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REVERSE INEQUALITY TO ARAKI'S INEQUALITY COMPARISON OF $A^pZ^pA^p$ AND $(AZA)^p$

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Dedicated to Françoise Piquard respectfully and affectionately

(*communicated by F. Hansen*)

Abstract. Let A and Z be n-by-n matrices. Suppose $A \geqslant 0$ (positive semi-definite) and $Z > 0$ with extremal eigenvalues a and b. Then, for each $p > 1$, there exist unitary matrices U and V such that

$$\frac{1}{K(a,b,p)} U(AZA)^p U^* \leqslant A^p Z^p A^p \leqslant K(a,b,p) V(AZA)^p V^*.$$

where $K(a,b,p)$ is the Ky Fan constant. The right inequality is both a generalization of Ky Fan's inequality

$$\langle h, Z^p h\rangle \leqslant K(a,b,p)\langle h, Zh\rangle^p,$$

where h is an arbitrary norm one vector, and a reverse inequality to Araki's inequality

$$\|(AZA)^p\| \leqslant \|A^p Z^p A^p\|.$$

for unitarily invariant norms $\|\cdot\|$.

1. Statements of results

Capital letters A, $B \dots Z$ mean n-by-n complex matrices, or operators on a finite dimensional Hilbert space $\mathscr{H}$; I stands for the identity. When A is positive semidefinite, resp. positive definite, we write $A \geqslant 0$, resp. $A > 0$. Let $\|\cdot\|$ be a general symmetric (or unitarily invariant) norm, i.e. $\|UAV\| = \|A\|$ for all A and all unitaries U, V. In [1] (see also [2 pp 258, 285]) Araki showed a trace inequality which entails the following inequality for symmetric norms:

THEOREM 1. *Let $A \geqslant 0$, $Z \geqslant 0$ and $p > 1$. Then, for every symmetric norm,*

$$\|(AZA)^p\| \leqslant \|A^p Z^p A^p\|. \tag{1}$$

For $0 < p < 1$, the above inequality is reversed.

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If we take for A a rank one projection $A = h \otimes h$, $\|h\| = 1$, then (1) reduces to Jensen's inequality for $t \longrightarrow t^p$,

$$\langle h, Zh\rangle^p \leqslant \langle h, Z^p h\rangle . \tag{2}$$

This inequality admits a reverse inequality. Ky Fan [7] (see also [5]) introduced the following constant, for a, $b > 0$ and integers $p > 1$,

$$K(a,b,p) = \frac{a^p b - ab^p}{(p-1)(a-b)}\left(\frac{p-1}{p}\frac{a^p - b^p}{a^p b - ab^p}\right)^p$$

and Furuta [6] showed the sharp reverse inequality of (2): If $Z > 0$ have extremal eigenvalues a and b, then for all $p > 1$,

$$\langle h, Z^p h\rangle \leqslant K(a,b,p)\langle h, Zh\rangle^p. \tag{3}$$

Furuta also considered the case $p < 0$. It is not difficult to see that it suffices to prove (3) for 2-by-2 matrices Z. This is done in [3] for the case $p = 2$, the resulting inequality can then be read as

$$\|Zh\| \leqslant \frac{a+b}{2\sqrt{ab}}\langle h, Zh\rangle . \tag{4}$$

In [5], Fujii-Seo-Tominaga extend (3) to an operator norm inequality,

$$\|A^p Z^p A^p\|_\infty \leqslant K(a,b,p)\|(AZA)^p\|_\infty \tag{5}$$

This is a reverse inequality to Araki's inequality for operator norm and letting $A = h \otimes h$ we recapture (3). In this note we show that the previous inequality holds for all symmetric norms. In fact a stronger result holds: we show that (5) can be extended to all eigenvalues. Given two positive operators X and Y, recall that the eigenvalues of Y dominates the corresponding eigenvalues of X iff there exists a unitary operator V such that $X \leqslant VYV^*$. We have the following reverse inequality for (1):

THEOREM 2. *Let $A \geqslant 0$ and let $Z > 0$ with extremal eigenvalues a and b. Then, for every $p > 1$, there exist unitary operators U and V such that*

$$\frac{1}{K(a,b,p)}U(AZA)^p U^* \leqslant A^p Z^p A^p \leqslant K(a,b,p)V(AZA)^p V^*. \tag{6}$$

The Ky Fan constant $K(a,b,p)$ and its inverse are optimal.

If we take $p = 2$, we get

$$\frac{4ab}{(a+b)^2}U(AZA)^2 U^* \leqslant A^2 Z^2 A^2 \leqslant \frac{(a+b)^2}{4ab}V(AZA)^2 V^*.$$

Equivalently, there exist unitary operators U_0 and V_0 such that

$$\frac{2\sqrt{ab}}{a+b}U_0(AZA)U_0^* \leqslant (A^2Z^2A^2)^{1/2} \leqslant \frac{a+b}{2\sqrt{ab}}V_0(AZA)V_0^*.$$

Replacing A by $A^{1/2}$ and denoting by $\mathrm{Sing}(X)$ (resp. $\mathrm{Eig}(X)$) the singular values of X (resp. the real eigenvalues of X) arranged in decreasing order, the previous inequalities can be stated as:

PROPOSITION 1. *Let $A \geqslant 0$ and let $Z > 0$ with extremal eigenvalues a and b. Then,*

$$\frac{2\sqrt{ab}}{a+b}\,\mathrm{Eig}(AZ) \leqslant \mathrm{Sing}(AZ) \leqslant \frac{a+b}{2\sqrt{ab}}\,\mathrm{Eig}(AZ). \tag{7}$$

Replacing A^p by A in Theorem 2, we obtain the folowing equivalent statement:

THEOREM 2'. *Let $A \geqslant 0$ and let $Z > 0$ with extremal eigenvalues a and b. Then, for every $1 > q > 0$, there exist unitary operators U and V such that*

$$\frac{1}{K^q(a,b,1/q)}\,UA^qZ^qA^qU^* \leqslant (AZA)^q \leqslant K^q(a,b,1/q)VA^qZ^qA^qV^*.$$

The Ky Fan constant $K(a,b,1/q)$ and its inverse are optimal.

2. Proofs of results

Partial proof of Proposition 1. Here we give the proof of the right hand side inequality of (7). It is this result which leads us to research a reverse inequality to (1) in terms of singular values. For the largest eigenvalue and singular value of AZ, the right hand side inequality of (7) can be read as

$$\|AZ\|_\infty \leqslant \frac{a+b}{2\sqrt{ab}}\,\rho(AZ). \tag{8}$$

where $\rho(\cdot)$ stands for the spectral radius. This inequality is proved in [3] as a consequence of (4). We shall derive from (8) the general inequality between the other singular values $\mu_k(AZ)$ and eigenvalues $\lambda_k(AZ)$. By the minimax principle,

$$\mu_k(AZ) = \mu_k(ZA) \leqslant \|ZAE\|_\infty$$

for every projection E, $\mathrm{corank}E = k-1$. Let F be the projection onto the range of $A^{1/2}E$. Then, applying (8),

$$\begin{aligned}\|ZAE\|_\infty &= \|ZA^{1/2}FA^{1/2}E\|_\infty\\ &\leqslant \|ZA^{1/2}FA^{1/2}\|_\infty\\ &\leqslant \frac{a+b}{2\sqrt{ab}}\,\rho(ZA^{1/2}FA^{1/2})\\ &= \frac{a+b}{2\sqrt{ab}}\,\|FA^{1/2}ZA^{1/2}F\|_\infty.\end{aligned}$$

Now, by a limit argument we may assume that A is invertible. Consequently, we may choose E in such a way that F satisties

$$\|FA^{1/2}ZA^{1/2}F\|_\infty = \mu_k(A^{1/2}ZA^{1/2})$$

so that

$$\mu_k(AZ) \leqslant \frac{a+b}{2\sqrt{ab}}\,\mu_k(A^{1/2}ZA^{1/2}) = \frac{a+b}{2\sqrt{ab}}\,\lambda_k(AZ)$$

establishing the right hand side inequality of (7). □

Proof of Theorem 2. It is a melding of the above proof and the original proof of (4). We begin with the right hand side inequality. By the minimax principle, there exists a subspace $\mathscr{F}$ of codimension $k-1$ such that

$$\begin{aligned}\mu_k((AZA)^p) &= \max_{y\in\mathscr{F},\,\|y\|=1} \langle\, y, (AZA)^p y\rangle \\ &= \max_{y\in\mathscr{F},\,\|y\|=1} \langle\, y, AZAy\rangle^{\,p}. \end{aligned} \tag{9}$$

On the other hand, still by the minimax principle, for every subspace $\mathscr{E}$ of codimension $k-1$, we have

$$\begin{aligned}\mu_k(A^pZ^pA^p) &\leqslant \max_{x\in\mathscr{E},\,\|x\|=1} \langle\, x, A^pZ^pA^px\rangle \;=\; \max_{x\in\mathscr{E},\,\|x\|=1} \langle\, A^px, Z^pA^px\rangle \\ &\leqslant \max_{x\in\mathscr{E},\,\|x\|=1} K(a,b,p)\langle\, \frac{A^px}{\|A^px\|}, Z\frac{A^px}{\|A^px\|}\rangle^{\,p}\|A^px\|^2 \\ &= \max_{x\in\mathscr{E},\,\|x\|=1} K(a,b,p)\langle A^px, ZA^px\rangle^{\,p}\|A^px\|^{2-2p} \\ &= \max_{x\in\mathscr{E},\,\|x\|=1} K(a,b,p)\langle\, \frac{A^{p-1}x}{\|A^{p-1}x\|}, AZA\frac{A^{p-1}x}{\|A^{p-1}x\|}\rangle^{\,p}\|A^{p-1}x\|^{2p}\|A^px\|^{2-2p}. \end{aligned}$$

Now, observe that

$$\begin{aligned}\|A^{p-1}x\|^{2p}\|A^px\|^{2-2p} &= \langle\, x, A^{2p-2}x\rangle^{\,p}\langle\, x, A^{2p}x\rangle^{\,1-p} \\ &= \langle\, x, (A^{2p})^{\frac{2p-2}{2p}}x\rangle^{\,p}\langle\, x, A^{2p}x\rangle^{\,1-p} \\ &\leqslant \langle\, x, A^{2p}x\rangle^{\,p-1}\langle\, x, A^{2p}x\rangle^{\,1-p} = 1 \end{aligned}$$

by concavity of $t \longrightarrow t^{\frac{2p-2}{2p}}$. Therefore

$$\mu_k(A^pZ^pA^p) \leqslant \max_{x\in\mathscr{E},\,\|x\|=1} K(a,b,p)\langle\, \frac{A^{p-1}x}{\|A^{p-1}x\|}, AZA\frac{A^{p-1}x}{\|A^{p-1}x\|}\rangle^{\,p}.$$

Now we may assume that A is invertible and we may choose $\mathscr{E}$ such that

$$\left\{\frac{A^{p-1}x}{\|A^{p-1}x\|}\ :\ x\in\mathscr{E},\ \|x\|=1\right\} = \{y\in\mathscr{F},\ \|y\|=1\}.$$

Hence

$$\mu_k(A^pZ^pA^p) \leqslant \max_{y\in\mathscr{F},\,\|y\|=1} K(a,b,p)\langle\, y, AZAy\rangle^{\,p}$$

and comparing with (9) we obtain the result.

To prove the left hand side inequality it then suffices to take the inverse of the right one (by a limit argument we may assume A invertible) so that

$$A^{-p}Z^{-p}A^{-p} \geqslant K(a,b,p)V(A^{-1}Z^{-1}A^{-1})^pV^*$$

Next, we replace A and Z by their inverses and we use

$$K(a,b,p) = K(a^{-1},b^{-1},p)$$

which follows from $K(a,b,p) = K(b,a,p)$ and $K(\lambda a,\lambda b,p) = K(a,b,p)$ for all $\lambda > 0$.

It remains to check optimality of the constants. For the right hand side inequality it is obvious since (3) is sharp. Optimality of the left hand side then follows since these two inequalities are equivalent. □

3. Related results

The starting point of our investigation was the following result [3] (see also [4, Chapter 2]) which entails (4) and (8).

THEOREM 3. *Let A, B such that $AB \geqslant 0$ and let $Z > 0$ with extremal eigenvalues a and b. Then, for every symmetric norm, the following sharp inequality holds*

$$\|ZAB\| \leqslant \frac{a+b}{2\sqrt{ab}}\,\|BZA\|.$$

In a forthcomming project we will show the following extension of Proposition 1.

THEOREM 4. *Let $A \geqslant 0$ and let $Z > 0$ with extremal eigenvalues a and b. Then, for every scalars s, $t > 0$, there exist unitary operators U and V such that*

$$\frac{2\sqrt{ab}}{a+b}\,U|ZA^{s+t}|U^* \leqslant |A^sZA^t| \leqslant \frac{a+b}{2\sqrt{ab}}\,V|ZA^{s+t}|V^*.$$

The reverse inequality (2) is connected to some operator operator inequality [6]:

THEOREM 5. (Furuta) *Let $0 < A \leqslant B$, let a and b be the extremal eigenvalues of A, and let α and β those of B. Then, for every $p > 1$,*

$$A^p \leqslant K(a,b,p)B^p$$

and

$$A^p \leqslant K(\alpha,\beta,p)B^p$$

where the Ky Fan constants $K(a,b,p)$ and $K(\alpha,\beta,p)$ are optimal.

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REFERENCES

[1] H. ARAKI, *On an inequality of Lieb and Thirring*, Letters in Math. Phys., **19**, (1990), 167–170.
[2] R. BHATIA, *Matrix Analysis*, Springer, Germany, 1996.
[3] J.-C. BOURIN, *Symmetric norms and reverse inequalities to Davis and Hansen-Pedersen characterizations of operator convexity*, Math. Ineq. Appl., (submitted).
[4] J.-C. BOURIN, *Compressions, dilations and Matrix Inequalities*, RGMIA monograph, Victoria university, Melbourne 2004 (http://rgmia.vu.edu.au/monograph)
[5] J. I. FUJII, Y. SEO AND M. TOMINAGA, *Kantorovich type inequalities for operator norm*, Math. Ineq. Appl., **8**, 3 (2005), 529–535.
[6] T. FURUTA, *Operator inequalities associated with Holder-McCarthy and Kantorovich inequalities,* J. Inequal. Appl., **2**, (1998), 137–148.
[7] KY FAN, *Some matrix inequalities*, Abh. Math. Sem. Univ., Hamburg **29**, (1966), 185–196.

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