Mathematical
Inequalities
& Applications
Volume 8, Number 3 (2005), 389–402

ON APPROXIMATE t–CONVEXITY

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(*communicated by Z. Daróczy*)

Abstract. A real valued function f defined on an open convex set D is called (ε,δ,p,t)-convex if it satisfies

$$f\left(tx+(1-t)y\right)\leqslant tf(x)+(1-t)f(y)+\delta+\varepsilon|x-y|^p \qquad \text{for} \quad x,y\in D.$$

The main result of the paper states that if f is locally bounded from above at a point of D and (ε,δ,p,t)-convex (where $0\leqslant p<1$ and $t\leqslant 1/2$) then it satisfies the convexity-type inequality

$$f\Big(\lambda x+(1-\lambda)y\Big)\leqslant \lambda f(x)+(1-\lambda)f(y)+\delta/t+\varepsilon\varphi(\lambda)|x-y|^p \qquad \text{for} \quad x,y\in D,\ \lambda\in[0,1],$$

where $\varphi:[0,1]\to\mathbb{R}$ is a continuous function satisfying

$$\varphi(\lambda)=\max\left\{\frac{1}{t^p-t};\frac{1}{(1/2-t/2)^p-(1-t)^{1-p}(1/2-t)^p}\right\}(\lambda(1-\lambda))^p.$$

In the case $p=1, t=1/2$ analogous results were obtained in [2].

1. Introduction

In this paper we intend to investigate the following general approximate convexity concept. A function $f:D\to\mathbb{R}$ is said to be (ε,δ,p,t)-convex if

$$f\left(tx+(1-t)y\right)\leqslant tf(x)+(1-t)f(y)+\delta+\varepsilon|x-y|^p$$

for every $x,y\in D$, where $\varepsilon,\delta\geqslant 0, p>0$ and $t\in]0,1[$. If this inequality holds for every $t\in[0,1]$, then the function f is called (ε,δ,p)-convex. Some particular cases were investigated in several papers.

In the case $\varepsilon=0$, the following theorem was proved by Páles in [5]:

THEOREM 1. *Let X be a real topological vector space. Let $D\subset X$ be an open and convex set and $f:D\to\mathbb{R}$ be $(0,\delta,p,t)$-convex. If f is locally bounded from above at a point of D, then*

$$f\left(\lambda x+(1-\lambda)y\right)\leqslant \lambda f(x)+(1-\lambda)f(y)+\max\left\{\frac{1}{t},\frac{1}{1-t}\right\}\delta$$

for all $x,y\in D$ and $\lambda\in[0,1]$.

Mathematics subject classification (2000): 26A51, 26B25.

Key words and phrases: convexity, approximately convexity, (ε,δ,p,t)-convexity.

This research has been supported by the Hungarian Scientific Research Fund (OTKA) Grant T-030082 and by the Higher Education, Research and Development Fund (FKFP) Grant 0215/2001.

The case $\varepsilon = 0$ and $t = 1/2$ was investigated by Nikodem and Ng [4], the specialization $\varepsilon = \delta = 0$ yields the theorem of Bernstein and Doetsch [1].

The case $p = 1$ and $t = 1/2$ was investigated in Házy and Páles [2]. In this case the following theorem was proved:

THEOREM 2. *Let D be an open convex subset of a real normed space $(X, |\cdot|)$. Let ε, δ be nonnegative constants. If $f : D \to \mathbb{R}$ is locally bounded from above at a point of D and $(\varepsilon, \delta, 1, 1/2)$-convex function on D, then*

$$f(\lambda x + (1-\lambda)y) \leqslant \lambda f(x) + (1-\lambda)f(y) + 2\delta + 2\varepsilon\varphi(\lambda)|x-y|$$

for all $x, y \in D$ and $\lambda \in [0,1]$, where φ is the Takagi function is defined by

$$\varphi(\lambda) = \sum \frac{\operatorname{dist}(2^n\lambda, \mathbb{Z})}{2^n}.$$

Since the Takagi function satisfies the inequality $\varphi(\lambda) \leqslant 1.4\phi(\lambda)$, where ϕ is defined by

$$\phi(\lambda) := \max\big(-\lambda \log_2 \lambda, -(1-\lambda)\log_2(1-\lambda)\big) = \begin{cases} -\lambda \log_2 \lambda & 0 \leqslant \lambda \leqslant \frac{1}{2}, \\ -(1-\lambda)\log_2(1-\lambda) & \frac{1}{2} \leqslant \lambda \leqslant 1, \end{cases}$$

therefore

$$f(\lambda x + (1-\lambda)y) \leqslant \lambda f(x) + (1-\lambda)f(y) + 2\delta + 2.8\varepsilon\phi(\lambda)|x-y|$$

holds for all $x, y \in D$ and $\lambda \in [0,1]$.

The aim of this paper is to extend Theorem 2 for $(\varepsilon, \delta, p, t)$-convex functions. Due to the symmetry, we may assume that $0 < t \leqslant 1/2$ in the sequel.

2. A functional equation and related functional inequalities

For fixed $p \geqslant 0$ and $t \in]0, 1/2]$, introduce the operator $T_{p,t}$ by

$$(T_{p,t}\Phi)(\lambda) = \begin{cases} \min\left\{(1-t)\Phi\left(\frac{\lambda}{1-t}\right) + \left(\frac{\lambda}{1-t}\right)^p; \; t\Phi\left(\frac{\lambda}{t}\right) + \left(\frac{\lambda}{t}\right)^p\right\} & 0 \leqslant \lambda \leqslant t, \\ \min\left\{(1-t)\Phi\left(\frac{\lambda}{1-t}\right) + \left(\frac{\lambda}{1-t}\right)^p; \; (1-t)\Phi\left(\frac{1-\lambda}{1-t}\right) + \left(\frac{1-\lambda}{1-t}\right)^p\right\} & t < \lambda < 1-t, \\ \min\left\{t\Phi\left(\frac{1-\lambda}{t}\right) + \left(\frac{1-\lambda}{t}\right)^p; \; (1-t)\Phi\left(\frac{1-\lambda}{1-t}\right) + \left(\frac{1-\lambda}{1-t}\right)^p\right\} & 1-t \leqslant \lambda \leqslant 1. \end{cases}$$

In this section we deal with the functional equation

$$\varphi(\lambda) = (T_{p,t}\varphi)(\lambda) \qquad (\lambda \in [0,1]). \tag{1}$$

In our first result we construct a nonnegative bounded solution $\varphi : [0,1] \to \mathbb{R}$ of the functional equation (1). This solution will be denoted by $\varphi_{p,t}$.

PROPOSITION 1. *Let the sequence $(\varphi_n) : [0,1] \to \mathbb{R}$ be defined by*

$$\begin{aligned} \varphi_1 &:= 0, \\ \varphi_{n+1}(\lambda) &:= (T_{p,t}\varphi_n)(\lambda). \end{aligned} \tag{2}$$

Then φ_n is an increasing sequence of continuous nonnegative functions and the function φ defined by

$$\varphi(\lambda) := \lim_{n\to\infty} \varphi_n(\lambda) \qquad (\lambda \in [0,1]), \tag{3}$$

satisfies (1). Furthermore this function is continuous on $[0,1]$, and symmetric with respect to $\lambda = 1/2$, i.e., $\varphi(\lambda) = \varphi(1-\lambda)$ for all $\lambda \in [0,1]$.

In our proof we use the following obvious lemma:

LEMMA 1. *Let a,b,c,d be arbitrary real numbers. Then*

$$|\min\{a,b\} - \min\{c,d\}| \leqslant \max\{|a-c|, |b-d|\}.$$

Proof Proof of Proposition 1..

It is easy to see, by induction, that φ_n is an increasing sequence of functions indeed, and φ_n is continuous, and symmetric with respect to $1/2$.

Applying Lemma 1, we prove, that the function $T_{p,t}$ is a contraction on the space of bounded real-valued functions defined on $I := [0,1]$ equipped with the usual supremum norm. If $\lambda \in [0,t]$, then

$$\begin{aligned} &|T_{p,t}\Phi(\lambda) - T_{p,t}\Psi(\lambda)| \\ &\quad = \left|\min\left\{(1-t)\Phi\left(\frac{\lambda}{1-t}\right) + \left(\frac{\lambda}{1-t}\right)^p ; t\Phi\left(\frac{\lambda}{t}\right) + \left(\frac{\lambda}{t}\right)^p\right\}\right. \\ &\qquad \left. - \min\left\{(1-t)\Psi\left(\frac{\lambda}{1-t}\right) + \left(\frac{\lambda}{1-t}\right)^p ; t\Psi\left(\frac{\lambda}{t}\right) + \left(\frac{\lambda}{t}\right)^p\right\}\right| \\ &\quad \leqslant \max\left\{\left|(1-t)\Phi\left(\frac{\lambda}{1-t}\right) - (1-t)\Psi\left(\frac{\lambda}{1-t}\right)\right|, \left|t\Phi\left(\frac{\lambda}{t}\right) - t\Psi\left(\frac{\lambda}{t}\right)\right|\right\} \\ &\quad \leqslant \max\{(1-t)||\Phi - \Psi||, t||\Phi - \Psi||\} = (1-t)||\Phi - \Psi||. \end{aligned}$$

The cases $\lambda \in [t, 1-t]$ and $\lambda \in [1-t, 1]$ can be dealt with similarly. Hence we get

$$||T_{p,t}\Phi - T_{p,t}\Psi|| = \max_{\lambda\in[0,1]} |T_{p,t}\Phi(\lambda) - T_{p,t}\Psi(\lambda)| \leqslant (1-t)||\Phi - \Psi||,$$

where $1 - t < 1$, therefore, by the Banach fixed point theorem, there exists a unique fixed point of the contraction $T_{p,t}$. Also, by this fixed point theorem, the sequence φ_n uniformly tends to the fixed point of $T_{p,t}$. Therefore φ is also continuous, nonnegative, and symmetric.

Now we investigate the functional inequalities

$$\Psi(\lambda) \leqslant (T_{p,t}\Psi)(\lambda) \qquad (\lambda \in [0,1]), \tag{4}$$

$$\Phi(\lambda) \geqslant (T_{p,t}\Phi)(\lambda) \qquad (\lambda \in [0,1]). \tag{5}$$

PROPOSITION 2. *Let $\Psi : [0,1] \to \mathbb{R}$ be an upper bounded solution of the functional inequality (4) and $\Phi : [0,1] \to \mathbb{R}$ be a lower bounded solution of the functional inequality (5). Then $\Psi \leqslant \varphi_{p,t} \leqslant \Phi$, where $\varphi_{p,t}$ is defined in (3).*

Proof. Let the sequence $(\psi_n) : [0,1] \to \mathbb{R}$ be defined by

$$\begin{aligned} \psi_1 &:= 1/t, \\ \psi_{n+1}(\lambda) &:= (T_{p,t}\psi_n)(\lambda) \end{aligned} \tag{6}$$

and

$$K := \sup_{\lambda\in[0,1]} \Psi(\lambda), \quad L := \inf_{\lambda\in[0,1]} \Phi(\lambda).$$

Then $\lim_{n\to\infty} \psi_n(\lambda) = \varphi(\lambda)$ because $T_{p,t}$ is a contraction and, by (4) and (5), we have that

$$\Psi(\lambda) \leqslant \left\{ \begin{array}{ll} tK + \left(\dfrac{\lambda}{t}\right)^p \leqslant tK + 1, & 0 \leqslant \lambda \leqslant t \\ (1-t)K + \left(\dfrac{\lambda}{1-t}\right)^p \leqslant (1-t)K + 1, & 0 \leqslant \lambda \leqslant 1-t \\ (1-t)K + \left(\dfrac{1-\lambda}{1-t}\right)^p \leqslant (1-t)K + 1, & t \leqslant \lambda \leqslant 1 \\ tK + \left(\dfrac{1-\lambda}{t}\right)^p \leqslant tK + 1, & 1-t \leqslant \lambda \leqslant 1 \end{array} \right\} \leqslant (1-t)K+1$$

and

$$\Phi(\lambda) \geqslant \left\{ \begin{array}{ll} tL + \left(\dfrac{\lambda}{t}\right)^p \geqslant tL, & 0 \leqslant \lambda \leqslant t \\ (1-t)L + \left(\dfrac{\lambda}{1-t}\right)^p \geqslant (1-t)L, & 0 \leqslant \lambda \leqslant 1-t \\ (1-t)L + \left(\dfrac{1-\lambda}{1-t}\right)^p \geqslant (1-t)L, & t \leqslant \lambda \leqslant 1 \\ tL + \left(\dfrac{1-\lambda}{t}\right)^p \geqslant tL, & 1-t \leqslant \lambda \leqslant 1 \end{array} \right\} \geqslant tL.$$

Therefore

$$K = \sup_{\lambda\in[0,1]} \Psi(\lambda) \leqslant (1-t)K + 1 \qquad \text{and} \qquad L = \inf_{\lambda\in[0,1]} \Phi(\lambda) \geqslant tL,$$

whence we get that

$$K \leqslant 1/t \qquad \text{and} \qquad L \geqslant 0.$$

That is,

$$\Psi \leqslant \psi_1 \qquad \text{and} \qquad \Phi \geqslant \varphi_1.$$

One can easily see, by induction, that $\Psi \leqslant \psi_n$ and $\Phi \geqslant \varphi_n$ for all n. By taking the limit $n \to \infty$, this implies that $\Phi \geqslant \varphi_{p,t}$ and $\Psi \leqslant \varphi_{p,t}$. □

In Proposition 3 below, we will compare $\varphi_{p,t}$ with function $\phi_p : [0,1] \to \mathbb{R}$ defined by the following formula:

$$\phi_p(\lambda) := (\lambda(1-\lambda))^p. \tag{7}$$

In order to prove this proposition, we need the following lemma.

LEMMA 2. *Let $0 \leqslant p < 1$ be an arbitrary constant and $\gamma_{p,t} : [0,t] \to \mathbb{R}$ be defined by*

$$\gamma_{p,t}(\lambda) = (1-\lambda)^p - t^{1-2p}(t-\lambda)^p.$$

Then $\gamma_{p,t}$ is a positive, monotone increasing function.

Proof. Since $t < 1$ and $0 < 1-p < 1$, therefore

$$\gamma_{p,t}(0) = 1 - t^{1-p} > 0.$$

The function $\gamma_{p,t}$ is differentiable and

$$\gamma'_{p,t}(\lambda) = -p(1-\lambda)^{p-1} + t^{1-2p}p\,(t-\lambda)^{p-1} = t^{-2p}p\Big(-(1-\lambda)^{p-1}t^{2p} + t(t-\lambda)^{p-1}\Big).$$

Since $t^{-2p}p > 0$, therefore is enough to prove that

$$t(t-\lambda)^{p-1} > (1-\lambda)^{p-1}t^{2p},$$

i.e.,

$$\left(\frac{t-\lambda}{1-\lambda}\right)^{p-1} > t^{2p-1}. \tag{8}$$

Let

$$g_{p,t}(\lambda) = \left(\frac{t-\lambda}{1-\lambda}\right)^{p-1}, \qquad (\lambda \in [0,t]).$$

Then $g_{p,t}(0) = t^{p-1} > t^{2p-1}$ and $g_{p,t}$ is a differentiable function and

$$g'_{p,t}(\lambda) = (p-1)\left(\frac{t-\lambda}{1-\lambda}\right)^{p-2}\frac{t-1}{(1-\lambda)^2} > 0,$$

which implies that $g_{p,t}$ is a monotone increasing function and the inequality (8) holds. So we get that the function $\gamma_{p,t}$ is monotone increasing too, which implies $\gamma_{p,t}(\lambda) > 1 - t^{1-p}$ for all $0 \leqslant \lambda \leqslant t$. □

The functions ϕ_p and $\varphi_{p,t}$ have the following property:

PROPOSITION 3. *If $0 < t \leqslant 1/2$ and $0 < p < 1$, then*

$$\begin{aligned}\frac{1}{(t(1-t))^p}\phi_p(\lambda) &\leqslant \varphi_{p,t}(\lambda)\\ &\leqslant \max\left\{\frac{1}{t^p - t}; \frac{1}{(1/2-t/2)^p - (1-t)^{1-p}(1/2-t)^p}\right\}\phi_p(\lambda)\end{aligned} \tag{9}$$

for all $\lambda \in [0,1]$.

Proof. In the first step we prove that the function $\Psi = \dfrac{1}{(t(1-t))^p}\phi_p$ is a solution of the functional inequality (4).

In this case we need to prove that $\Psi(\lambda) \leqslant T_{p,t}(\Psi)(\lambda)$, i.e.,

$$\Psi(\lambda) \leqslant \begin{cases} \min\left\{(1-t)\Psi\left(\dfrac{\lambda}{1-t}\right) + \left(\dfrac{\lambda}{1-t}\right)^p; \right. \\ \qquad\qquad \left. t\Psi\left(\dfrac{\lambda}{t}\right) + \left(\dfrac{\lambda}{t}\right)^p\right\} & 0 \leqslant \lambda \leqslant t, \\ \min\left\{(1-t)\Psi\left(\dfrac{\lambda}{1-t}\right) + \left(\dfrac{\lambda}{1-t}\right)^p; \right. \\ \qquad\qquad \left. (1-t)\Psi\left(\dfrac{1-\lambda}{1-t}\right) + \left(\dfrac{1-\lambda}{1-t}\right)^p\right\} & t < \lambda < 1-t, \\ \min\left\{t\Psi\left(\dfrac{1-\lambda}{t}\right) + \left(\dfrac{1-\lambda}{t}\right)^p; \right. \\ \qquad\qquad \left. (1-t)\Psi\left(\dfrac{1-\lambda}{1-t}\right) + \left(\dfrac{1-\lambda}{1-t}\right)^p\right\} & 1-t \leqslant \lambda \leqslant 1, \end{cases}$$

which means that

$$\Psi(\lambda) \leqslant \begin{cases} t\Psi\left(\dfrac{\lambda}{t}\right) + \left(\dfrac{\lambda}{t}\right)^p, & \text{if } 0 \leqslant \lambda \leqslant t, \\ (1-t)\Psi\left(\dfrac{\lambda}{1-t}\right) + \left(\dfrac{\lambda}{1-t}\right)^p, & \text{if } 0 \leqslant \lambda < 1-t, \\ t\Psi\left(\dfrac{1-\lambda}{t}\right) + \left(\dfrac{1-\lambda}{t}\right)^p, & \text{if } 1-t \leqslant \lambda \leqslant 1, \\ (1-t)\Psi\left(\dfrac{1-\lambda}{1-t}\right) + \left(\dfrac{1-\lambda}{1-t}\right)^p, & \text{if } t < \lambda \leqslant 1. \end{cases} \tag{10}$$

If $0 \leqslant \lambda \leqslant t$, then applying the previous lemma, we get that

$$\frac{1}{(t(1-t))^p}\gamma_{p,t}(\lambda) \leqslant \frac{1}{(t(1-t))^p}\gamma_{p,t}(t) = \frac{1}{(t(1-t))^p}(1-t)^p = \frac{1}{t^p},$$

which implies

$$\frac{1}{(t(1-t))^p}\left[(1-\lambda)^p - t^{1-p}\left(1-\frac{\lambda}{t}\right)^p\right] \leqslant \frac{1}{t^p}.$$

Transforming this inequality, we get

$$\Psi(\lambda)=\frac{1}{(t(1-t))^p}\lambda^p(1-\lambda)^p \leqslant t\frac{1}{(t(1-t))^p}\frac{\lambda^p}{t^p}\left(1-\frac{\lambda}{t}\right)^p+\frac{\lambda^p}{t^p}=t\Psi\left(\frac{\lambda}{t}\right)+\left(\frac{\lambda}{t}\right)^p,$$

which is the first inequality of (10).

If $0 \leqslant \lambda \leqslant 1-t$, applying the previous lemma with $1-t$ instead of t, we get that

$$\frac{1}{(t(1-t))^p}\gamma_{p,1-t}(\lambda) \leqslant \frac{1}{(t(1-t))^p}\gamma_{p,1-t}(1-t) = \frac{1}{(t(1-t))^p}t^p = \frac{1}{(1-t)^p},$$

that is,

$$\frac{1}{(t(1-t))^p}\left[(1-\lambda)^p - (1-t)^{1-p}\left(1-\frac{\lambda}{1-t}\right)^p\right] \leqslant \frac{1}{(1-t)^p},$$

which implies that

$$\frac{1}{(t(1-t))^p}\lambda^p(1-\lambda)^p \leqslant (1-t)\frac{1}{(t(1-t))^p}\frac{\lambda^p}{(1-t)^p}\left(1-\frac{\lambda}{1-t}\right)^p + \frac{\lambda^p}{(1-t)^p}.$$

Thus we get the second inequality of (10).
Since ϕ_p is symmetric with respect to $\lambda = 1/2$, i.e., $\phi_p(\lambda) = \phi_p(1-\lambda)$ for all $\lambda \in [0,1]$, therefore Ψ is also symmetric. Therefore in the case $t < \lambda \leqslant 1$ and $1-t \leqslant \lambda \leqslant 1$ we get

$$\frac{1}{(t(1-t))^p}\lambda^p(1-\lambda)^p \leqslant (1-t)\frac{1}{(t(1-t))^p}\left(\frac{\lambda-t}{1-t}\right)\left(\frac{1-\lambda}{1-t}\right)^p + \left(\frac{1-\lambda}{1-t}\right)^p$$

and

$$\frac{1}{(t(1-t))^p}\lambda^p(1-\lambda)^p \leqslant t\frac{1}{(t(1-t))^p}\left(\frac{\lambda+t-1}{t}\right)^p\left(\frac{1-\lambda}{t}\right)^p + \left(\frac{1-\lambda}{t}\right)^p,$$

i.e., the third and the fourth inequalities of (10) hold. Thus $\frac{1}{(t(1-t))^p}\phi_p$ is a bounded solution of inequality (4), which, by Lemma 2, implies that

$$\frac{1}{(t(1-t))^p}\phi_p \leqslant \varphi_{p,t}.$$

Now, we prove that the constant

$$c_{p,t} = \max\left\{\frac{1}{t^p - t};\ \frac{1}{(1/2-t/2)^p - (1-t)^{1-p}(1/2-t)^p}\right\}$$

is such that the function $\Phi = c_{p,t}\phi_p$ is a solution of the functional inequality (5).

In this case we need to prove that $\Phi(\lambda) \geqslant T_{p,t}(\Phi)(\lambda)$, i.e.,

$$\Phi(\lambda) \geqslant \begin{cases} \min\left\{(1-t)\Phi\left(\frac{\lambda}{1-t}\right) + \left(\frac{\lambda}{1-t}\right)^p;\right. \\ \qquad \left. t\Phi\left(\frac{\lambda}{t}\right) + \left(\frac{\lambda}{t}\right)^p\right\} & 0 \leqslant \lambda \leqslant t, \\ \min\left\{(1-t)\Phi\left(\frac{\lambda}{1-t}\right) + \left(\frac{\lambda}{1-t}\right)^p;\right. \\ \qquad \left. (1-t)\Phi\left(\frac{1-\lambda}{1-t}\right) + \left(\frac{1-\lambda}{1-t}\right)^p\right\} & t < \lambda < 1-t, \\ \min\left\{t\Phi\left(\frac{1-\lambda}{t}\right) + \left(\frac{1-\lambda}{t}\right)^p;\right. \\ \qquad \left. (1-t)\Phi\left(\frac{1-\lambda}{1-t}\right) + \left(\frac{1-\lambda}{1-t}\right)^p\right\} & 1-t \leqslant \lambda \leqslant 1, \end{cases}$$

If $0 \leqslant \lambda \leqslant t$, then the functions $\dfrac{1}{t^p\gamma_{p,t}(\lambda)}$ and $\dfrac{1}{(1-t)^p\gamma_{p,1-t}(\lambda)}$ are monotone decreasing and

$$\frac{1}{t^p\gamma_{p,t}(0)} = \frac{1}{t^p - t},$$
$$\frac{1}{(1-t)^p\gamma_{p,1-t}(0)} = \frac{1}{(1-t)^p-(1-t)}.$$

As $t \leqslant 1/2$, therefore $t^p - t \geqslant (1-t)^p - (1-t)$, thus

$$c_{p,t} \geqslant \frac{1}{t^p-t} = \min\left\{\frac{1}{t^p-t}; \frac{1}{(1-t)^p-(1-t)}\right\}$$
$$= \max_{0\leqslant\lambda\leqslant t}\left\{\min\left\{\frac{1}{t^p\gamma_{p,t}(\lambda)}; \frac{1}{(1-t)^p\gamma_{p,1-t}(\lambda)}\right\}\right\}.$$

Thus

$$c_{p,t} \geqslant \min\left\{\frac{1}{t^p\gamma_{p,t}(\lambda)}; \frac{1}{(1-t)^p\gamma_{p,1-t}(\lambda)}\right\}$$

for all $0 \leqslant \lambda \leqslant t$, which implies that either

$$c_{p,t}\gamma_{p,t}(\lambda) = c_{p,t}\left[(1-\lambda)^p - t^{1-2p}(t-\lambda)^p\right] \geqslant \frac{1}{t^p},$$

or

$$c_{p,t}\gamma_{p,1-t}(\lambda) = c_{p,t}\left[(1-\lambda)^p - (1-t)^{1-2p}(1-t-\lambda)^p\right] \geqslant \frac{1}{(1-t)^p}$$

hold for all $0 \leqslant \lambda \leqslant t$. It means that either

$$c_{p,t}\lambda^p(1-\lambda)^p \geqslant tc_{p,t}\frac{\lambda^p}{t^p}\left(1-\frac{\lambda}{t}\right)^p + \frac{\lambda^p}{t^p},$$

or

$$c_{p,t}\lambda^p(1-\lambda)^p \geqslant (1-t)c_{p,t}\frac{\lambda^p}{(1-t)^p}\left(1-\frac{\lambda}{1-t}\right)^p + \frac{\lambda^p}{(1-t)^p}$$

for all $0 \leqslant \lambda \leqslant t$. Therefore

$$c_{p,t}\lambda^p(1-\lambda)^p \geqslant \min\left\{(1-t)c_{p,t}\left(\frac{\lambda}{1-t}\right)^p\left(1-\frac{\lambda}{1-t}\right) + \left(\frac{\lambda}{1-t}\right)^p;\right.$$
$$\left. tc_{p,t}\left(\frac{\lambda}{t}\right)^p\left(1-\frac{\lambda}{t}\right)^p + \left(\frac{\lambda}{t}\right)^p\right\},$$

which implies

$$\Phi(\lambda) \geqslant \min\left\{(1-t)\Phi\left(\frac{\lambda}{1-t}\right) + \left(\frac{\lambda}{1-t}\right)^p; t\Phi\left(\frac{\lambda}{t}\right) + \left(\frac{\lambda}{t}\right)^p\right\}, \quad (0 \leqslant \lambda \leqslant t).$$

If $t \leqslant \lambda \leqslant 1-t$, then the function $\dfrac{1}{(1-t)^p\gamma_{p,1-t}(\lambda)}$ is monotone decreasing and $\dfrac{1}{(1-t)^p\gamma_{p,1-t}(1-\lambda)}$ is monotone increasing and

$$\frac{1}{(1-t)^p\gamma_{p,1-t}(1/2)} = \frac{1}{(1/2-t/2)^p - (1-t)^{1-p}(1/2-t)^p}.$$

Therefore

$$
\begin{aligned}
c_{p,t} &\geqslant \frac{1}{(1/2-t/2)^p-(1-t)^{1-p}(1/2-t)^p} \\
&= \max_{t\leqslant\lambda\leqslant 1-t}\left\{\begin{array}{ll} \dfrac{1}{(1-t)^p\gamma_{p,1-t}(1-\lambda)}, & \text{if } t\leqslant\lambda\leqslant 1/2 \\ \dfrac{1}{(1-t)^p\gamma_{p,1-t}(\lambda)}, & \text{if } 1/2\leqslant\lambda\leqslant 1-t \end{array}\right\} \\
&= \max_{t\leqslant\lambda\leqslant 1-t}\left\{\min\left\{\frac{1}{(1-t)^p\gamma_{p,1-t}(\lambda)};\frac{1}{(1-t)^p\gamma_{p,1-t}(1-\lambda)}\right\}\right\}.
\end{aligned}
$$

These inequalities hold if

$$
c_{p,t}\geqslant \min\left\{\frac{1}{(1-t)^p\gamma_{p,1-t}(\lambda)};\frac{1}{(1-t)^p\gamma_{p,1-t}(1-\lambda)}\right\}\qquad (t\leqslant\lambda\leqslant 1-t),
$$

which implies, for all $t\leqslant\lambda\leqslant 1-t$, that either

$$
c_{p,t}\gamma_{p,1-t}(\lambda)=c_{p,t}\left[(1-\lambda)^p-(1-t)^{1-2p}\left((1-t)-\lambda\right)^p\right]\geqslant\frac{1}{(1-t)^p},
$$

or

$$
c_{p,t}\gamma_{p,1-t}(1-\lambda)=c_{p,t}\left[\lambda^p-(1-t)^{1-2p}\left(\lambda-t\right)^p\right]\geqslant\frac{1}{(1-t)^p},
$$

that is, for all $t\leqslant\lambda\leqslant 1-t$, either

$$
c_{p,t}\lambda^p(1-\lambda)^p\geqslant(1-t)c_{p,t}\frac{\lambda^p}{(1-t)^p}\frac{(1-t-\lambda)^p}{(1-t)^p}+\frac{\lambda^p}{(1-t)^p},
$$

or

$$
c_{p,t}\lambda^p(1-\lambda)^p\geqslant(1-t)c_{p,t}\frac{(1-\lambda)^p}{(1-t)^p}\frac{(\lambda-t)^p}{(1-t)^p}+\frac{(1-\lambda)^p}{(1-t)^p}.
$$

Therefore

$$
\begin{aligned}
c_{p,t}\lambda^p(1-\lambda)^p\geqslant\min\Bigg\{&(1-t)c_{p,t}\left(\frac{\lambda}{1-t}\right)^p\left(1-\frac{\lambda}{1-t}\right)+\left(\frac{\lambda}{1-t}\right)^p; \\
&(1-t)c_{p,t}\left(\frac{1-\lambda}{1-t}\right)^p\left(1-\frac{1-\lambda}{1-t}\right)^p+\left(\frac{1-\lambda}{1-t}\right)^p\Bigg\}
\end{aligned}
$$

for all $t\leqslant\lambda\leqslant 1-t$, thus

$$
\Phi(\lambda)\geqslant\min\left\{(1-t)\Phi\left(\frac{\lambda}{1-t}\right)+\left(\frac{\lambda}{1-t}\right)^p;(1-t)\Phi\left(\frac{1-\lambda}{1-t}\right)+\left(\frac{1-\lambda}{1-t}\right)^p\right\}
$$

for all $t\leqslant\lambda\leqslant 1-t$.

Since ϕ_p is symmetric with respect to $\lambda=1/2$, i.e., $\phi_p(\lambda)=\phi_p(1-\lambda)$ for all $\lambda\in[0,1]$, therefore Φ is also symmetric. Thus in the case $1-t\leqslant\lambda\leqslant 1$ we get

$$
c_{p,t}\geqslant\max_{t\leqslant\lambda\leqslant 1}\left\{\min\left\{\frac{1}{t^p\gamma_{p,t}(\lambda)};\frac{1}{(1-t)^p\gamma_{p,1-t}(\lambda)}\right\}\right\}=\max_{0\leqslant\lambda\leqslant t}\left\{\frac{1}{t^p\gamma_{p,t}(\lambda)}\right\}=\frac{1}{t^p-t},
$$

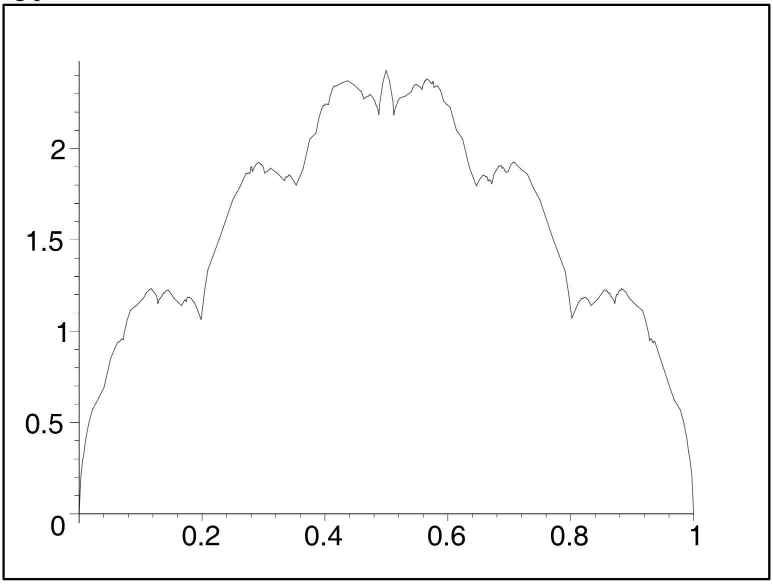
that is

$$\Phi(\lambda) \geqslant \min\left\{t\Phi\left(\frac{1-\lambda}{t}\right)+\left(\frac{1-\lambda}{t}\right)^p;(1-t)\Phi\left(\frac{1-\lambda}{1-t}\right)+\left(\frac{1-\lambda}{1-t}\right)^p\right\}$$

for all $1-t \leqslant \lambda \leqslant 1$.

Therefore if $c_{p,t} = \max\left\{\frac{1}{t^p-t};\frac{1}{(1/2-t/2)^p-(1-t)^{1-p}(1/2-t)^p}\right\}$, then $c_{p,t}\phi_p$ is a bounded solution of the inequality (5), which implies that $\varphi_{p,t} \leqslant c_{p,t}\phi_p$.

In the case $t = 0.2, p = 0.5$ the graph of function $\varphi_{p,t}$ is demonstrated by the following picture:



3. Regularity properties of (ε,δ,p,t)-convex functions

In our next results, we deal with boundedness and continuity properties of (ε,δ,p,t)-convex functions. The proofs are analogous to what was followed for (ε,δ)-midconvexity in [2].

THEOREM 3. *Let D be an open convex subset of a real normed space $(X,|\cdot|)$. Let ε, δ, p be nonnegative constants and $t \in]0,1[$. If $f : D \to \mathbb{R}$ is (ε,δ,p,t)-convex and locally bounded from above at a point of D, then f is locally bounded on D.*

The next two theorems essentially weaken the local boundedness assumption if the underlying space is of finite dimension (that is based on Steinhaus' and Piccard's theorems (cf. [7], [6])).

THEOREM 4. *Let D be an open convex subset of $\mathbb{R}^n$ and $f : D \to \mathbb{R}$ be an (ε,δ,p,t)-convex function. Assume that there exists a Lebesgue-measurable set $S \subset D$*

of positive measure and a Lebesgue-measurable function $g : S \to \mathbb{R}$ such that $f \leqslant g$ on S. Then f is locally bounded on D.

THEOREM 5. *Let D be an open convex subset of $\mathbb{R}^n$ and $f : D \to \mathbb{R}$ be an $(\varepsilon, \delta, p, t)$-convex function. Assume that there exists a Baire-measurable set $S \subset D$ of second category and a Baire-measurable function $g : S \to \mathbb{R}$ such that $f \leqslant g$ on S. Then f is locally bounded on D.*

Clearly, if $f = g + h$ where g is a $(\varepsilon, 0, p, t)$-convex function and $|h| \leqslant \delta/2$, then f is $(\varepsilon, \delta, p, t)$-convex. Therefore, if $\delta > 0$ then no continuity properties of $(\varepsilon, \delta, p, t)$-convex function can be stated. Thus, in order to reach a meaningful situation, we need to restrict our attention to the case $\delta = 0$.

THEOREM 6. *Let D be an open convex subset of a real normed space $(X, |\cdot|)$. Let ε and p be nonnegative constants. If $f : D \to \mathbb{R}$ is $(\varepsilon, 0, p, t)$-convex and locally bounded from above at a point of D, then f is continuous.*

4. Main Results

The following result offers a generalization of the theorems of Bernstein and Doetsch [1], Ng and Nikodem [4], Páles [5] and the results of Házy and Páles [2].

THEOREM 7. *Let D be an open convex subset of a real normed space $(X, |\cdot|)$. Let $\varepsilon, \delta, p, t$ be nonnegative constants, where $t \in]0, 1/2]$. If $f : D \to \mathbb{R}$ is $(\varepsilon, \delta, p, t)$-convex and locally bounded from above at a point of D, then*

$$f(\lambda x + (1-\lambda)y) \leqslant \lambda f(x) + (1-\lambda)f(y) + \delta/t + \varepsilon\varphi_{p,t}(\lambda)|x-y|^p \tag{11}$$

for all $x, y \in D$ and $\lambda \in [0, 1]$, where $\varphi_{p,t}$ is the function constructed in Proposition 1.

Proof. If $\varepsilon = 0$, then the statement follows from the result of Páles [5] mentioned as Theorem 1 in the introduction. Therefore, we may assume that $\varepsilon > 0$. Let $x, y \in D, x \neq y$ be fixed and introduce the function $g : [0, 1] \to \mathbb{R}$ by the following form:

$$g(\lambda) := f(\lambda x + (1-\lambda)y) - \lambda f(x) - (1-\lambda)f(y) \qquad (\lambda \in [0, 1]).$$

The function g has the following properties

- (a) g is bounded from above.
- (b) g is $(\varepsilon|x-y|^p, \delta, p, t)$-convex, i.e.,
  $g(t\lambda + (1-t)\mu) - (tg(\lambda) + (1-t)g(\mu)) \leqslant \delta + \varepsilon|\lambda - \mu|^p|x-y|^p.$
- (c) $g(0) = g(1) = 0.$

Using the assumptions on f it is simple to prove these properties. The local upper boundedness of f implies that g is bounded from above, i.e., (a) holds. Using

(ε,δ,p,t)-convexity of f, we get

$$\begin{aligned}
&g(t\lambda+(1-t)\mu)-(tg(\lambda)+(1-t)g(\mu))\\
&\quad=f\Big((t\lambda+(1-t)\mu)x+(1-(t\lambda+(1-t)\mu))y\Big)\\
&\qquad-(t\lambda+(1-t)\mu)f(x)-(1-(t\lambda+(1-t)\mu))f(y)\\
&\qquad-t\Big(f(\lambda x+(1-\lambda)y)-\lambda f(x)-(1-\lambda)f(y)\Big)\\
&\qquad-(1-t)\Big(f(\mu x+(1-\mu)y)-\mu f(x)-(1-\mu)f(y)\Big)\\
&\quad=f\Big(t(\lambda x+(1-\lambda)y)+(1-t)f(\mu x+(1-\mu))y\Big)\\
&\qquad-tf(\lambda x+(1-\lambda)y)-(1-t)f(\mu x+(1-\mu)y)\\
&\quad\leqslant\delta+\varepsilon\Big|\lambda x+(1-\lambda)y-\mu x-(1-\mu)y\Big|^p=\delta+\varepsilon|x-y|^p\,|\lambda-\mu|^p.
\end{aligned}$$

Thus (b) is proved. The property (c) is trivial.
Define the function Φ by the following form:

$$\Phi(\lambda)=\frac{g(\lambda)-\delta/t}{\varepsilon|x-y|^p}.$$

We are going to show that Φ satisfies the functional inequality (4).

Assume that $0\leqslant\lambda\leqslant t$. Then λ can be written in the following two ways:

$$\lambda=t0+(1-t)\frac{\lambda}{1-t}$$

and

$$\lambda=(1-t)0+t\frac{\lambda}{t}.$$

In the first case, using the $(\varepsilon|x-y|^p,\delta,p,t)$-convexity of g, we obtain

$$g(\lambda)\leqslant tg(0)+(1-t)g\left(\frac{\lambda}{1-t}\right)+\delta+\varepsilon|x-y|^p\left|0-\frac{\lambda}{1-t}\right|^p.$$

Therefore, using $g(\lambda)=\Phi(\lambda)\varepsilon|x-y|^p+\delta/t$ and $g(0)=0$, we get

$$\begin{aligned}
&\Phi(\lambda)\varepsilon|x-y|^p+\frac{1}{t}\delta\\
&\qquad\leqslant(1-t)\left(\Phi\left(\frac{\lambda}{1-t}\right)\varepsilon|x-y|^p+\frac{1}{t}\delta\right)+\delta+\varepsilon|x-y|^p\left(\frac{\lambda}{1-t}\right)^p.
\end{aligned}$$

Subtracting $\frac{1}{t}\delta$ from both sides of inequality and dividing by $\varepsilon|x-y|^p$, we get

$$\Phi(\lambda)\leqslant(1-t)\Phi\left(\frac{\lambda}{1-t}\right)+\left(\frac{\lambda}{1-t}\right)^p.$$

In the second case, using the $(\varepsilon|x-y|^p,\delta,p,t)$-convexity of g, we obtain

$$g(\lambda)\leqslant(1-t)g(0)+tg\left(\frac{\lambda}{t}\right)+\delta+\varepsilon|x-y|^p\left|0-\frac{\lambda}{t}\right|^p.$$

Therefore, using $g(\lambda) = \Phi(\lambda)\varepsilon|x-y|^p + \frac{1}{t}\delta$ and $g(0) = 0$, we get

$$\Phi(\lambda)\varepsilon|x-y|^p + \frac{1}{t}\delta \leqslant t\left(\Phi\left(\frac{\lambda}{t}\right)\varepsilon|x-y|^p + \frac{1}{t}\delta\right) + \delta + \varepsilon|x-y|^p\left(\frac{\lambda}{t}\right)^p.$$

We get

$$\Phi(\lambda) \leqslant t\Phi\left(\frac{\lambda}{t}\right) + \left(\frac{\lambda}{t}\right)^p + \frac{\left(2-\frac{1}{t}\right)-\delta}{\varepsilon|x-y|^p} \leqslant t\Phi\left(\frac{\lambda}{t}\right) + \left(\frac{\lambda}{t}\right)^p.$$

We obtained that if $\lambda \in [0,t]$ then

$$\Phi(\lambda) \leqslant \min\left\{(1-t)\Phi\left(\frac{\lambda}{1-t}\right) + \left(\frac{\lambda}{1-t}\right)^p; t\Phi\left(\frac{\lambda}{t}\right) + \left(\frac{\lambda}{t}\right)^p.\right\}$$

Similarly, if $\lambda \in [t, 1-t]$ then λ enjoys the representations

$$\lambda = t0 + (1-t)\frac{\lambda}{1-t}$$

and

$$\lambda = t1 + (1-t)\frac{\lambda - t}{1-t}$$

hence arguing similarly as above we get

$$\Phi(\lambda) \leqslant \min\left\{(1-t)\Phi\left(\frac{\lambda}{1-t}\right) + \left(\frac{\lambda}{1-t}\right)^p; (1-t)\Phi\left(\frac{1-\lambda}{1-t}\right) + \left(\frac{1-\lambda}{1-t}\right)^p\right\}.$$

Finally if $\lambda \in [1-t, 1]$ then

$$\lambda = t1 + (1-t)\frac{\lambda - t}{1-t}$$

and

$$\lambda = (1-t)1 + t\frac{\lambda + t - 1}{t}$$

and we similarly get

$$\Phi(\lambda) \leqslant \min\left\{t\Phi\left(\frac{1-\lambda}{t}\right) + \left(\frac{1-\lambda}{t}\right)^p; (1-t)\Phi\left(\frac{1-\lambda}{1-t}\right) + \left(\frac{1-\lambda}{1-t}\right)^p\right\}.$$

Since g is bounded from above, hence Φ is an upper bounded function, which implies that Φ is a solution of functional inequality (4). Applying Proposition 2, we get, that $\Phi \leqslant \varphi_{p,t}$. By the construction of Φ, we get that $g(\lambda) \leqslant \varepsilon|x-y|^p\varphi_{p,t}(\lambda) + (1/t)\delta$. Thus, using the definition of g, we obtain

$$f(\lambda x + (1-\lambda)y) \leqslant \lambda f(x) + (1-\lambda)f(y) + \delta/t + \varepsilon\varphi_{p,t}(\lambda)|x-y|^p.$$

Applying the statement of Proposition 3, we immediately get the following result.

COROLLARY 1. *Let D be an open, convex subset of real normed space $(X,|\cdot|)$. Let ε,δ be nonnegative constants, $0\leqslant p<1$. If $f:D\to\mathbb{R}$ is (ε,δ,p,t)-convex (where $t\in\,]0,1/2]$) and locally bounded from above at a point of D, then*

$$f(\lambda x+(1-\lambda)y)\leqslant \lambda f(x)+(1-\lambda)f(y)+\delta/t +\varepsilon\max\left\{\frac{1}{t^p-t};\frac{1}{(1/2-t/2)^p-(1-t)^{1-p}(1/2-t)^p)}\right\}(\lambda(1-\lambda))^p|x-y|^p$$

for all $x,y\in D$ and $\lambda\in[0,1]$.

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(Received February 24, 2004)

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