $z > 1$

$$2[\beta(z) + \gamma(z)] - \delta(z) \geqslant B[V + RT]P.$$

Proof. From (10) and (11) we have

$$\begin{aligned} 2(\beta + \gamma) - \delta &= B[2(G+T)^2 - 2U - (H+T)^2]P \\ &= B[2(G^2 - H^2) + 2(TG - GT) + 4GT + H^2 - 2U - 2HT + T^2]P \\ &= B[2K + 2L + 2HT + H^2 + T^2 - 2U]P, \end{aligned}$$

by (9) and (7). Hence by the definitions of V and R we obtain

$$\begin{aligned} 2(\beta + \gamma) - \delta &= B[V + RT + \{D - 2U + CT + T^2\}]P \\ &\geqslant B[V + RT]P \end{aligned}$$

by Lemma 1, completing the proof. □

4. The Main Result

We now draw together our preliminary results and Lemma 2 to establish our convexity result.

THEOREM 1. *Suppose the diagonal entries of the matrices*

$$\begin{aligned} W &:= 4V - 4(M+1)H - (M-1)^2 I, \\ Z &:= R - (M+1)I \end{aligned}$$

are all nonnegative and that either at least one diagonal element of W *is positive or* $Z_{j,j} > 0$ *for at least one value of* j *with* $2 \leqslant j \leqslant N$. *Then* $\alpha(z)/(z^M - 1)$ *is strictly convex for all* $z > 1$.

Proof. We have that

$$z^2 \frac{d^2}{dz^2} \left(\frac{\alpha(z)}{z^M - 1} \right) = \frac{Q(z^M)}{(z^M - 1)^3},$$

where $Q(x)$ is the quadratic form

$$Q(x) := x^2 [-2M\beta + \gamma + M(M+1)\alpha] + x[-2\gamma + 2M\beta + M(M-1)\alpha] + \gamma$$

in x. This quadratic has discriminant

$$M^2 \left[4\beta^2 - 8\alpha\gamma + 4(M-1)\alpha\beta + (M-1)^2\alpha^2\right],$$

which by (12) is less than or equal to

$$M^2\alpha \left[-4(2\beta + 2\gamma - \delta) + 4(M+1)\beta + (M-1)^2\alpha\right].$$

From Lemma 2, (8) and the first relation in (4), this in turn is less than or equal to

$$\begin{aligned} M^2\alpha B[-4(V+RT) + 4(M+1)(H+T) + (M-1)^2 I]P \\ = M^2\alpha B[-4W - 4ZT]P. \end{aligned}$$

Since B and P have all positive entries and $T_{i,i} > 0$ for $2 \leqslant i \leqslant N$, the conditions of the enunciation suffice to ensure that the discriminant is negative. Hence $Q(x) > 0$ for all real x and in particular $Q(z^M) > 0$ for all $z > 1$. Thus the second derivative of $\alpha(z)/(z^M - 1)$ is positive for $z > 1$ and the theorem is established. $\square$

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