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STABILITY OF ANGLE–PRESERVING MAPPINGS ON THE PLANE

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Abstract. We prove that for the mappings of the plane the property of preserving the angle between vectors is stable. We apply this result to prove some kind of stability of the Wigner equation on the plane.

1. Introduction

The class of functions between Hilbert spaces preserving the absolute value of the inner product has found some applications in quantum phisics. In the book of E.P. Wigner [6] the class of such operators was described. Therefore the functional equation

$$|\langle f(x)|f(y)\rangle| = |\langle x|y\rangle|$$

postulated for all x, y from the domain is called the *Wigner equation*.

Speaking of the *stability* of a functional equation we follow the question of S. Ulam: "when is it true that the solution of an equation differing slightly from a given one, must of necessity be close to the solution of the given equation?" (see [5], p. 63) As the words "differing slightly" and "be close" may have various meanings, different kinds of stability can be dealt with (cf. [3]).

Different types of stability of the Wigner equation have been considered by the author. In particular, the following result, based on [2] and [4], was established in [1]:

THEOREM 1. *Let E and F be real or complex Hilbert spaces and let $\varepsilon > 0$ and $p \in \mathbb{R} \setminus \{1\}$ be fixed. Then for a function $f : E \to F$ satisfying*

$$\big|\,|\langle f(x)|f(y)\rangle| - |\langle x|y\rangle|\,\big| \leqslant \varepsilon \, \|x\|^p \, \|y\|^p, \quad x, y \in E_p \tag{1}$$

($E_p = E$ for $p \geqslant 0$ and $E_p = E \setminus \{0\}$ for $p < 0$) there exists, a unique up to phase-equivalency, function $I : E \to F$ satisfying the Wigner equation and such that

$$\|f(x) - I(x)\| \leqslant \sqrt{\varepsilon}\, \|x\|^p, \quad x \in E_p. \tag{2}$$

Moreover, if $E = F = \mathbb{R}^n$ ($n \in \mathbb{N}$) the so called superstability holds: any function f satisfying (1) *must be an exact solution of the Wigner equation on E_p.*

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The stability problem for the case $p = 1$ is generally unsolved. Some partial results concerning this case with an example showing that there is no superstability then can be found in [1]. In the present paper we consider this case for the euclidean, two-dimensional space $\mathbb{R}^2$ with its standard inner product $\langle\cdot|\cdot\rangle$. The results can be generalized to the case of arbitrary two-dimensional real Hilbert spaces E and F. We are considering the class of approximate solutions of the Wigner equation

$$|\langle f(x)|f(y)\rangle| = |\langle x|y\rangle| \quad \text{for } x, y \in \mathbb{R}^2 \tag{W}$$

coinsisting of mappings $f : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying the functional inequality

$$\big||\langle f(x)|f(y)\rangle| - |\langle x|y\rangle|\big| \leqslant \varepsilon \|x\| \|y\|, \quad x, y \in \mathbb{R}^2. \tag{$\star$}$$

Proving the stability of the Wigner equation we use a geometrical property of preservation of the angle and its stability. By $\cos(u,v)$ we mean the cosine of the angle between non-zero vectors u and v. Note that $|\cos(u,v)|$ is the cosine of the angle between the lines spanned by u and v (directions of u and v). We are dealing with mappings which preserves (accurately or approximately) the angle between the directions of vectors. Thus we consider the following system:

$$\begin{cases} U(x) = 0 \iff x = 0 \\ |\cos(U(x), U(y))| = |\cos(x,y)| \quad \text{for } x, y \in \mathbb{R}^2 \setminus \{0\}. \end{cases} \tag{3}$$

The class of mappings satisfying the system (3) consists of mappings $U(x) = \sigma(x)I(x)$ where $I : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear isometry and $\sigma : \mathbb{R}^2 \to \mathbb{R}$ is an arbitrary function.

We consider also some perturbation of (3):

$$\begin{cases} f(x) = 0 \iff x = 0 \\ \big||\cos(f(x), f(y))| - |\cos(x,y)|\big| \leqslant \varepsilon \quad x, y \in \mathbb{R}^2 \setminus \{0\}. \end{cases} \tag{$\star\star$}$$

2. Results and proofs

The following lemma (cf. [1]) is easy to obtain.

LEMMA 1. *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies $(\star)$ (with $0 \leqslant \varepsilon < 1$). Define*

$$g(x) := \begin{cases} \dfrac{f(x)\,\|x\|}{\|f(x)\|}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then:

(a) $\big||\langle g(x)|g(y)\rangle| - |\langle x|y\rangle|\big| \leqslant 2\varepsilon \|x\| \|y\|, \ x, y \in \mathbb{R}^2;$
(b) $\|g(x)\| = \|x\|, \ x \in \mathbb{R}^2;$
(c) $\|f(x) - g(x)\| \leqslant \left(1 - \sqrt{1-\varepsilon}\right) \|x\|, \ x \in \mathbb{R}^2.$

Now we get the stability of angle-preserving mappings.

THEOREM 2. *For an arbitrary constant $\delta > 0$ there exists $\varepsilon > 0$ such that for any function $f : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying $(\star\star)$ there exists a function $U : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying (3) and such that*

$$|\cos(f(x), U(x))| \geqslant 1 - \delta \quad \textit{for } x \in \mathbb{R}^2 \setminus \{0\}. \tag{4}$$

Proof. Let $e_1 := (1,0)$, $e_2 := (0,1)$ and $e_0 := (1,1)$. For $x = (x_1,x_2) \in \mathbb{R}^2$ we denote: $\overline{x} := (x_1,-x_2)$.

Let us fix an arbitrary $\delta > 0$. We choose $0 < \varepsilon < \delta_4 < \delta_3 < \delta_2 < \delta_1 < \delta$ such that for all $u,v,w,z,y \in \mathbb{R}^2 \setminus \{0\}$ the following implications hold.

$$\left\{\begin{array}{l} |\cos(u,z)| > 1-\delta_1 \\ \quad\text{and} \\ |\cos(v,z)| > 1-\delta_1 \end{array}\right. \implies |\cos(u,v)| \geqslant 1-\delta. \tag{5}$$

$$\left\{\begin{array}{l} |\cos(u,e_1)| > 1-\delta_2 \\ \quad\text{or} \\ |\cos(u,e_2)| > 1-\delta_2 \end{array}\right. \implies |\cos(u,\overline{u})| > 1-\delta_1. \tag{6}$$

$$\left\{\begin{array}{l} \big||\cos(w,e_0)| - |\cos(u,e_0)|\big| < \delta_3 \\ \text{and} \\ |\cos(w,\overline{u})| > 1-\delta_3 \end{array}\right. \implies \left\{\begin{array}{l} |\cos(u,e_1)| > 1-\delta_2 \\ \text{or} \\ |\cos(u,e_2)| > 1-\delta_2. \end{array}\right. \tag{7}$$

$$\left\{\begin{array}{l} \big||\cos(w,\overline{e_0})| - |\cos(u,e_0)|\big| < \delta_3 \\ \text{and} \\ |\cos(w,u)| > 1-\delta_3 \end{array}\right. \implies \left\{\begin{array}{l} |\cos(u,e_1)| > 1-\delta_2 \\ \text{or} \\ |\cos(u,e_2)| > 1-\delta_2. \end{array}\right. \tag{8}$$

(Implication (8) follows from (7) as $|\cos(x,\overline{y})| = |\cos(\overline{x},y)|$.)

$$\left\{\begin{array}{l} \big||\cos(w,z)| - |\cos(u,v)|\big| < \delta_4 \\ \text{and} \\ |\cos(z,y)| > 1-\delta_4 \end{array}\right. \implies \big||\cos(w,y)| - |\cos(u,v)|\big| < \delta_3. \tag{9}$$

$$\big||\cos(u,e_1)| - |\cos(v,e_1)|\big| \leqslant \varepsilon \implies \left\{\begin{array}{l} |\cos(u,v)| > 1-\delta_4 \\ \text{or} \\ |\cos(u,\overline{v})| > 1-\delta_4. \end{array}\right. \tag{10}$$

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a function satisfying $(\star\star)$. Additionally, in the first part of the proof, we assume that $f(e_1)$ and e_1 are colinear vectors, i.e.,

$$|\cos(f(e_1),e_1)| = 1. \tag{11}$$

This implies, for an arbitrary $x \in \mathbb{R}^2 \setminus \{0\}$,

$$|\cos(f(x),f(e_1))| = |\cos(f(x),e_1)|.$$

Therefore, we have from $(\star\star)$

$$\big||\cos(f(x),e_1)| - |\cos(x,e_1)|\big| \leqslant \varepsilon, \quad x \in \mathbb{R}^2 \setminus \{0\}.$$

According to (10), for an arbitrary $x \in \mathbb{R}^2 \setminus \{0\}$, the following two cases are possible:

(i) $|\cos(f(x),x)| > 1-\delta_4$;

(*ii*) $|\cos(f(x),\overline{x})| > 1-\delta_4$.
In particular, for e_0, we have two possibilities

$$\begin{array}{lll} & 1^\circ & |\cos(f(e_0),e_0)| > 1-\delta_4 \\ \text{or} & 2^\circ & |\cos(f(e_0),\overline{e_0})| > 1-\delta_4. \end{array}$$

Let us suppose now that the case 1° holds. Then we define a mapping

$$U(x) := x \ \text{ for } x \in \mathbb{R}^2.$$

which obviously satisfies (3) and it remains to prove (4). As it is obvious for $x=0$, let us fix $x \in \mathbb{R}^2 \setminus \{0\}$. If x satisfies (i) then

$$|\cos(f(x),U(x))| = |\cos(f(x),x)| > 1-\delta_4 > 1-\delta,$$

i.e., (4) holds. Suppose now that x satisfies (ii). As we have

$$|\cos(f(e_0),e_0)| > 1-\delta_4 \quad \text{and} \quad \big|\,|\cos(f(x),f(e_0))| - |\cos(x,e_0)|\,\big| \leqslant \varepsilon < \delta_4,$$

then from (9)

$$\big|\,|\cos(f(x),e_0)| - |\cos(x,e_0)|\,\big| < \delta_3.$$

The last inequality, together with

$$|\cos(f(x),\overline{x})| > 1-\delta_4 > 1-\delta_3$$

yields, because of (7),

$$|\cos(x,e_1)| > 1-\delta_2 \quad \text{or} \quad |\cos(x,e_2)| > 1-\delta_2$$

which implies (according to (6))

$$|\cos(x,\overline{x})| > 1-\delta_1.$$

The above inequality, the fact that

$$|\cos(f(x),\overline{x})| > 1-\delta_4 > 1-\delta_1$$

and (5) gives us

$$|\cos(f(x),x)| \geqslant 1-\delta$$

which proves (4).

In the complementary case 2° we define

$$U(x) := \overline{x}, \quad x \in \mathbb{R}^2$$

which obviously satisfies (3). The proof of (4) runs similarly as in the case 1°. It is obvious for $x=0$, so let us fix $x \in \mathbb{R}^2 \setminus \{0\}$. If x satisfies (ii) then we have

$$|\cos(f(x),U(x))| = |\cos(f(x),\overline{x})| > 1-\delta_4 > 1-\delta$$

which proves (4). Suppose now that x satisfies (i). Since we have

$$|\cos(f(e_0),\overline{e_0})| > 1-\delta_4 \quad \text{and} \quad \big|\,|\cos(f(x),f(e_0))| - |\cos(x,e_0)|\,\big| \leqslant \varepsilon < \delta_4,$$

from (9) we get

$$||\cos(f(x),\overline{e_0})| - |\cos(x,e_0)|| < \delta_3.$$

This inequality, together with

$$|\cos(f(x),x)| > 1-\delta_4 > 1-\delta_3,$$

(8) and (6) yields

$$|\cos(x,\overline{x})| > 1-\delta_1.$$

The above inequality with

$$|\cos(f(x),x)| > 1-\delta_4 > 1-\delta_1$$

and (5) gives us

$$|\cos(f(x),\overline{x})| \geqslant 1-\delta$$

which proves (4).

Now, we consider the case when (11) is not assumed. Then, there exists R — a rotation about the origin, such that $R(f(e_1))$ is colinear with e_1, i.e., $|\cos(R(f(e_1)),e_1)| = 1$. Then the function $h := R\circ f$ satisfies the condition $(\star\star)$ and also (11). Thus, due to the first part of the proof, there exists $U' : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying (3) and

$$|\cos(h(x),U'(x))| \geqslant 1-\delta.$$

Defining $U := R^{-1}\circ U'$ we obtain that U satisfies (3) and

$$\begin{aligned}|\cos(f(x),U(x))| &= |\cos(R^{-1}(R(f(x))),R^{-1}(U'(x)))|\\ &= |\cos(h(x),U'(x))|\\ &\geqslant 1-\delta.\end{aligned}$$

This finishes the proof. □

The correspondence between δ and ε gives us a function $\mathbb{R}_+ \ni \delta \to \varepsilon(\delta) \in \mathbb{R}_+$. Moreover, we obtained that $\varepsilon(\delta) < \delta$ which proves that $\lim_{\delta\to 0}\varepsilon(\delta) = 0$. Without loss of generality we may assume that the function $\varepsilon = \varepsilon(\delta)$ is increasing (we may assume that all the correspondences $\delta \to \delta_1,\dots,\delta_4 \to \varepsilon$ considered in (5)-(10) are increasing). Therefore we can define the inverse function $\delta = \delta(\varepsilon)$ defined on an interval $(0,\varepsilon_0)$ for some $\varepsilon_0 > 0$. We can reformulate our result.

COROLLARY 1. *There exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0,\varepsilon_0)$ if $f : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies $(\star\star)$ then there exists a function $U : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying* (3) *and such that*

$$|\cos(f(x),U(x))| \geqslant 1-\delta(\varepsilon) \quad \textit{for } x \in \mathbb{R}^2\setminus\{0\}.$$

with some function $\delta : (0,\varepsilon_0) \to \mathbb{R}_+$ satisfying the condition $\lim_{\varepsilon\to 0}\delta(\varepsilon) = 0$.

Now we can establish the stability result for the Wigner equation.

THEOREM 3. *For an arbitrary $\delta > 0$ there exists $\varepsilon > 0$ such that for an arbitrary function $f : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying $(\star)$ there exists a function $I : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying the Wigner equation* (W) *and such that*

$$\|f(x) - I(x)\| \leqslant \delta\,\|x\| \quad \textit{for } x \in \mathbb{R}^2.$$

Proof. Let us fix $\delta > 0$ and put $\delta' := \frac{\delta^2}{8}$. Applying Theorem 2, we obtain for δ' suitably constant ε'. Let $\varepsilon'' < \min\{\varepsilon', 2\}$ be such that $1 - \sqrt{1 - \frac{\varepsilon''}{2}} \leqslant \frac{\delta}{2}$ and define $\varepsilon := \frac{\varepsilon''}{2}$. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be an arbitrary function satisfying $(\star)$. Define the function $g : \mathbb{R}^2 \to \mathbb{R}^2$ as in Lemma 1. Therefore

$$\big|\,|\langle g(x)|g(y)\rangle| - |\langle x|y\rangle|\,\big| \leqslant 2\varepsilon\,\|x\|\,\|y\| = \varepsilon''\,\|x\|\,\|y\|, \;\; x, y \in \mathbb{R}^2$$

and

$$\|g(x)\| = \|x\|, \;\; x \in \mathbb{R}^2.$$

This gives us

$$g(x) = 0 \;\;\Leftrightarrow\;\; x = 0$$

and

$$\big|\,|\cos(g(x), g(y))| - |\cos(x, y)|\,\big| \leqslant \varepsilon' \quad \text{for } x, y \in \mathbb{R}^2 \setminus \{0\}.$$

Thus g satisfies $(\star\star)$. Then, in virtue of Theorem 2, there exists $U : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying (3) and such that

$$|\cos(g(x), U(x))| \geqslant 1 - \delta' \quad \text{for } x \in \mathbb{R}^2 \setminus \{0\}.$$

Define

$$J(x) := \begin{cases} \dfrac{U(x)\,\|x\|}{\|U(x)\|}, & x \neq 0, \\ 0, & x = 0 \end{cases}$$

and let $I(x) := \sigma(x)J(x)$ where $\sigma(0) = 1$ and for $x \in \mathbb{R}^2 \setminus \{0\}$ the value $\sigma(x) \in \{-1, 1\}$ is such that $\cos(g(x), \sigma(x)J(x)) \geqslant 0$. Since

$$|\cos(g(x), J(x))| = |\cos(g(x), U(x))| \geqslant 1 - \delta',$$

we conclude that $\cos(g(x), I(x)) \geqslant 1 - \delta'$. We have also, for arbitrary $x, y \in \mathbb{R}^2 \setminus \{0\}$,

$$|\cos(I(x), I(y)| = |\cos(U(x), U(y))| = |\cos(x, y)|$$

and

$$\|I(x)\| = \|x\|.$$

Therefore

$$|\langle I(x)|I(y)\rangle| = |\langle x|y\rangle| \quad \text{for } x, y \in \mathbb{R}^2.$$

Moreover, we have for $x \in \mathbb{R}^2 \setminus \{0\}$,

$$\begin{aligned} \|I(x) - g(x)\|^2 &= \|I(x)\|^2 + \|g(x)\|^2 - 2\|I(x)\|\,\|g(x)\|\cos(I(x), g(x)) \\ &= 2\|x\|^2\,(1 - \cos(I(x), g(x))) \\ &\leqslant 2\|x\|^2\delta' \\ &= \frac{\delta^2}{4}\|x\|^2, \end{aligned}$$

$$\text{i.e.,} \qquad \|I(x) - g(x)\| \leqslant \frac{\delta}{2}\|x\|.$$

Finally, we have

$$\begin{aligned}\|I(x)-f(x)\| &\leqslant \|I(x)-g(x)\|+\|g(x)-f(x)\| \\ &\leqslant \frac{\delta}{2}+\left(1-\sqrt{1-\frac{\varepsilon''}{2}}\right)\|x\| \\ &\leqslant \delta\|x\|\end{aligned}$$

for all $x \in \mathbb{R}^2$. The proof is finished. □

Following Corollary 1 we get the final result.

COROLLARY 2. *There exists* $\varepsilon_0 > 0$ *such that for any* $\varepsilon \in (0, \varepsilon_0)$ *if* $f : \mathbb{R}^2 \to \mathbb{R}^2$ *satisfies the inequality*

$$\big|\,|\langle f(x)|f(y)\rangle| - |\langle x|y\rangle|\,\big| \leqslant \varepsilon\,\|x\|\,\|y\|, \quad x, y \in \mathbb{R}^2,$$

then there exists a mapping $I : \mathbb{R}^2 \to \mathbb{R}^2$ *satisfying the Wigner equation on* $\mathbb{R}^2$ *and such that*

$$\|f(x) - I(x)\| \leqslant \delta(\varepsilon)\|x\|, \quad x \in \mathbb{R}^2$$

for some function $\delta : (0, \varepsilon_0) \to \mathbb{R}_+$ *satisfying the condition* $\lim_{\varepsilon\to 0} \delta(\varepsilon) = 0$.

It is an open problem to extend the assertions of Theorems 2 and 3 to the n-dimensional case.

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