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STABILITY OF AN INCOMPLETE GAMMA–TYPE FUNCTIONAL EQUATION

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(*communicated by T. M. Rassias*)

Abstract. We investigate the Hyers-Ulam-Rassias stability of an incomplete gamma-type functional equation

$$f(\phi_1(x_1),\cdots,\phi_n(x_n),\psi_1(y_1),\cdots,\psi_m(y_m)) = \theta(x_1,\cdots,x_n,y_1,\cdots,y_m)f(x_1,\cdots,x_n,y_1,\cdots,y_m) + \lambda(x_1,\cdots,x_n,y_1,\cdots,y_m)$$

with a restricted domain. By this result we obtain the stability of the incomplete gamma functional equation

$$f(x+1,y) = xf(x,y) + e^{-y}(y)^x$$

with a restricted domain.

1. Introduction

In 1940, S.M. Ulam [see 3, 6, 23] raised the following problem : *Under what conditions does there exist an additive mapping near an approximately additive mapping?* In 1941, this problem was solved by D.H. Hyers [5, 6]. Therefore we usually say that the equation $E_1(g) = E_2(g)$ has the Hyers-Ulam stability if for an approximate solution f such as following type : $|\ E_1(f) - E_2(f)\ | \leqslant \delta$ there exists a function g such that $E_1(g) = E_2(g)$ and $|\ f(x) - g(x)\ | \leqslant \epsilon$. This stability problem has been further generalized by Th.M. Rassias [15] in 1978, then it is said the Hyers-Ulam-Rassias stability. Later, many Rassias type theorem concerning the stability of different functional equations were obtained by numerous authors (see, for instance [12-22]).

Now we consider the incomplete gamma function. The incomplete gamma function is given by

$$\Gamma(x,y) = \int_y^\infty e^{-t}t^{x-1}dt$$

where x is positive and y is nonnegative. P. Natalini and B. Palumbo [14] proved some inequalities for the incomplete gamma function which follow from monotonicity

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properties. It is well known that $\Gamma(x,0)$ is a gamma function. Note that $f(x+1) = xf(x)$ (for all $x > 0$) is the gamma functional equation. S. -M. Jung [9, 10] obtained stability theorems of the gamma functional equation. K.W. Jun, G.H. Kim and the author [8, 14] also obtained stability theorems of the beta functional equation.

In this paper we define an incomplete gamma functional equation as follows : for all $x > 0$ and $y \geqslant 0$,

$$f(x+1,y) = xf(x+y) + e^{-y}y^x. \tag{1}$$

Note that the incomplete gamma function is a solution of the equation (1). Also we define a incomplete gamma-type functional equation as follows:

$$\begin{aligned} &f(\phi_1(x_1),\cdots,\phi_n(x_n),\psi_1(y_1),\cdots,\psi_m(y_m)) \\ &\quad = \theta(x_1,...,x_n,y_1,...,y_m)f(x_1,...,x_n,y_1,...,y_m) + \lambda(x_1,...,x_n,y_1,...,y_m). \end{aligned} \tag{2}$$

Throughout this paper, let $\phi_i : (0,\infty) \to R$ be a strictly increasing function for each $i = 1,2,\cdots,n$, let $\psi_j : S \subseteq R \to R$ be a function for each $j = 1,2,\cdots,m$, let θ, $\lambda : (0,\infty)^n \times S^m \to R$ be functions and let c_i be a given nonnegative real numbers for each $i = 1,2,\cdots,n$. The aim of the present note is to give the Hyers-Ulam-Rassias stability theorems of the equation (1) and (2) with a restricted domain.

2. Hyers-Ulam-Rassias stability of the equation (2)

In the following theorem we investigate the Hyers-Ulam-Rassias stability of the equation (2) with a restricted domain. Note that for some function φ, $\varphi^0(x) = x$ and $\varphi^n(x) = \varphi(\varphi^{n-1}(x))$ for all x.

THEOREM 2.1. *If a function $g : (0,\infty)^n \times S^m \to R$ satisfies the inequality*

$$\begin{aligned} &|g(\phi_1(x_1),\cdots,\phi_2(x_n),\psi_1(y_1),\cdots,\psi_m(y_m)) \\ &\quad - \theta(x_1,\cdots,x_n,y_1,\cdots,y_m)g(x_1,\cdots,x_n,y_1,\cdots,y_m) \\ &\quad - \lambda(x_1,\cdots,x_n,y_1,\cdots,y_m)\,| \\ &\quad < \epsilon(x_1,\cdots,x_n,y_1,\cdots,y_m) \end{aligned} \tag{3}$$

for all $x_i > c_i$ with each $c_i \in (0,\infty)$, all $y_i \in S$ and some function $\theta : (0,\infty)^n \times S^m \to R$ with

$$\omega(x_1,...,x_n,y_1,...,y_m) := \sum_{k=0}^{\infty} \frac{\epsilon(\varphi_1^k(x_1),...,\varphi_n^k(x_n),\psi_1^k(y_1),...,\psi_m^k(x_m))}{\prod_{j=0}^{k} |\ \theta(\phi_1^j(x_1),...,\phi_n^j(x_n),\psi_1^j(y_1),...,\psi_m^j(x_m))\ |} < \infty$$

for all $x_i > c_i$ $(i = 1,2,\cdots,n)$ and $y_j \in S$ $(j = 1,2,\cdots,m)$, then there exists a unique solution $f : (0,\infty)^n \times s^m \to R$ such that

$$\begin{aligned} &f(\phi_1(x_1),...,\phi_n(x_n),\psi_1(y_1),...,\psi_m(y_m)) \\ &\quad = \theta(x_1,...,x_n,y_1,...,y_m)f(x_1,...,x_n,y_1,...,y_m) + \lambda(x_1,...,x_n,y_1,...,y_m) \end{aligned}$$

for all $x_i > 0$ $(i = 1, 2, \cdots, n)$ *and* $y_i \in S$ *and*

$$| g(x_1, \cdots, x_n, y_1, \cdots, y_m) - f(x_1, \cdots, x_n, y_1, \cdots, y_m) | < \omega(x_1, \cdots, x_n, y_1, \cdots, y_m)$$

for all $x_i > c_i$ $(i = 1, 2, \cdots, n)$ *and* $y_j \in S$ $(j = 1, 2, \cdots, m)$.

Proof. For the convenience let

$$\begin{aligned} &X = (x_1, \cdots, x_n), \\ &Y = (y_1, \cdots, y_m), \\ &\Phi(X) = (\phi_1(x_1), \cdots, \phi_n(x_n)), \\ &\Phi^k(X) = (\phi_1^k(x_1), \cdots, \phi_m^k(x_m)), \\ &\Psi(Y) = (\psi_1(y_1), \cdots, \psi_n(x_n)), \\ &\Psi^k(Y) = (\psi_1^k(y_1), \cdots, \psi_m^k(x_m)), \\ &C = (c_1, \cdots, c_n) \end{aligned}$$

and $x > C$ means $x_i > c_i$ for each $i = 1, 2, \cdots, n$. Then the condition (3) implies

$$| g(\Phi(X), \Psi(Y)) - \theta(X, Y)g(X, Y) - \lambda(X, Y) | < \epsilon(X, Y) \tag{4}$$

for all $X > C$ and $Y \in S^m$. Let $\omega_l : (0, \infty)^n \times S^m \to (0, \infty)$ and $f_l : (0, \infty)^n \times S^m \to R$ be functions defined by

$$\omega_l(X, Y) := \sum_{k=0}^{l-1} \frac{\epsilon(\Phi^k(X), \Psi^k(Y))}{\prod_{j=0}^{k} | \theta(\Phi^j(X), \Psi^j(Y)) |}$$

for each positive integer l, for all $X \in (0, \infty)^n$ and $Y \in S^m$ and

$$f_l(X, Y) = \frac{g(\Phi^n(X), \Psi^n(Y))}{\prod_{j=0}^{l-1} | \theta(\Phi^j(X), \Psi^j(Y)) |} - \sum_{k=0}^{l-1} \frac{\lambda(\Phi^k(X), \Psi^k(Y))}{\prod_{j=0}^{k} | \theta(\Phi^j(X), \Psi^j(Y)) |}$$

for each positive integer l, for all $X \in (0, \infty)^n$ and $Y \in S^m$, respectively. By (4) we have

$$\begin{aligned} &| f_{l+1}(X, Y) - f_l(X, Y) | \\ &\quad = \frac{1}{\prod_{j=0}^{l} | \theta(\Phi^j(X), \Psi^j(Y)) |} \, | g(\Phi^{l+1}(X), \Psi^{l+1}(Y)) \\ &\qquad - \theta(\Phi^l(X), \Psi^l(Y))g(\Phi^l(X), \Psi^l(Y)) - \lambda(\Phi^l(X), \Psi^l(Y)) | \\ &\quad \leqslant \frac{\epsilon(\Phi^l(X), \Psi^l(Y))}{\prod_{j=0}^{l} | \theta(\Phi^j(X), \Psi^j(Y)) |} \end{aligned}$$

for each positive integer l, for all $X > C$ and $Y \in S^m$.

Now we use induction on l to prove

$$| f_l(X, Y) - g(X, Y) | \leqslant \omega_n(X, Y)$$

for all positive integer n, for all $X > C$ and $Y \in S^m$. For the case $l = 1$ the above inequality is an immediate consequence of (4). Assume that it holds for some l. Then

$$\begin{aligned} | f_{l+1}(X,Y) - g(X,Y) | &\leqslant | f_{l+1}(X,Y) - f_l(X,Y) | + | f_l(X,Y) - g(X,Y) | \\ &\leqslant \omega_{l+1}(X,Y) \end{aligned}$$

for all $X > C$ and $Y \in S^m$. We claim that $\{f_l(X,Y)\}$ is a Cauchy sequence. Indeed, for $n > m$, $x > C$ and $Y \in S^m$ we have

$$\begin{aligned} | f_n(X,Y) - f_m(X,Y) | &\leqslant \sum_{j=m}^{n-1} | f_{j+1}(X,Y) - f_j(X,Y) | \\ &\leqslant \sum_{j=m}^{n-1} \frac{\epsilon(\Phi^k(X), \Psi^k(Y))}{\prod_{j=0}^{k} | \theta(\Phi^j(X), \Psi^j(Y)) |} \\ &= \omega_n(X,Y) - \omega_m(X,Y) \to 0 \end{aligned}$$

as $m \to \infty$. Hence we can define a function $\tilde{f} : (c_1, \infty) \times \cdots \times (c_n, \infty) \times S^m \to R$ by

$$\tilde{f}(X,Y) = \lim_{l \to \infty} f_l(X,Y)$$

for all $x > C$ and $Y \in S^m$. Since $f_l(\Phi(X), \Psi(Y)) = \theta(X,Y) f_{l+1}(X,Y) + \lambda(X,Y)$, we have

$$\tilde{f}(\Phi(X), \Psi(Y)) = \theta(X,Y)\tilde{f}(X,Y) + \lambda(X,Y)$$

for all $x > C$ and $Y \in S^m$ and

$$\begin{aligned} | \tilde{f}(X,Y) - g(X,Y) | &= \lim_{l \to \infty} | f_l(X,Y) - g(X,Y) | \\ &\leqslant \lim_{l \to \infty} \omega_{l+1}(X,Y) \\ &= \omega(X,Y) \end{aligned}$$

for all $x > C$ and $Y \in S^m$.

Now we extend the function $\tilde{f}$ to $(0,\infty)^n \times S^m$. We define a function $f : (0,\infty)^n \times S^m \to R$ such that if $x > C$ and $Y \in S^m$,

$$f(X,Y) = \tilde{f}(X,Y)$$

and if $0 < x \leqslant c_i$ for some $i = 1, 2, \cdots, n$ and $Y \in S^m$

$$f(X,Y) = \frac{\tilde{f}(\Phi^k(X), \Psi^k(Y))}{\prod_{j=0}^{k-1} | \theta(\Phi^j(X), \Psi^j(Y)) |} - \sum_{j=0}^{k-1} \frac{\lambda(\Phi^j(X), \Psi^j(Y))}{\prod_{i=0}^{j} | \theta(\Phi^i(X), \Psi^i(Y)) |}$$

where k is the smallest natural number satisfying the inequality $\Phi^k(X) > C$, that is $\phi_i^k(x) > c_i$ for all $i = 1, 2, \cdots, n$.

In the latter case, we have

$$
\begin{aligned}
f(\Phi(X),\Psi(Y)) &= \frac{\tilde{f}(\Phi^{k+1}(X),\Psi^{k+1}(Y))}{\prod_{j=1}^{k} \mid \theta(\Phi^j(X),\Psi^j(Y)) \mid} - \sum_{j=1}^{k} \frac{\lambda(\Phi^j(X),\Psi^j(Y))}{\prod_{i=1}^{j} \mid \theta(\Phi^i(X),\Psi^i(Y)) \mid} \\
&= \frac{\tilde{f}(\Phi^{k}(X),\Psi^{k}(Y))}{\prod_{j=1}^{k-1} \mid \theta(\Phi^j(X),\Psi^j(Y)) \mid} - \sum_{j=1}^{k-1} \frac{\lambda(\Phi^j(X),\Psi^j(Y))}{\prod_{i=1}^{j} \mid \theta(\Phi^i(X),\Psi^i(Y)) \mid} \\
&= \theta(X,Y)f(X,Y) + \lambda(X,Y).
\end{aligned}
$$

Thus for every $X \in (0,\infty)^n$ and $Y \in S^m$ we have

$$
f(\Phi(X),\Psi(Y)) = \theta(X,Y)f(X,Y) + \lambda(X,Y).
$$

If $h : (0,\infty)^n \times S^m \to R$ is a function which satisfies

$$
h(\Phi(X),\Psi(Y)) = \theta(X,Y)h(X,Y) + \lambda(X,Y).
$$

for all $X \in (0,\infty)^n$ and $Y \in S^m$ and

$$
\mid h(X,Y) - g(X,Y) \mid \leqslant \omega(X,Y)
$$

for all $x > C$ and $Y \in S^m$, then

$$
\begin{aligned}
\mid f(X,Y) - h(X,Y) \mid &= \mid f(\Phi(X),\Psi(Y)) - h(\Phi(X),\Psi(Y)) \mid \frac{1}{\mid \theta(X,Y) \mid} \\
&= \mid f(\Phi^l(X),\Psi^l(Y)) - h(\Phi^l(X),\Psi^l(Y)) \mid \frac{1}{\prod_{j=0}^{l-1} \theta(\Phi^j(X),\Psi^j(Y))} \\
&\leqslant \frac{1}{\prod_{j=0}^{l-1} \mid \theta(\Phi^j(X),\Psi^j(Y)) \mid} (\mid f(\Phi^l(X),\Psi^l(Y)) - g(\Phi^l(X),\Psi^l(Y)) \mid \\
&\quad + \mid g(\Phi^l(X),\Psi^l(Y)) - h(\Phi^l(X),\Psi^l(Y)) \mid) \\
&\leqslant \frac{2\omega(\Phi^l(X),\Psi^l(Y))}{\prod_{j=0}^{l-1} \mid \theta(\Phi^l(X),\Psi^l(Y)) \mid} \\
&= 2[\omega(X,Y) - \omega_n(X,Y)] \to 0
\end{aligned}
$$

as $l \to \infty$. This implies the uniqueness of f. $\square$

3. Applications

As an application of Theorem 2.1 we can derive the Hyers-Ulam stability of the incomplete gamma functional equation as follows;

COROLLARY 3.1. *If a function* $g:(0,\infty)\times[0,\infty)\to R$ *satisfies the inequality*

$$|\, g(x+1,y)-xg(x,y)-e^{-y}y^{x}\,|\leqslant\delta$$

for some $\delta>0$*, for all* $x>c>0$ *and* $y\geqslant 0$*, then there exists a unique solution* $f:(0,\infty)\times[0,\infty)\to R$ *of the equation (1) with*

$$|\, f(x,y)-g(x,y)\,|\leqslant\frac{e\delta}{x}$$

for all $x>c$ *and* $y\geqslant 0$.

Proof. We apply Theorem 2.1 with $\phi_1(x)=x+1$, $\psi_1(y)=y$, $\theta(x,y)=x$, $\lambda(x,y)=e^{-y}y^{x}$ and $\epsilon(x,y)=\delta$ in the two variables. Then

$$\begin{aligned}\omega(x,y)&=\sum_{k=0}^{\infty}\frac{\epsilon(\phi_1^k(x),\psi_1^k(y))}{\prod_{j=0}^{k}|\,\theta(\phi^j(x),\psi^j(y))\,|}\\&=\sum_{k=0}^{\infty}\frac{\delta}{\prod_{j=0}^{k}(x+j)}\\&\leqslant\frac{e\delta}{x}\end{aligned}$$

for all $x>c$ and $y\geqslant 0$. □

The function $f(x,y)=\int_y^1 e^t t^x dt$ is a solution of an incomplete gamma-type functional equation

$$f(x+1,y)=-(x+1)f(x,y)-e^{y}y^{x+1}+e.$$

We can obtain the Hyers-Ulam stability of this equation as follows;

COROLLARY 3.2. *Let* $\delta>0$. *Suppose that a function* $g:(0,\infty)\times[0,1]\to R$ *satisfies the inequality*

$$|\, g(x+1,y)+(x+1)g(x,y)+e^{y}y^{x+1}-e\,|\leqslant\delta$$

for all $x>c>0$ *and* $y\in[0,1]$. *Then there exists a unique function* $f:(0,\infty)\times[0,1]\to R$ *such that*

$$f(x+1,y)+(x+1)f(x,y)+e^{y}y^{x+1}-e=0$$

for all $x\in(0,\infty)$ *and* $y\in[0,1]$ *and*

$$|\, f(x,y)-g(x,y)\,|\leqslant\delta(e-1)$$

for all $x\in(c,\infty)$ *and* $y\in[0,1]$.

Proof. We apply Theorem 2.1 with $\phi_1(x)=x+1$, $\psi_1(y)=y$, $\theta(x,y)=-(x+1)$, $\lambda(x,y)=e-e^{y}y^{x+1}$ and $\epsilon(x,y)=\delta$ in the two variables. For all $x\in(c,\infty)$ and

$y \in [0,1]$,

$$\begin{aligned}\omega(x,y) &= \sum_{k=0}^{\infty} \frac{\delta}{\prod_{j=0}^{k} \mid \theta(\phi_1^j(x), \psi_1^j(y)) \mid} \\ &= \sum_{k=0}^{\infty} \frac{\delta}{\prod_{j=0}^{k}(x+j+1)} \\ &= \frac{\delta}{x+1} + \frac{\delta}{(x+1)(x+2)} + \frac{\delta}{(x+1)(x+2)(x+3)} + \cdots \\ &\leqslant \delta(e-1).\end{aligned}$$

Thus we lead to the conclusion. □

The function $f(x,y) = \int_y^e (\ln t)^x dt$ is a solution of a incomplete gamma-type functional equation

$$f(x+1,y) + (x+1)f(x,y) + (x+1)e = y(\ln y)^{x+1}.$$

We obtain the Hyers-Ulam stability of this equation by Theorem 2.1 with two variables.

COROLLARY 3.3. *Let* $\delta > 0$. *Suppose that a function* $g : (0,\infty) \times [1,e] \to R$ *satisfies the inequality*

$$\mid g(x+1,y) + (x+1)g(x,y) + (x+1)e - y(\ln y)^{x+1} \mid < \delta$$

for all $x > c > 0$ *and* $y \in [1,e]$. *Then there exists a unique function* $f : (0,\infty) \times [1,e] \to R$ *such that*

$$f(x+1,y) + (x+1)f(x,y) + (x+1)e - y(\ln y)^{x+1} = 0$$

for all $x \in (0,\infty)$ *and* $y \in [1,e]$ *and*

$$\mid f(x,y) - g(x,y) \mid \leqslant \delta(e-1)$$

for all $x > c > 0$ *and* $y \in [1,e]$.

Proof. We apply Theorem 2.1 with $\phi_1(x) = x+1$, $\psi_1(y) = y$, $\theta(x,y) = -(x+1)$, $\lambda(x,y) = -(x+1)e + y(\ln y)^{x+1}$ and $\epsilon(x,y) = \delta$. For all $x \in (c,\infty)$ and $y \in [1,e]$,

$$\begin{aligned}\omega(x,y) &= \sum_{k=0}^{\infty} \frac{\delta}{\prod_{j=0}^{k} \mid \theta(\phi_1^j(x), \psi_1^j(y)) \mid} \\ &= \sum_{k=0}^{\infty} \frac{\delta}{\prod_{j=0}^{k}(x+j+1)} \\ &= \delta(e-1).\end{aligned}$$

Thus we lead to the conclusion. □

Note that the functional equation

$$g(x+1,y+1)=\frac{xy}{(x+y)(x+y+1)}g(x,y)$$

for all $x,y>0$ is called *"the beta functional equation"*. Since

$$\sum_{j=0}^{\infty}\prod_{i=0}^{j}\frac{(x+i)(y+i)}{(x+y+2i)(x+y+2i+1)}<\infty$$

and

$$\sum_{j=0}^{\infty}\prod_{i=0}^{j}\frac{(x+y+2i)(x+y+2i+1)}{(x+i)(y+i)}=\infty$$

for each $x,y>0$, we must investigate the stability of the same beta functional equation

$$g(x+1,y+1)^{-1}=\frac{(x+y)(x+y+1)}{xy}g(x,y)^{-1}$$

for all $x,y>0$. It is well known that the beta function

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt$$

is a solution of the beta functional equation.

COROLLARY 3.4. *If a mapping* $g:(0,\infty)\times(0,\infty)\to(0,\infty)$ *satisfies the inequality*

$$|\,g(x+1,y+1)^{-1}-\frac{(x+y)(x+y+1)}{xy}g(x,y)^{-1}\,|\leqslant\delta$$

for all $x>n_1$ *and* $y>n_2$*, then there exists a unique function* $f:(0,\infty)\times(0,\infty)\to(0,\infty)$ *such that* f *is a solution of the beta functional equation and satisfies the inequality*

$$|\,g(x,y)^{-1}-f(x,y)^{-1}\,|\leqslant\frac{xy}{(x+y)(x+y+1)}\delta$$

for all $x>n_1$ *and* $y>n_2$.

Proof. We apply Theorem 2.1 with $\phi_1(x)=x+1$, $\phi_2(y)=y+1$, $\lambda(x,y)=0$ and $\theta(x,y)=\frac{(x+y)(x+y+1)}{xy}$. For any $x,y>0$ we have

$$\sum_{k=0}^{\infty}\prod_{j=0}^{k}\frac{\delta}{\theta(\phi_1^j(x),\phi_2^j(y))}\leqslant\frac{\delta}{\theta(x,y)}\sum_{k=0}^{\infty}\frac{1}{2^k}=\frac{2\delta}{\theta(x,y)}.$$

By Theorem 2.1, there exists a unique function $F:(0,\infty)\times(0,\infty)\to(0,\infty)$ such that

$$F(x+1,y+1)=\frac{(x+y)(x+y+1)}{xy}F(x,y)$$

for all $x, y > 0$, and

$$| g(x,y)^{-1} - F(x,y) | \leqslant \frac{xy}{(x+y)(x+y+1)} 2\delta$$

for all $x > n_1$ and $y > n_2$. Let $F(x,y) = f(x,y)^{-1}$ for all $x, y > 0$. Then we complete the proof of Corollary. $\square$

Consider the Schröder functional equation with two variables

$$f(\phi_1(x), \phi_2(y)) = kf(x,y).$$

For example, $f(x,y) = e^{x+y}$ is a solution of a Schröder functional equation

$$f(x+1, y+1) = e^2 f(x,y).$$

As an application of Theorem 2.1, we can derive the following corollary, concerning the Hyers-Ulam stability of the Schröder functional equation.

COROLLARY 3.5. *Let $k > 1$, $\delta > 0$ and ϕ_1, $\phi_2 : (0,\infty) \to (0,\infty)$ be strictly increasing functions. If a function $g : (0,\infty) \times (0,\infty) \to R$ satisfies the inequality*

$$| g(\phi_1(x), \phi_2(y)) - kg(x,y) | < \delta$$

for all $x > n_1$ and $y > n_2$, then there exists a unique function $f : (0,\infty) \times (0,\infty) \to R$ such that

$$f(\phi_1(x), \phi_2(y)) - kf(x,y) = 0$$

for all $x, y > 0$ and

$$| g(x,y) - f(x,y) | \leqslant \frac{\delta}{k-1}$$

for all $x > n_1$ and $y > n_2$.

Proof. By Theorem 2.1 with $\lambda(x,y) = 0$, $\theta(x,y) = k$, and $\epsilon(x,y) = \delta$, we have

$$w(x,y) = \delta \sum_{i=0}^{\infty} \frac{1}{k^{i+1}} = \frac{\delta}{k-1}$$

for all $x, y > 0$. $\square$

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