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BOUNDARY VALUE PROBLEMS ASSOCIATED WITH PERTURBED NONLINEAR SYLVESTER SYSTEMS – EXISTENCE AND UNIQUENESS

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Abstract. This paper is concerned with the existence and uniqueness of solutions to the boundary value problem associated with the general first-order linear/nonlinear Sylvester system

$$R'(t) = A(t)R(t) + R(t)B(t) + F(t, R(t)),$$

where the matrices involved are of appropriate dimensions and are continuous on some interval $[0,T]$, and $F \in C[[0,T] \times R^{n\times n},\ R^{n\times n}]$. The boundary conditions considered are of the form

$$UR = \alpha,$$

where $U : C[0,T] \mapsto R^{n\times n}$, and $C[0,T]$ is the space of all continuous bounded functions $f : [0,T] \mapsto R^{n\times n}$.

1. Introduction

In this paper, we shall be concerned with the existence and uniqueness of solutions to the boundary value problem associated with the general first-order linear/nonlinear Sylvester system

$$R'(t) = A(t)R(t) + R(t)B(t) + F(t, R(t)), \tag{1.1}$$

where A and B are $(n \times n)$ continuous matrices on some interval $[0,T]$, and $F \in C[[0,T] \times R^{n\times n},\ R^{n\times n}]$. We assume for the sake of convenience that $F(t,0) \equiv 0$, so that the system (1.1) admits a zero solution. We seek a solution of (1.1) satisfying the general boundary conditions

$$UR = \alpha, \tag{1.2}$$

where $U : C[0,T] \mapsto R^{n\times n}$, and $C[0,T]$ is the space of all continuous bounded functions $f : [0,T] \mapsto R^{n\times n}$.

The consideration of general boundary value problems for the Sylvester equation is motivated by applications to such important areas as convection-diffusion problems, augmented regulator problems, nonlinear control problems, and pole assignment problems for descriptor systems. The significance of this formulation is its integral equation

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representation as given in equation (1.5), and its straight-forward solution procedure as indicated in section 2.

We shall use the notation $\|f\|_\infty = \sup_{t\in[0,T]} \|f(t)\|$. Let $\Phi(t)$ and $\Psi(t)$ denote fundamental matrix solutions of the systems $R'(t) = A(t)R(t)$ and $R'(t) = B^*(t)R(t)$, respectively. Then any solution of the homogeneous system

$$R'(t) = A(t)R(t) + R(t)B(t) \tag{1.3}$$

is of the form $R(t) = \Phi(t)\zeta\Psi^*(t)$ [4], where $\zeta \in R^{n\times n}$ is a constant matrix. If $R(t)$ is any solution of (1.1) and $\bar{R}(t)$ is a particular solution of (1.1), then $R(t) - \bar{R}(t)$ is a solution of (1.3). Thus

$$R(t) = \bar{R}(t) + \Phi(t)\zeta\Psi^*(t).$$

A particular solution $\bar{R}(t)$ of (1.1) is given by [4]

$$\bar{R}(t) = \Phi(t)\Big[\int_{t_0}^{t} \Phi^{-1}(s)F(s,R(s)))\Psi^{*-1}(s)ds\ \Big]\ \Psi^*(t).$$

Thus the general solution of (1.1) is given by

$$R(t) = \Phi(t)\zeta\Psi^*(t) + \Phi(t)\Big[\int_{t_0}^{t} \Phi^{-1}(s)F(s,R(s)))\Psi^{*-1}(s)ds\ \Big]\ \Psi^*(t). \tag{1.4}$$

It may be noted that when $B^* = A$ and F is Hermetian, the system (1.1) is called a Lyapunov system; and then $\Psi^* = \Phi^*$ in the solution (1.4).

The general solution of the homogeneous equation (1.3) satisfies the general boundary condition matrix (1.2) if and only if

$$U(\Phi(\cdot)\zeta\Psi^*(\cdot)) = \chi\zeta\vartheta^*$$

for every $\zeta \in R^{n\times n}$, where χ is the matrix whose columns are the values of U on the corresponding columns of Φ and ϑ^* is the matrix whose rows are the values of U on the corresponding rows of Ψ^*. Therefore the general solution of (1.1) can be written as

$$R(t) = \Phi(t)\zeta\Psi^*(t) + P(t,R), \tag{1.5}$$

where

$$P(t,R) = \Phi(t)\Big[\int_{t_0}^{t} \Phi^{-1}(s)F(s,R(s)))\Psi^{*-1}(s)ds\ \Big]\ \Psi^*(t).$$

This solution satisfies the boundary condition matrix (1.2) if and only if

$$UR = \alpha = \chi\zeta\vartheta^* + UP(\cdot,R).$$

This equation in ζ has a unique solution for some $\alpha \in R^{n\times n}$ if and only if

$$\zeta = \chi^{-1}[\alpha - UP(\cdot,R)]\vartheta^{*-1}. \tag{1.6}$$

Thus the boundary value problem $(1.1), (1.2)$ will have a solution on $[0,T]$, in view of (1.5) and (1.6), if a function $R(t)$ can be found that satisfies the integral equation

$$R(t) = \Phi(t)\chi^{-1}[\alpha - UP(\cdot,R)]\vartheta^{*-1}\Psi^*(t) + P(t,R). \tag{1.7}$$

We are now in a position to prove the following existence theorem for the boundary value problem (1.1), (1.2) using the Schauder-Tychonov fixed point theorem.

THEOREM 1.1. *Define* $Q : [0,T] \mapsto R^{n\times n}$ *by*

$$Q(t) = \max_{\|R(t)\|\leqslant\beta}\{\|\Phi^{-1}(t)F(t,R(t)))\Psi^{*-1}(t)\|\},$$

and define the operator κ *by*

$$\kappa f = \chi^{-1}[\alpha - UP(\cdot,R)]\vartheta^{*-1}$$

for every $f \in B^{\beta}$, *where* B^{β} *is the closed ball of* $C[0,T]$ *centered at the origin with radius* $\beta > 0$. *Now let*

$$L_1 = \max_{t\in[0,T]}\|\Phi(t)\|,\ \ L_2 = \max_{t\in[0,T]}\|\Psi(t)\|,\ \ M = \sup_{f\in B^{\beta}}\|\kappa f\|,\ \text{and}\ N = \max_{t\in[0,T]}\int_0^t Q(s)ds.$$

Then the boundary value problem (1.1), (1.2) *will have at least one solution on* $[0,T]$ *if* $L_1(M+N)L_2 \leqslant \beta$.

Proof. We first show that the operator $T : B^{\beta} \mapsto C[0,T]$, defined by

$$(TR)(t) = \Phi(t)[\,\kappa f + \int_{t_0}^{t}\Phi^{-1}(s)F(s,R(s)))\Psi^{*-1}(s)ds\ ]\ \Psi^{*}(t),$$

has a unique fixed point in B^{β}. For this purpose, let $t, t_1 \in [0,T]$ be given. Then consider

$$\begin{aligned}\|(TR(t) - (TR)(t_1)\| &= \|\Phi(t)[\kappa f + \int_0^t \Phi^{-1}(s)F(s,R(s)))\Psi^{*-1}(s)ds\ ]\ \Psi^{*}(t)\\ &\quad - \Phi(t_1)[\kappa f + \int_0^{t_1}\Phi^{-1}(s)F(s,R(s)))\Psi^{*-1}(s)ds\ ]\ \Psi^{*}(t_1)\|\\ &\leqslant (M+N)\|\Phi(t)-\Phi(t_1)\| + L_1\|\int_t^{t_1}Q(s)ds\|L_2\\ &\leqslant (M+N)\|\Phi(t)-\Phi(t_1)\| + L\|\int_t^{t_1}Q(s)ds\|,\end{aligned}$$

where $L = L_1L_2$.

Let $\epsilon > 0$ be given. Then there exists a $\delta(\epsilon) > 0$ satisfying

$$\|\Phi(t) - \Phi(t_1)\| < \frac{\epsilon}{2(M+N)} \tag{1.8}$$

and

$$\|\int_0^{t_1}Q(s)ds\| < \frac{\epsilon}{2L}, \tag{1.9}$$

for every $t, t_1 \in [0,T]$ with $|t - t_1| < \delta(\epsilon)$. This is clear because Φ and Ψ are uniformly continuous on $R^{n\times n}$, and the function

$$H(t) \equiv \int_0^t Q(s)ds$$

is uniformly continuous on the interval $[0, T]$. Inequalities (1.8) and (1.9) imply the equicontinuity of the set TB^β. The inclusion relation $TB^\beta \subset B^\beta$ follows from $L(M+N) \leqslant \beta$. Hence TB^β is relatively compact.

Next, we show that T is continuous on B^β. Let $\{R_m\}_{m=1}^\infty \subset B^\beta$ and $R \in B^\beta$ be such that

$$\|R_m - R\|_\infty \longrightarrow 0 \quad \text{as} \quad m \longrightarrow \infty.$$

It follows that

$$\begin{aligned}
\|TR_m - TR\|_\infty \leqslant L\|\kappa R_m - \kappa R\| \\
&+ \| \int_0^t \Phi^{-1}(s)[F(s, R_m(s)) - F(s, R(s))]\Psi^{*-1}(s)ds\| \\
&\leqslant L(L(\|\Phi^{-1}\| \, \|\Psi^{*-1}\| \, \|R\| + 1))\times \\
&\times \| \int_0^t \chi^{-1}[F(s, R_m(s)) - F(s, R(s))]\vartheta^{*-1} ds\|.
\end{aligned}$$

The second integral in (1.10) converges uniformly to zero. Hence

$$\|TR_m - TR\|_\infty \longrightarrow 0 \quad \text{as} \quad m \longrightarrow \infty.$$

By the Schauder-Tychonov fixed point theorem, there exists a fixed point $\xi \in B^\beta$ of the operator T. This fixed point $\xi(t)$, $t \in [0, T]$, is a solution of the boundary value problem $(1.1), (1.2)$. □

Under alternate hypotheses, we use Brouwer's fixed point theorem to establish another existence and uniqueness result for the boundary value problem $(1.1), (1.2)$.

THEOREM 1.2. *Let B^β be the closed ball of $C[0, T]$ centered at the origin with radius $\beta > 0$. Assume that there exists a constant $K > 0$ such that for every $R_0 \in B^\beta$, the solution $R(t, 0, \zeta)$ of (1.1) with $R(0) = \zeta$ exists on $[0, T]$, is unique, and satisfies*

$$\sup_{t \in [0,T]} \|R(t, 0, \zeta)\| \leqslant K.$$

Let $Q(t), N,$ and $L = L_1 L_2$ be as in Theorem 1.1, and assume that

$$\|\chi^{-1}\|(\|\alpha\| + \|U\| LN) \leqslant \beta.$$

Then the boundary value problem $(1.1), (1.2)$ has a unique solution on any interval $[0, T]$.

Proof. Consider the operator T defined by

$$TR = \chi^{-1}(\alpha - UP_1(\cdot, R_0)]\vartheta^{*-1},$$

where

$$P_1(t, R) = \int_0^t \Phi(t)\Phi^{-1}(s)F(s, R(s, 0, R_0))\Psi^{*-1}(s)\Psi^*(t)ds.$$

We first note that $TB^{\beta} \subset B^{\beta}$. To prove the continuity of T, we show the continuity of $R(t,0,R_0)$ with respect to R_0. Let $\{R_m\}_{m=1}^{\infty} \subset B^{\beta}$ and $R \in B^{\beta}$ be such that

$$\|R_m - R\|_{\infty} \longrightarrow 0 \quad \text{as} \quad m \longrightarrow \infty,$$

and let Φ_m and Φ be solutions of the system of equations

$$R'(t) = A(t)R(t) + R(t)B(t) + F(t,R(t)), R(0) = R_m,$$
$$R'(t) = A(t)R(t) + R(t)B(t) + F(t,R(t)), R(0) = R,$$

respectively. Our assumptions imply that there exists a $K > 0$ such that $\|R_m(t)\|_{\infty} \leqslant K$ (for $m = 1,2,3,\dots$) and $\|R(t)\|_{\infty} \leqslant K$ for all $t \in [0,T]$. Now consideration of the inequality

$$\|R'_m(t)\|_{\infty} \leqslant K \sup_{t\in[0,T]} \{\|A(t)\| + \|B(t)\|\} \quad + \sup_{\|R(t)\|_{\infty} \leqslant K,\ t\in[0,T],} \|F(t,R(t))\|$$

proves that the sequence of functions $\{R_m(t)\}$ is equicontinuous and uniformly bounded, because by Ascoli's Theorem, there exists a subsequence $\{R_{m_j}(t)\}_{j=1}^{\infty}$ of $\{R_m(t)\}_{m=1}^{\infty}$ such that $R_{m_j}(t) \longrightarrow \bar{R}(t)$ as $j \longrightarrow \infty$, uniformly on $[0,T]$, where $\bar{R}(t) \in C[0,T]$. Taking the limit as $j \longrightarrow \infty$ in

$$R_{m_j}(t) = R_{m_j}(0) + \int_0^t A(s)R_{m_j}(s)ds + \int_0^t R_{m_j}(s)B(s)ds + \int_0^t F(s,R_{m_j}(s))ds,$$

we obtain

$$\bar{R}(t) = R_0 + \int_0^t A(s)\bar{R}(s)ds + \int_0^t \bar{R}(s)B(s)ds + \int_0^t F(s,\bar{R}(s))ds.$$

Therefore we have that $\bar{R}(t) \equiv R(t)$, by the uniqueness of solutions of initial value problems.

Hence we have actually shown the following. Since we could have started with any subsequence of $\{R_m(t)\}$ instead of $\{R_m(t)\}$ itself, we have that every subsequence of $\{R_m(t)\}$ contains a subsequence which converges uniformly to $R(t)$ on $[0,T]$. Thus the convergence of $\{R_m(t)\}$ to $R(t)$ is uniform on $[0,T]$.

Note that if $R_m \in B^{\beta}$ (for $m = 1,2,3,\dots$) and $R \in B^{\beta}$ satisfy $\|R_m - R\|_{\infty} \longrightarrow 0$ as $m \longrightarrow \infty$, then

$$\|R(t,0,R_m) - R(t,0,R_0)\| \longrightarrow \infty \quad \text{uniformly on} \quad [0,T].$$

This proves the continuity of the function $R(t,0,R_0)$ with respect to $R_0 \in B^{\beta}$, and the uniform continuity of $R(t,0,R_0)$ with respect to $t \in [0,T]$.

Let $\{R_m\}$ and R be as above. Then

$$\|TR_m - TR\|_{\infty} \leqslant \|\chi^{-1}\|\|U\|L\|\vartheta *^{-1}\| \times$$
$$\times \int_0^T \|\Phi^{-1}(s)\|\|F(s,R(s,0,R_m)) - F(s,R(s,0,R_0))\|ds.$$

The above integrand tends to zero uniformly as $m \longrightarrow \infty$. Therefore $\|TR_m - TR\|_\infty \longrightarrow 0$ as $m \longrightarrow \infty$, which proves the continuity of T on B^β. Now Brower's fixed point theorem implies the existence of some $R_0 \in B^\beta$ such that $TR_0 = R_0$. This vector R_0 is the solution of the boundary value problem $(1.1), (1.2)$, which completes the proof of the theorem. $\square$

2. Applications to Classical Two-Point Boundary Value Problems

In this section, we consider the classical two-point boundary value problem

$$R'(t) = A(t)R(t) + R(t)B(t) + F(t, R(t)), \quad a \leqslant t \leqslant b, \tag{2.1}$$

and

$$MR(a)M_1 + NR(b)N_1 = \alpha, \tag{2.2}$$

where $M, M_1, N,$ and N_1 are constant square matrices of order n, and all scalars are assumed to be real. Any solution of (2.1) can be written in the form

$$R(t) = Y(t)[\int_a^t Y^{-1}(s)F(s, R(s)))Z^{*-1}(s)ds]Z^*(t) + Y(t)\zeta Z^*(t),$$

where $Y(t)$ and $Z(t)$ are fundamental matrix solutions of the systems $R'(t) = A(t)R(t)$ and $R'(t) = B^*(t)R(t)$, respectively, and $\zeta \in R^{n\times n}$ is a constant matrix. Substituting this general solution into the boundary condition matrix (2.2), we obtain

$$\begin{aligned} &MY(a)\zeta Z^*(a)M_1 + NY(b)\zeta Z^*(b)N_1 \\ &\qquad + NY(b)[\int_a^b Y^{-1}(s)F(s, R(s)))Z^{*-1}(s)ds]Z^*(b)N_1 = \alpha, \end{aligned}$$

which is equivalent to

$$A_1\zeta B_1 + A_2\zeta B_2 = X, \tag{2.3}$$

where

$$A_1 = MY(a), \quad A_2 = NY(b), \quad B_1 = Z^*(a)M_1, \quad B_2 = Z^*(b)N_1,$$

and

$$X = \alpha - NY(b)[\int_a^b Y^{-1}(s)F(s, R(s))Z^{*-1}(s)ds]Z^*(b)N_1$$

are all known matrices of order $(n \times n)$. We use the Kronecker product representation to solve the symplectic system (2.3).

If $A \in C^{p\times q}$ and $B \in C^{m\times n}$, then the Kronecker product (or tensor product) of A and B, denoted by $A \otimes B$, is defined by

$$(A \otimes B) = (a_{ij}B) \in C^{pm\times qn} \ \ (i = 1, \ldots, p;\ j = 1, \ldots, q).$$

With this in mind, if $G = (A_1 \otimes B_1^T + A_2 \otimes B_2^T)$, then we can easily verify that equation (2.3) is equivalent to the system

$$Gc = x, \tag{2.4}$$

where $c \in C^{n^2\times 1}$ is defined by $c = vec(\zeta^T)$, and $x = vec(X^T)$. In fact, equation (2.4) corresponds to a system of n^2 scalar equations for the elements of ζ. If G is nonsingular, then $c = G^{-1}x$. In general, if G is not invertible, in order to determine existence and uniqueness of the solution of the hybrid system (2.4), we need information about the eigenvalues of G. We denote the set of all eigenvalues of G by $\sigma(G)$, the spectrum of G.

Case 1: Suppose that A_1 and B_1 are both nonsingular, then

$$\zeta - A\zeta B = Y, \tag{2.5}$$

where $A = -A_1^{-1}A_2$, $B = B_2B_1^{-1}$, and $Y = A_1^{-1}XB_1^{-1}$. Using properties of the Kronecker product as above, we see that equation (2.5) is equivalent to

$$[(I\otimes I)-(A\otimes B^T)]c = y \iff Ic - Hc = y,$$

where $y = vec(Y^T)$ and $H = (A\otimes B^T)$. Now, we can substitute $\zeta = Y + A\zeta B$ in the second term of (2.5), or $c = y + Hc$ in the second term of the equivalent expression $c - Hc = y$, to obtain

$$\zeta - A(Y + A\zeta B)B = Y \iff c - H(y + Hc) = y,$$

or

$$\zeta - A^2\zeta B^2 = Y + AYB \iff c - H^2c = y + Hy.$$

Proceeding in a similar manner, we see that in general,

$$\zeta - A^n\zeta B^n = Y + AYB + A^2YB^2 + \cdots + A^nYB^n \iff c - H^nc = y + Hy + H^2y + \cdots + H^ny.$$

If the spectral radii of A and B, denoted by $\rho(A)$ and $\rho(B)$, are such that $\rho(A)\rho(B) < 1$, then we have that $A^n\zeta B^n \longrightarrow 0$ as $n \longrightarrow \infty$. In such a case,

$$\zeta = \sum_{j=0}^{\infty} A^jYB^j = A_1^{-1}XB_1^{-1} + \sum_{j=1}^{\infty}(A_1^{-1}A_2)^j(A_1^{-1}XB_1^{-1})(B_2B_1^{-1})^j.$$

Substituting this expression for ζ into the general solution of (2.1), we obtain the iterative relationship

$$R^{(i)}(t) = Y(t)(A_1^{-1}XB_1^{-1})Z^*(t) + Y(t)[\sum_{j=1}^{\infty}(A_1^{-1}A_2)^j(A_1^{-1}XB_1^{-1})(B_2B_1^{-1})^j]Z^*(t)$$
$$+ Y(t)[\int_a^t Y^{-1}(s)F(s,R^{(i-1)}(s))Z^{*-1}(s)ds]Z^*(t).$$

Now we assume that F satisfies a Lipschitz condition with respect to the second variable, with Lipschitz constant $\Lambda > 0$. Using the expression for X following equation (2.3), and the condition numbers of Y and Z^*, denoted by $\kappa(Y)$ and $\kappa(Z^*)$, and defined by

$$\kappa(Y) = \sup_{a\leqslant t\leqslant b}\|Y(t)\| \sup_{a\leqslant t\leqslant b}\|Y^{-1}(t)\| \quad \text{and} \quad \kappa(Z^*) = \sup_{a\leqslant t\leqslant b}\|Z^*(t)\| \sup_{a\leqslant t\leqslant b}\|Z^{*-1}(t)\|,$$

we have that

$$\begin{aligned}\|R^{(i)}(t)-R^{(i-1)}(t)\| &\leqslant \Lambda\kappa(Y)\kappa(Z^*)\Big(\int_a^b \|R^{(i-1)}(s)-R^{(i-2)}(s)\|ds\Big)\\ &\quad\times\Big\{\|A_1^{-1}\|\|B_1^{-1}\|\|NY(b)N_1\|\|Z^*(b)\|\\ &\quad\times\Big[1+\sum_{j=1}^{\infty}\|(A_1^{-1}A_2)^j\|\|(B_2B_1^{-1})^j\|\Big]+1\Big\}\\ &\leqslant \delta(b-a)\|R^{(i-1)}-R^{(i-2)}\|,\end{aligned}$$

where

$$\delta=\Lambda\kappa(Y)\kappa(Z^*)\Big\{\|A_1^{-1}\|\|B_1^{-1}\|\|NY(b)N_1\|\|Z^*(b)\|\times \times\Big[1+\sum_{j=1}^{\infty}\|(A_1^{-1}A_2)^j\|\|(B_2B_1^{-1})^j\|\Big]+1\Big\}.$$

By repeating this process, we obtain the relationship at the $i-th$ stage of iteration as

$$\|R^{(i)}(t)-R^{(i-1)}(t)\|\leqslant\delta^{i-1}(b-a)^{i-1}\|R^{(1)}-R^{(0)}\|.$$

Thus we have that R is a contraction mapping whenever $\delta(b-a)<1$, and hence, R has a unique fixed point by the Banach fixed point theorem. This fixed point is the unique solution of the two point boundary value problem $(2.1),(2.2)$. This approach can be generalized to multipoint boundary value problems.

Case 2: Suppose that A_1 and B_2 are both nonsingular, then (2.3) takes the form

$$K\zeta+\zeta L=Y, \tag{2.6}$$

where $K=A_1^{-1}A_2$, $L=B_1B_2^{-1}$, and $Y=A_1^{-1}XB_2^{-1}$. One of the most effective methods of solving the matrix equation (2.6) is the Bartels-Stewart algorithm [6]. Key to this technique is the orthogonal reduction of K and L to triangular form using the QR algorithm. The method for finding the general solution for ζ is as follows. Let $K\in R^{n\times n}$ and $L\in R^{n\times n}$ be given matrices and define the linear transformation $\eta: R^{n\times n}\mapsto R^{n\times n}$ by

$$\eta(\zeta)=K\zeta+\zeta L. \tag{2.7}$$

This linear transformation is nonsingular if and only if K and $-L$ have no eigenvalues in common; i.e., if λ is an eigenvalue of K with corresponding eigenvector u and μ is an eigenvector of L with corresponding eigenvector v, then

$$Kuv^T+uv^TL^T=(\lambda+\mu)uv^T.$$

Thus $\lambda+\mu$ is an eigenvalue of the system (2.6), which can be solved if and only if

$$\lambda_i+\mu_j\neq 0$$

for all $i,j \in \{1,2,...,n\}$. When K and L are reduced to diagonal form by similarity transformations; i.e., when there exist matrices U and V such that

$$U^{-1}KU = diag(\lambda_1,\lambda_2,...,\lambda_n) \equiv K_1,$$
$$V^{-1}LV = diag(\mu_1,\mu_2,...,\mu_n) \equiv L_1,$$

then (2.6) is equivalent to

$$(U^{-1}KU)(U^{-1}\zeta V) + (U^{-1}\zeta V)(V^{-1}LV) = U^{-1}YV.$$

This system can be solved via the following four-step procedure.

Step 1 : Use similarity transformations to obtain the diagonal matrices K_1 and L_1.

Step 2 : Solve $UE = YV$ for E.

Step 3 : Solve the transformed system $K_1\zeta_1 + \zeta_1 L_1 = E$ for ζ_1.

Step 4 : Solve the system $\zeta V = UX$ for ζ.

Using these results, the solution of system (2.6) is obtained as

$$\zeta = U\zeta_1 V^{-1},$$

where $(\zeta_1)_{ij} = \frac{e_{ij}}{(\lambda_i+\mu_j)}$ and $E = (e_{ij}) = U^{-1}YV$. Now, substituting the general form of ζ in the variation of parameters formula (1.4), we obtain

$$R(t) = \Phi(t)U\zeta_1 V^{-1}\Psi^*(t) + \Phi(t)\Big[\int_{t_0}^{t} \Phi^{-1}(s)F(s,R(s)))\Psi^{*-1}(s)ds\ \Big]\ \Psi^*(t).$$

Assuming that F satisfies a Lipschitz condition with respect to the second variable, with Lipschitz constant $\Lambda > 0$, as before we have that

$$\begin{aligned}
\|R^{(i)}(t) - R^{(i-1)}(t)\| &\leqslant \Lambda\kappa(\Phi)\kappa(\Psi^*)\Big(\int_a^b \|R^{(i-1)}(s) - R^{(i-2)}(s)\|ds\Big) \\
&\leqslant \Lambda\kappa(\Phi)\kappa(\Psi^*)(b-a)\|R^{(i-1)} - R^{(i-2)}\| \\
&\vdots \\
&\leqslant \Lambda^{i-1}\kappa^{i-1}(\Phi)\kappa^{i-1}(\Psi^*)(b-a)^{i-1}\|R^{(1)} - R^{(0)}\|.
\end{aligned}$$

Again we have that R is a contraction mapping whenever $\delta(b-a) < 1$, where $\delta = \Lambda\kappa(\Phi)\kappa(\Psi^*)$. Hence by the Banach fixed point theorem, R has a unique fixed point which is the solution to the two point boundary value problem $(2.1), (2.2)$.

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