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# GENERAL INCLUSION THEOREMS FOR ABSOLUTE SUMMABILITY OF ORDER $k \geqslant 1$

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*Abstract.* We establish a general inclusion theorem for absolute summability of order $k \geqslant 1$, involving two lower triangular matrices. As corollaries we obtain a number of other inclusion theorems.

Let $\sum a_n$ be a given series with partial sums $s_n$, $(C,\alpha)$ the Cesàro matrix of order $\alpha$. If $\sigma_n^\alpha$ denotes the nth term of the $(C,\alpha)$-transform of $\{s_n\}$, then Flett [3] defined absolute summability of order $k \geqslant 1$ as follows. A series $\sum a_n$ is said to be summable $|C,\alpha|_k, k \geqslant 1$ if

$$\sum_{n=1}^{\infty} n^{k-1}|\sigma_n^\alpha - \sigma_{n-1}^\alpha|^k := \sum_{n=1}^{\infty} n^{k-1}|\Delta_n \sigma_{n-1}^\alpha|^k < \infty. \tag{1}$$

In an effort to extend (1) to other classes of matrices, some authors have interpreted the $n$ in (1) to represent the reciprocal of the nth diagonal entry of the matrix.

For example, in [2], with $Z_n$ denoting the nth term of the weighted mean transform of a sequence $\{s_n\}$, i.e.,

$$Z_n = \frac{1}{Pn}\sum_{k=0}^{n} p_k s_k, \tag{2}$$

their version of (1) becomes

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}|\Delta Z_{n-1}|^k < \infty. \tag{3}$$

Detailed arguments showing that (3) is not an appropriate extension of (1) appear in [6], and so will not be repeated here.

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For an arbitrary lower triangular matrix $T$, we shall say that a series $\sum a_n$ is summable $|T|_k, k \geqslant 1$, if

$$\sum_{n=1}^{\infty} n^{k-1}|\Delta t_{n-1}|^k < \infty, \tag{4}$$

where

$$t_n := \sum_{k=0}^{n} t_{nk}s_k. \tag{5}$$

Such an extension is used, for example, in [1].

The purpose of this paper is to establish a general absolute inclusion theorem involving a pair of triangles. We obtain, as corollaries, inclusion theorems for special classes of triangles.

Let $A$ be a lower triangular matrix. Associated with $A$ are two lower triangular matrices $\overline{A} = (\bar{a}_{nk})$ and $\hat{A} = (\hat{a}_{nk})$ with entries defined by

$$\bar{a}_{nk} = \sum_{i=k}^{n} a_{ni} \quad \text{and} \quad \hat{a}_{nk} = \bar{a}_{nk} - \bar{a}_{n-1,k}, \tag{6}$$

respectively.

With $y_n$ denoting the $n$-th term of the $A$-transform of a sequence $\{s_n\}$, we have, using (6),

$$y_n = \sum_{k=0}^{n} a_{nk}s_k = \sum_{k=0}^{n} a_{nk} \sum_{i=k}^{n} a_i = \sum_{i=0}^{n} a_i \sum_{k=i}^{n} a_{ni} = \sum_{i=0}^{n} \bar{a}_{ni}a_i,$$

and

$$Y_n := y_n - y_{n-1} = \sum_{i=0}^{n} \bar{a}_{ni}a_i - \sum_{i=0}^{n-1} \bar{a}_{n-1,i}a_i = \sum_{i=0}^{n} \hat{a}_{ni}a_i, \quad \text{since} \quad \bar{a}_{n-1,n} = 0. \tag{7}$$

A lower triangular matrix $A$ is called a triangle if $a_{nn} \neq 0$ for each $n$. Then $A$ has a unique two-sided inverse, which we shall denote by $A' = (a'_{nk})$. Clearly, if $A$ is a triangle, then $\hat{A}'$ exists, and is a triangle, since $\hat{a}_{nn} = a_{nn}$.

The notation $\Delta_\nu \hat{a}_{n\nu}$ means $\hat{a}_{n\nu} - \hat{a}_{n,\nu+1}$.

Theorem 1 of this paper represents the first time that two arbitrary triangles have been used in an absolute inclusion theorem, using either definition (3) or (4). Theorem 1 also represents one of the most general such inclusion theorems that one can expect to obtain.

Theorem 2 is an inclusion theorem in which the first matrix involved is an arbitrary triangle. This is also the first inclusion theorem of this type.

By restricting $A$ and $B$ to specific classes of matrices, we obtain, as corollaries, most of the known inclusion theorems, using (4), as special cases.

THEOREM 1. *Let $A$ and $B$ be triangles satisfying*

(*i*) $\dfrac{|a_{nn}|}{|b_{nn}|} = O(1)$,

(*ii*) $\left|\dfrac{b_{n+1,n} - b_{nn}}{b_{nn}b_{n+1,n+1}}\right| = O(1)$,

(*iii*) $\sum_{\nu=0}^{n-1} |\Delta_\nu \hat{a}_{n\nu}| = O(|a_{nn}|)$,

(*iv*) $\sum_{n=\nu+1}^{\infty} (n|a_{nn}|)^{k-1}|\Delta_\nu \hat{a}_{n\nu}| = O(\nu^{k-1}|a_{\nu\nu}|^k)$,

(*v*) $\sum_{\nu=0}^{n-1} |a_{\nu\nu}||\hat{a}_{n,\nu+1}| = O(|a_{nn}|)$,

(*vi*) $\sum_{n=\nu+1}^{\infty} (n|a_{nn}|)^{k-1}|\hat{a}_{n,\nu+1}| = O((\nu|a_{\nu\nu}|)^{k-1})$,

(*vii*) $\sum_{r=0}^{n-2} |a_{nr}|\left|\sum_{\nu=2}^{n} \hat{a}_{n\nu}\hat{b}'_{\nu r}\right| = O(|a_{nn}|)$,

*and*

(*viii*) $\sum_{n=r+2}^{\infty} (n|a_{nn}|)^{k-1}\left|\sum_{\nu=r+2}^{n} \hat{a}_{n\nu}\hat{b}'_{\nu r}\right| = O((r|a_{rr}|)^{k-1})$.

*Then, if $\sum a_n$ is summable $|B|_k$, it is summable $|A|_k, k \geqslant 1$.*

*Proof.* If $x_n$ denotes the nth term of the $B$-transform of a sequence $s_n$, then, as in (7),

$$X_n := x_n - x_{n-1} = \sum_{i=0}^{n} \hat{b}_{ni}a_i. \tag{8}$$

Since $\hat{B}'$ is a triangle, we may solve (8) for $a_n$ to get

$$a_n = \sum_{r=0}^{n} \hat{b}'_{nr}X_r. \tag{9}$$

Substituting (9) into (7) yields

$$\begin{aligned} Y_n &= \sum_{\nu=0}^{n} \hat{a}_{n\nu} \sum_{r=0}^{\nu} \hat{b}'_{\nu r}X_r = \sum_{\nu=0}^{n} \hat{a}_{n\nu}\Big(\hat{b}'_{\nu\nu}X_\nu + \hat{b}'_{\nu,\nu-1}X_{\nu-1} + \sum_{r=0}^{\nu-2} \hat{b}'_{\nu r}X_r\Big) \\ &= \sum_{\nu=0}^{n} \hat{a}_{n\nu}\hat{b}'_{\nu\nu}X_\nu + \sum_{\nu=0}^{n} \hat{a}_{n\nu}\hat{b}'_{\nu,\nu-1}X_{\nu-1} + \sum_{\nu=0}^{n} \hat{a}_{n\nu} \sum_{r=0}^{\nu-2} \hat{b}'_{\nu r}X_r \\ &= \hat{a}_{nn}b'_{nn}X_n + \sum_{\nu=0}^{n-1} \hat{a}_{n\nu}b'_{\nu\nu}X_\nu + \sum_{\nu=0}^{n-1} \hat{a}_{n,\nu+1}\hat{b}_{\nu+1,\nu}X_\nu + \sum_{\nu=0}^{n} \hat{a}_{n\nu} \sum_{r=0}^{\nu-2} \hat{b}_{\nu r}X_r \\ &= \frac{a_{nn}X_n}{b_{nn}} + \sum_{\nu=0}^{n-1} \frac{(\Delta_\nu \hat{a}_{n\nu})}{b_{\nu\nu}}X_\nu + \sum_{\nu=0}^{n-1} \hat{a}_{n,\nu+1}(\hat{b}'_{\nu\nu} + \hat{b}'_{\nu+1,\nu})X_\nu + \sum_{\nu=0}^{n-1} \hat{a}_{n\nu} \sum_{r=0}^{\nu-2} \hat{b}'_{\nu r}X_r. \end{aligned} \tag{10}$$

Using the fact that

$$\sum_{j=k}^{n} \hat{b}_{nj}\hat{b}'_{jk} = \delta_k^n,$$

and using $(6)$,

$$\begin{aligned}\hat{b}'_{\nu\nu} + \hat{b}'_{\nu+1,\nu} &= \frac{1}{b_{\nu\nu}} - \frac{\hat{b}_{\nu+1,\nu}}{b_{\nu\nu}b_{\nu+1,\nu+1}} = \frac{1}{b_{\nu\nu}}\Big(1 - \frac{\bar{b}_{\nu+1,\nu} - \bar{b}_{\nu\nu}}{b_{\nu+1,\nu+1}}\Big) \\ &= \frac{1}{b_{\nu\nu}}\Big(1 - \frac{b_{\nu+1,\nu} + b_{\nu+1,\nu+1} - b_{\nu\nu}}{b_{\nu+1,\nu+1}}\Big) = \frac{1}{b_{\nu\nu}}\Big(\frac{b_{\nu\nu} - b_{\nu+1,\nu}}{b_{\nu+1,\nu+1}}\Big).\end{aligned} \tag{11}$$

Substituting $(11)$ into $(10)$ we have

$$\begin{aligned}Y_n &= \frac{a_{nn}X_n}{b_{nn}} + \sum_{\nu=0}^{n-1} \frac{\Delta_\nu \hat{a}_{n\nu} X_\nu}{b_{\nu\nu}} + \sum_{\nu=0}^{n-1} \hat{a}_{n,\nu+1}\Big(\frac{b_{\nu\nu} - b_{\nu+1,\nu}}{b_{\nu\nu}b_{\nu+1,\nu+1}}\Big)X_\nu + \sum_{\nu=2}^{n} \hat{a}_{n\nu} \sum_{r=0}^{\nu-2} \hat{b}'_{\nu r} X_r \\ &= T_{n1} + T_{n2} + T_{n3} + T_{n4}, \quad \text{say}.\end{aligned}$$

From $(4)$, by Minkowski's inequality, it is sufficient to prove that

$$\sum_{n=1}^{\infty} n^{k-1}|T_{ni}|^k < \infty, \qquad i = 1,2,3,4.$$

Using $(i)$,

$$\sum_{n=1}^{\infty} n^{k-1}|T_{n1}|^k = \sum_{n=1}^{\infty} n^{k-1}\Big|\frac{a_{nn}X_n}{b_{nn}}\Big|^k = O(1)\sum_{n=1}^{\infty} n^{k-1}|X_n|^k = O(1),$$

since $\sum a_n$ is summable $|B|_k$.

Using $(i)$, Hölder's inequality, $(iii)$, and $(iv)$,

$$\begin{aligned}\sum_{n=1}^{\infty} n^{k-1}|T_{n2}|^k &= \sum_{n=1}^{\infty} n^{k-1}\Big|\sum_{\nu=0}^{n-1} b_{\nu\nu}^{-1}(\Delta_\nu \hat{a}_{n\nu})X_\nu\Big|^k \leqslant \sum_{n=1}^{\infty} n^{k-1}\Big(\sum_{\nu=0}^{n-1} |b_{\nu\nu}|^{-1}|\Delta_\nu \hat{a}_{n\nu}||X_\nu|\Big)^k \\ &= O(1)\sum_{n=1}^{\infty} n^{k-1}\Big(\sum_{\nu=0}^{n-1} |a_{\nu\nu}|^{-1}|\Delta_\nu \hat{a}_{n\nu}||X_\nu|\Big)^k \\ &= O(1)\sum_{n=1}^{\infty} n^{k-1}\Big(\sum_{\nu=0}^{n-1} |a_{\nu\nu}|^{-k}|\Delta_\nu \hat{a}_{n\nu}||X_\nu|^k\Big)\Big(\sum_{\nu=0}^{n-1} |\Delta_\nu \hat{a}_{n\nu}|\Big)^{k-1} \\ &= O(1)\sum_{n=1}^{\infty} (n|a_{nn}|)^{k-1}\sum_{\nu=0}^{n-1} |a_{\nu\nu}|^{-k}|\Delta_\nu \hat{a}_{n\nu}||X_\nu|^k \\ &= O(1)\sum_{\nu=0}^{\infty} |a_{\nu\nu}|^{-k}|X_\nu|^k \sum_{n=\nu+1}^{\infty} (n|a_{nn}|)^{k-1}|\Delta_\nu \hat{a}_{n\nu}| \\ &= O(1)\sum_{\nu=0}^{\infty} |a_{\nu\nu}|^{-k}|X_\nu|^k \nu^{k-1}|a_{\nu\nu}|^k = O(1)\sum_{\nu=1}^{\infty} \nu^{k-1}|X_\nu|^k = O(1).\end{aligned}$$

Using $(ii)$, $(v)$, and $(vi)$,

$$\begin{aligned}\sum_{n=1}^{\infty} n^{k-1}|T_{n3}|^k &= \sum_{n=1}^{\infty} n^{k-1}\Big|\sum_{\nu=0}^{n-1} \hat{a}_{n,\nu+1}\Big(\frac{b_{\nu\nu}-b_{\nu+1,\nu}}{b_{\nu\nu}b_{\nu+1,\nu+1}}\Big)X_\nu\Big|^k \\ &= O(1)\sum_{n=1}^{\infty} n^{k-1}\Big(\sum_{\nu=0}^{n-1}|\hat{a}_{n,\nu+1}||X_\nu|\Big)^k \\ &= O(1)\sum_{n=1}^{\infty} n^{k-1}\Big(\sum_{\nu=0}^{n-1}|a_{\nu\nu}|^{1-k}|\hat{a}_{n,\nu+1}||X_\nu|^k\Big)\Big(\sum_{\nu=0}^{n-1}|a_{\nu\nu}||\hat{a}_{n,\nu+1}|\Big)^{k-1} \\ &= O(1)\sum_{n=1}^{\infty}(n|a_{nn}|)^{k-1}\sum_{\nu=0}^{n-1}|a_{\nu\nu}|^{1-k}|\hat{a}_{n,\nu+1}||X_\nu|^k \\ &= O(1)\sum_{\nu=0}^{\infty}|a_{\nu\nu}|^{1-k}|X_\nu|^k\sum_{n=\nu+1}^{\infty}(n|a_{nn}|)^{k-1}|\hat{a}_{n,\nu+1}| \\ &= O(1)\sum_{\nu=0}^{\infty}|a_{\nu\nu}|^{1-k}|X_\nu|^k\nu^{k-1}|a_{\nu\nu}|^{k-1} = O(1)\sum_{\nu=0}^{\infty}\nu^{k-1}|X_\nu|^k = O(1).\end{aligned}$$

From conditions $(vii)$ and $(viii)$,

$$\begin{aligned}\sum_{n=1}^{\infty} n^{k-1}|T_{n4}|^k &= \sum_{n=1}^{\infty} n^{k-1}\Big|\sum_{\nu=2}^{n}\hat{a}_{n\nu}\sum_{r=0}^{\nu-2}X_r\Big|^k = \sum_{n=1}^{\infty} n^{k-1}\Big|\sum_{r=0}^{n-2}X_r\sum_{\nu=r+2}^{n}\hat{a}_{n\nu}\hat{b}'_{\nu r}\Big|^k \\ &\leqslant \sum_{n=1}^{\infty} n^{k-1}\sum_{r=0}^{n-2}|a_{rr}|^{1-k}|X_r|^k\Big|\sum_{\nu=r+2}^{n}\hat{a}_{n\nu}\hat{b}'_{\nu r}\Big|\Big(\sum_{r=0}^{n-2}|a_{rr}|\Big|\sum_{\nu=r+2}^{n}\hat{a}_{n\nu}\hat{b}'_{\nu r}\Big|\Big)^{k-1} \\ &= O(1)\sum_{n=1}^{\infty}\sum_{r=0}^{n-2}|a_{rr}|^{1-k}|X_r|^k\Big|\sum_{\nu=r+2}^{n}\hat{a}_{n\nu}\hat{b}'_{\nu r}\Big|(n|a_{nn}|)^{k-1} \\ &= O(1)\sum_{r=0}^{\infty}|a_{rr}|^{1-k}|X_r|^k\sum_{n=r+2}^{\infty}(n|a_{nn}|)^{k-1}\Big|\sum_{\nu=r+2}^{n}\hat{a}_{n\nu}\hat{b}'_{\nu r}\Big| \\ &= O(1)\sum_{r=0}^{\infty}|a_{rr}|^{1-k}|X_r|^k(r|a_{rr}|)^{k-1} = O(1)\sum_{r=1}^{\infty}r^{k-1}|X_r|^k = O(1).\end{aligned}$$

□

A triangle $B$ is called factorable if its nonzero entries $b_{nk}$ can be written in the form $c_n d_k$ for each $n$ and $k$.

COROLLARY 1. *Let $A$ and $B$ be triangles, $B$ factorable, satisfying conditions $(i)$ - $(vi)$ of Theorem 1. Then, if $\sum a_n$ is summable $|B|_k$, it is summable $|A|_k, k \geqslant 1$.*

*Proof.* To verify conditions $(vii)$ and $(viii)$ of Theorem 1 we need the following lemma.

LEMMA 1. *A factorable triangle $B$ has an inverse that is bidiagonal.*

*Proof of Lemma.* Let $B'$ denote the inverse of $B$. Then, since

$$\sum_{j=n}^{n+k} b_{n+k,j} b'_{jn} = 0, \qquad k > 0,$$

$$\begin{aligned} b'_{n+1,n} &= \frac{-b_{n+1,n}}{b_{nn}b_{n+1,n+1}} = -\frac{c_{n+1}d_n}{c_n d_n c_{n+1} d_{n+1}} = \frac{-1}{c_n d_{n+1}}. \\ b'_{n+2,n} &= -\frac{1}{b_{n+2,n+2}}[b_{n+2,n}b'_{nn} + b_{n+2,n+1}b'_{n+1,n}] \\ &= -\frac{1}{b_{n+2,n+2}}\Big[\frac{c_{n+2}d_n}{c_n d_n} + c_{n+2}d_{n+1}\Big(-\frac{1}{c_n d_{n+1}}\Big)\Big] \\ &= 0. \end{aligned}$$

Suppose that $b'_{n+j,n} = 0$ for $2 < j \leqslant k$. Then

$$\begin{aligned} b'_{n+k+1,n} &= -\frac{1}{b_{n+k+1,n+k+1}}[b_{n+k+1,n}b'_{nn} + b_{n+k+1,n+1}b'_{n+1,n}] \\ &= -\frac{1}{b_{n+k+1,n+k+1}}\Big[\frac{c_{n+k+1}d_n}{c_n d_n} + c_{n+k+1}d_{n+1}\Big(-\frac{1}{c_n d_{n+1}}\Big)\Big] \\ &= 0, \end{aligned}$$

and $B'$ is bidiagonal. □

It is well known that, if $B$ is a weighted mean matrix, then $\widehat{B}$ has entries

$$\hat{b}_{nk} = \frac{P_{k-1}p_n}{P_n P_{n-1}}.$$

Hence $\widehat{B}$ is factorable. By Lemma 1, $\widehat{B}'$ is bidiagonal, so $\hat{b}'_{\nu r} = 0$ for each $0 \leqslant r \leqslant \nu - 2$, and conditions $(vii)$ and $(viii)$ are satisfied trivially. □

COROLLARY 2. *Let $\{p_n\}$ be a positive sequence, $A$ a triangle, satisfying*

$$(i) \qquad \frac{P_n}{p_n}|a_{nn}| = O(1)$$

*and conditions $(iii)$ - $(vi)$ of Theorem 1. Then, if $\sum a_n$ is summable $|\overline{N}, p_n|_k$, it is summable $|A|_k$, $k \geqslant 1$.*

*Proof.* With $B = (\overline{N}, p_n)$, condition $(i)$ of Theorem 1 reduces to condition $(i)$ of Corollary 2.

$$\Big|\frac{b_{n+1,n} - b_{nn}}{b_{nn}b_{n+1,n+1}}\Big| = \Big|\frac{p_n/P_{n+1} - p_n/P_n}{p_n p_{n+1}/P_n P_{n+1}}\Big| = \frac{p_n p_{n+1}}{p_n p_{n+1}} = 1.$$

and condition $(ii)$ of Theorem 1 is automatically satisfied.
Since a weighted mean matrix is factorable, the result follows from Corollary 1. □

Corollary 2 is the corrected form of the theorem of [6].

COROLLARY 3. *Let $D$ be a Rhaly generalized Cesàro matrix, $B$ be a triangle satisfying*

$$(i)\quad \frac{1}{(n+1)|b_{nn}|} = O(1),$$

$$(ii)\quad \left|\frac{b_{n+1,n}-b_{nn}}{b_{nn}b_{n+1,n+1}}\right| = O(1),$$

$$(iii)\quad \frac{t^n}{n(n+1)(1-t)}\sum_{r=0}^{n-2} t^{-r}\left|\sum_{\nu=2}^{n}[(n+1)t^{n-\nu}-nt^{n-\nu-1}-1]\hat{b}'_{\nu r}\right| = O(1),$$

*and*

$$(iv)\quad \sum_{n=r+2}^{\infty}\frac{1}{n(n+1)(1-t)}\left|\sum_{\nu=r+2}^{n}[(n+1)t^{n-\nu}-nt^{n-\nu-1}-1]\hat{b}'_{\nu r}\right|^k = O(1).$$

*Then, if $\sum a_n$ is summable $|B|_k$, it is summable $|D|_k, k \geqslant 1$.*

*Proof.* We shall restrict our attention to $t<1$, since, with $t=1$, the Rhaly generalized Cesàro matrix reduces to $(C,1)$.

Since $d_{nk} = t^{n-k}/(n+1)$, (see, e.g., [4]) conditions $(i)$, $(ii)$, $(vii)$, and $(viii)$ of Theorem 1 reduce to conditions $(i)$ - $(iv)$ of Corollary 3, respectively.

We shall now show that conditions $(iii)$ - $(vi)$ of Theorem 1 are satisfied.

For Rhaly generalized Cesàro matrices,

$$\Delta_\nu \hat{d}_{n\nu} = d_{n\nu} - d_{n-1,\nu} = \frac{t^{n-\nu}}{n+1} - \frac{t^{n-1-\nu}}{n}.$$

$$\begin{aligned}(n+1)\sum_{\nu=0}^{n-1}|\Delta_\nu \hat{d}_{n\nu}| &= (n+1)\Big[\sum_{\nu=0}^{n-1}\frac{t^{n-1-\nu}}{n} - \sum_{\nu=0}^{n-1}\frac{t^{n-\nu}}{n+1}\Big]\\ &= (n+1)\Big[\frac{t^{n-1}}{n}\sum_{\nu=0}^{n-1}t^{-\nu} - \frac{t^n}{n+1}\sum_{\nu=0}^{n-1}t^{-\nu}\Big]\\ &= (n+1)t^{n-1}\Big(\frac{1}{n}-\frac{t}{n+1}\Big)\Big[\frac{1-1/t^n}{1-1/t}\Big]\\ &= \frac{(n+1)t^n}{t-1}\Big(\frac{n+1-nt}{n(n+1)}\Big)\frac{t^n-1}{t^n}\\ &= \frac{n+1-nt}{n(1-t)}(1-t^n) = O(1).\end{aligned}$$

Therefore $(iii)$ is satisfied.

There exist positive constants $K_1$ and $K_2$ such that

$$\begin{aligned}&\frac{(\nu+1)^k}{\nu^{k-1}}\sum_{n=\nu+1}^{\infty}(n|d_{nn}|)^{k-1}|\Delta_\nu \hat{d}_{n\nu}|\\ &\qquad = O(1)\nu\sum_{n=\nu+1}^{\infty}\Big(\frac{n}{n+1}\Big)^{k-1}\Big(\frac{t^{n-1-\nu}}{n}-\frac{t^{n-\nu}}{n+1}\Big)\\ &\qquad = O(1)\nu\sum_{n=\nu+1}^{\infty}\Big(\frac{t^{n-1-\nu}}{n}-\frac{t^{n-\nu}}{n+1}\Big)\\ &\qquad = O(1)\nu(1/\nu-0) = O(1).\end{aligned}$$

and $(iv)$ is satisfied.

$$\begin{aligned}\hat{d}_{n,\nu+1} &= \bar{d}_{n,\nu+1} - \bar{d}_{n-1,\nu+1} = \sum_{i=\nu+1}^{n} d_{ni} - \sum_{i=\nu+1}^{n-1} d_{n-1,i} \\ &= \sum_{i=\nu+1}^{n} \frac{t^{n-i}}{n+1} - \sum_{i=\nu+1}^{n-1} \frac{t^{n-1-i}}{n} \\ &= \sum_{j=0}^{n-\nu-1} \frac{t^{n-j-\nu-1}}{n+1} - \frac{1}{n}\sum_{j=0}^{n-\nu-2} t^{n-1-j-\nu-1} \\ &= \frac{t^{n-\nu-1}}{n+1}\sum_{j=0}^{n-\nu-1} t^{-j} - \frac{t^{n-\nu-2}}{n}\sum_{j=0}^{n-\nu-2} t^{-j} \\ &= \frac{1-t^{n-\nu}}{(n+1)(1-t)} - \frac{1-t^{n-\nu-1}}{n(1-t)} \\ &= \frac{1}{n(n+1)(1-t)}[n - nt^{n-\nu} - (n+1) + (n+1)t^{n-\nu-1}] \\ &= \frac{(n+1)t^{n-\nu-1} - nt^{n-\nu} - 1}{n(n+1)(1-t)} = \frac{O(1)}{n(n+1)(1-t)}.\end{aligned}$$

$$\sum_{n=0}^{\nu-1} |d_{\nu\nu}||\hat{d}_{n,\nu+1}| = \sum_{n=1}^{\nu-1} \frac{O(1)}{(\nu+1)n(n+1)(1-t)} = O(1),$$

and condition $(v)$ is satisfied.

$$\begin{aligned}\frac{1}{(\nu|d_{\nu\nu}|)^{k-1}} &\sum_{n=\nu+1}^{\infty} (n|d_{nn}|)^{k-1}|\hat{d}_{n,\nu+1}| \\ &= O(1)\sum_{n=\nu+1}^{\infty} O(1)\frac{1}{n(n+1)(1-t)} \\ &= \frac{O(1)}{(1-t)(\nu+1)} = O(1),\end{aligned}$$

and $(vi)$ is satisfied. □

COROLLARY 4. *Let $\{p_n\}$ be a positive sequence, $D$ a Rhaly generalized Cesàro matrix satisfying*

$$(i) \qquad \frac{P_n}{(n+1)p_n} = O(1).$$

*Then, if $\sum a_n$ is summable $|\overline{N}, p_n|_k$, it is summable $D_k, k \geqslant 1$.*

*Proof.* Setting $B = (\overline{N}, p_n)$ in Corollary 3 yields condition $(i)$ of Corollary 4.

$$\left|\frac{b_{n+1,n} - b_{nn}}{b_{nn}b_{n+1,n+1}}\right| = \left|\frac{p_n/P_{n+1} - p_n/P_n}{p_np_{n+1}/P_nP_{n+1}}\right| = \left|\frac{-p_np_{n+1}}{p_np_{n+1}}\right| = O(1),$$

and condition $(ii)$ of Corollary 3 is satisfied.

As in the proof of Corollary 1, conditions $(iii)$ and $(iv)$ of Corollary 3 are automatically satisfied. □

COROLLARY 5. *If* $\sum a_n$ *is summable* $|C,1|_k$*, then it is summable* $|D|_k$*,* $k \geqslant 1$.

*Proof.* Use Corollary 4 with each $p_n = 1$. □

It is not possible to use Theorem 1 with $B$ a Rhaly generalized Cesàro matrix, since condition $(ii)$ of Theorem 1 is violated.

COROLLARY 6. *Let* $F$ *be a Rhaly p-Cesàro matrix,* $B$ *a triangle satisfying*

$(i)$ $\dfrac{1}{(n+1)^p|b_{nn}|} = O(1)$,

$(ii)$ $\left|\dfrac{b_{n+1,n} - b_{nn}}{b_{nn}b_{n+1,n+1}}\right| = O(1)$,

$(iii)$ $\displaystyle\sum_{r=0}^{n-2}\left|\sum_{\nu=2}^{n}\left[\frac{n-\nu+1}{(n+1)^p} - \frac{n-\nu}{n^p}\right]\hat{b}'_{\nu r}\right| = O(1)$.

*and*

$(iv)$ $\displaystyle\sum_{n=r+2}^{\infty}\left(\frac{n}{(n+1)^p}\right)^{k-1}\left|\sum_{\nu=r+2}^{n}\left[\frac{n-\nu+1}{(n+1)^p} - \frac{n-\nu}{n^p}\right]\hat{b}'_{\nu r}\right|$
$\displaystyle = O\left(\left(\frac{r}{(r+1)^p}\right)^{k-1}\right)$.

*Then, if* $\sum a_n$ *is summable* $|B|_k$*, then it is summable* $|F|_k, k \geqslant 1$.

*Proof.* Let $F$ denote the Rhaly p-Cesàro matrix (See, e.g., [5]). We shall assume that $p > 1$, since $p = 1$ reduces the matrix to $(C,1)$.

$$\Delta_\nu \hat{f}_{n\nu} = \bar{f}_{n\nu} - \bar{f}_{n-1,\nu} = \frac{1}{(n+1)^p} - \frac{1}{n^p}.$$

$$\begin{aligned}
(n+1)^p\sum_{\nu=0}^{n-1}|\Delta_\nu \hat{f}_{n\nu}| &= (n+1)^p\sum_{\nu=0}^{n-1}\left(\frac{1}{n^p} - \frac{1}{(n+1)^p}\right)\\
&= (n+1)^p n\left(\frac{1}{n^p} - \frac{1}{(n+1)^p}\right)\\
&= n\left(\left(\frac{n+1}{n}\right)^p - 1\right).
\end{aligned}$$

$$\begin{aligned}
\lim_n \frac{((n+1)/n)^p - 1}{1/n} &= \lim_n \frac{p((n+1)/n)^{p-1}(-1/n^2)}{-1/n^2}\\
&= \lim_n p\left(\frac{n+1}{n}\right)^{p-1} = p,
\end{aligned}$$

and condition $(iii)$ of Theorem 1 is satisfied.

Conditions $(i)$, $(ii)$, $(vii)$, and $(viii)$ of Theorem 1 are conditions $(i)$ - $(iv)$ of Corollary 6, respectively.

$$\begin{aligned}
J_4 :&= \frac{1}{\nu^{k-1}|f_{\nu\nu}|^k} \sum_{n=\nu+1}^{\infty} (n|f_{nn}|)^{k-1} |\Delta_\nu \hat{f}_{n\nu}| \\
&= \frac{(\nu+1)^{pk}}{\nu^{k-1}} \sum_{n=\nu+1}^{\infty} \Big(\frac{n}{(n+1)^p}\Big)^{k-1} \Big(\frac{1}{n^p} - \frac{1}{(n+1)^p}\Big) \\
&= O(1)\nu^{(p-1)k+1} \sum_{n=\nu+1}^{\infty} O(1) n^{(1-p)(k-1)} \Big(\frac{1}{n^p} - \frac{1}{(n+1)^p}\Big) \\
&= O(1)\nu \sum_{n=\nu+1}^{\infty} \Big(\frac{\nu}{n}\Big)^{k(p-1)} n^{p-1} \Big(\frac{1}{n^p} - \frac{1}{(n+1)^p}\Big).
\end{aligned}$$

*Claim.*

$$n^{p-1}\Big(\frac{1}{n^p} - \frac{1}{(n+1)^p}\Big) \leqslant p\Big(\frac{1}{n} - \frac{1}{n+1}\Big).$$

*Proof of the claim.* The above inequality is equivalent to

$$\frac{n^{p-1}[(n+1)^p - n^p]}{(n(n+1))^p} \leqslant \frac{p}{n(n+1)}$$

or

$$\frac{(n+1)^p - n^p}{p(n+1)^{p-1}} \leqslant 1.$$

$$\begin{aligned}
f(n) :&= \frac{(n+1)^p - n^p}{p(n+1)^{p-1}} = \frac{1}{p}\Big[n+1 - n\Big(\frac{n}{n+1}\Big)^{p-1}\Big] \\
f'(n) &= \frac{1}{p}\Big[1 - \Big(\frac{n}{n+1}\Big)^{p-1} - n(p-1)\Big(\frac{n}{n+1}\Big)^{p-2}\frac{1}{(n+1)^2}\Big] \\
&= \frac{1}{p}\Big[1 - \Big(\frac{n}{n+1}\Big)^{p-1} - \frac{(p-1)}{n+1}\Big(\frac{n}{n+1}\Big)^{p-1}\Big] \\
&= \frac{1}{p}\Big[1 - \Big(1 + \frac{p-1}{n+1}\Big)\Big(\frac{n}{n+1}\Big)^{p-1}\Big]. \\
f''(n) &= -\frac{1}{p}\Big[-\frac{(p-1)}{(n+1)^2}\Big(\frac{n}{n+1}\Big)^{p-1} + \Big(1 + \frac{p-1}{n+1}\Big)(p-1)\Big(\frac{n}{n+1}\Big)^{p-2}\frac{1}{(n+1)^2}\Big] \\
&= \frac{(p-1)}{p(n+1)^3}\Big(\frac{n}{n+1}\Big)^{p-2}[n - n - 1 - (p-1)] < 0.
\end{aligned}$$

Therefore $f'$ is decreasing in $n$. $\lim f'(n) = 0$, so $f$ is increasing in $n$. We may write

$$f(n) = \frac{1 - (n/(n+1))^p}{p/(n+1)}.$$

Thus

$$\lim f(n) = \lim \frac{-p(n/(n+1))^{p-1}(1/(n+1)^2)}{-p/((n+1)^2)} = \lim \Big(\frac{n}{n+1}\Big)^{p-1} = 1,$$

and $(iv)$ of Theorem 1 is satisfied.

$$\begin{aligned}
\hat{f}_{n,\nu+1} &= \bar{f}_{n,\nu+1} - \bar{f}_{n-1,\nu+1} = \sum_{i=\nu+1}^{n} f_{ni} - \sum_{i=\nu+1}^{n-1} f_{n-1,i} \\
&= \sum_{i=\nu+1}^{n} \frac{1}{(n+1)^p} - \sum_{i=\nu+1}^{n-1} \frac{1}{n^p} = \frac{n-\nu}{(n+1)^p} - \frac{n-\nu-1}{n^p}.
\end{aligned}$$

$$\begin{aligned}
J_5 :&= (n+1)^p \sum_{\nu=0}^{n-1} \frac{1}{(\nu+1)^p} \left| \frac{n-\nu}{(n+1)^p} - \frac{n-\nu-1}{n^p} \right| \\
&\leqslant (n+1)^p \left[ \sum_{\nu=0}^{n-1} \frac{(n-\nu)}{(\nu+1)^p} \left( \frac{1}{n^p} - \frac{1}{(n+1)^p} \right) + \sum_{\nu=0}^{n-1} \frac{1}{n^p(\nu+1)^p} \right] \\
&= (n+1)^p \left( \frac{1}{n^p} - \frac{1}{(n+1)^p} \right) \sum_{\nu=0}^{n-1} \frac{(n-\nu)}{(\nu+1)^p} + \left( \frac{n+1}{n} \right)^p \sum_{\nu=0}^{n-1} \frac{1}{(\nu+1)^p} \\
&\leqslant (n+1)^{p+1} \left( \frac{1}{n^p} - \frac{1}{(n+1)^p} \right) \sum_{\nu=0}^{n-1} \frac{1}{(\nu+1)^p} + O(1) \\
&= O(1) n \left( \left( \frac{n+1}{n} \right)^p - 1 \right) + O(1).
\end{aligned}$$

Since

$$\lim n \left( \left( \frac{n+1}{n} \right)^p - 1 \right) = p,$$

$(v)$ of Theorem 1 is satisfied.

$$\begin{aligned}
J_6 :&= \frac{1}{(\nu|f_{\nu\nu}|)^{k-1}} \sum_{n=\nu+1}^{\infty} (n|f_{nn}|)^{k-1} |\hat{f}_{n,\nu+1}| \\
&= \left( \frac{(\nu+1)^p}{\nu} \right)^{k-1} \sum_{n=\nu+1}^{\infty} \left( \frac{n}{(n+1)^p} \right)^{k-1} \left| \frac{n-\nu}{(n+1)^p} - \frac{n-\nu-1}{n^p} \right| \\
&\leqslant \sum_{n=\nu+1}^{\infty} (n-\nu) \left( \frac{1}{n^p} - \frac{1}{(n+1)^p} \right) + \sum_{n=\nu+1}^{\infty} \frac{1}{n^p} \\
&\leqslant \sum_{n=\nu+1}^{\infty} n \left( \frac{1}{n^p} - \frac{1}{(n+1)^p} \right) + O(1) \\
&= \sum_{n=\nu+1}^{\infty} \frac{1}{n^{p-1}} - \frac{n}{(n+1)^p} + O(1) \\
&= \sum_{n=\nu+1}^{\infty} \frac{1}{n^{p-1}} - \frac{1}{(n+1)^{p-1}} + \sum_{n=\nu+1}^{\infty} \frac{1}{(n+1)^p} + O(1) \\
&= \frac{1}{(\nu+1)^{p-1}} + O(1) = O(1),
\end{aligned}$$

and $(vi)$ of Theorem 1 is satisfied. □

COROLLARY 7. *Let* $\{p_n\}$ *be a positive sequence,* $F$ *a Rhaly p-Cesàro matrix satisfying*

$$\frac{P_n}{(n+1)^p p_n} = O(1).$$

*Then, if* $\sum a_n$ *is summable* $|\overline{N},p_n|_k$ *is it summable* $|F|_k, k \geqslant 1$.

*Proof.* Set $B = (\overline{N},p_n)$ in Corollary 6 to obtain the above condition. As in the proof of Corollary 2, conditions $(ii)$ - $(iv)$ of Corollary 6 are automatically satisfied. □

COROLLARY 8. *If* $\sum a_n$ *is summable* $|C,1|_k$*, then it is summable* $|F|_k$*, $k \geqslant 1$.*

To prove Corollary 8 observe that the condition of Corollary 7 is automatically satisfied.

It is not possible to use Theorem 1 with $B = F$, since condition $(ii)$ of Theorem 1 is violated.

A Nörlund matrix $(N,p_n)$ is a lower triangular matrix with nonzero entries $a_{nk} = p_{n-k}/P_n$.

COROLLARY 9. *Let* $\{p_n\}$ *be a positive sequence,* $A$ *a triangle satisfying*

$(i)$ $P_n|a_{nn}| = O(1)$,

$(ii)$ $p_1P_n - p_0P_{n+1} = O(1)$,

*and conditions* $(iii)$ - $(viii)$ *of Theorem 1. Then, if,* $\sum a_n$ *is summable* $|N,p_n|_k$ *it is summable* $|A|_k, k \geqslant 1$.

With $B = (N,p_n)$, the conditions of Theorem 1 reduce immediately to those of Corollary 9.

COROLLARY 10. *Let* $A$ *be a triangle satisfying*

$$n|a_{nn}| = O(1),$$

*and conditions* $(iii)$ - $(vi)$ *of Theorem 1. Then, if* $\sum a_n$ *is summable* $|C,1|_k$*, it is summable* $|A|_k, k \geqslant 1$.

*Proof.* In Corollary 2 set each $p_n = 1$. □

THEOREM 2. *Let* $\{q_n\}$ *be a positive sequence,* $B$ *be a triangle satisfying*

$(i)$ $\dfrac{q_n}{Q_n|b_{nn}|} = O(1)$,

$(ii)$ $\left|\dfrac{b_{n+1} - b_{nn}}{b_{nn}b_{n+1,n+1}}\right| = O(1)$,

$(iii)$ $\displaystyle\sum_{n=\nu+1}^{\infty} n^{k-1}\left(\frac{q_n}{Q_n}\right)^k \frac{1}{Q_{n-1}} = O\left(\frac{(\nu q_\nu)^{k-1}}{Q_\nu^k}\right)$,

$(iv)$ $\displaystyle\sum_{r=0}^{n-2} q_r\left|\sum_{\nu=2}^{n} \frac{Q_{\nu-1}}{Q_nQ_{n-1}}\hat{b}'_{\nu r}\right| = O(1)$, *and*

$$(v)\quad \sum_{n=r+2}^{\infty}\left(\frac{nq_n}{Q_n}\right)^{k-1}\left|\sum_{v=r+2}^{n}\frac{Q_{v-1}q_n}{Q_nQ_{n-1}}\hat{b}'_{vr}\right| = O\left(\left(\frac{rq_r}{Q_r}\right)^{k-1}\right).$$

*Then, if $\sum a_n$ is summable $|B|_k$, it is summable $|\overline{N},q_n|_k, k \geqslant 1$.*

*Proof.* In Theorem 1 set $A = (\overline{N},p_n)$. Then conditions $(i)$, $(ii)$, $(vii)$, and $(viii)$ of Theorem 1 become conditions $(i)$, $(ii)$, $(iv)$, and $(v)$ of Theorem 2, respectively. Conditions $(iv)$ and $(vi)$ of Theorem 1 reduce to condition $(iii)$ of Theorem 2.

For any matrix $A$, it is well known that $\Delta_v\hat{a}_{nv} = a_{nv} - a_{n-1,v}$. With $A = (\overline{N},q_n)$,

$$\sum_{v=0}^{n-1}|\Delta_v\hat{a}_{nv}| = \sum_{v=0}^{n-1}|a_{nv}-a_{n-1,v}| = \sum_{v=0}^{n-1}\left|\frac{q_v}{Q_n}-\frac{q_v}{Q_{n-1}}\right| = \frac{q_n}{Q_nQ_{n-1}}\sum_{v=0}^{n-1}q_v = \frac{q_n}{Q_n},$$

and condition $(iii)$ of Theorem 1 is satisfied.

Also,

$$\hat{a}_{n,v+1} = \frac{q_nQ_v}{Q_nQ_{n-1}},$$

so

$$\sum_{v=0}^{n-1}|a_{vv}\hat{a}_{n,v+1}| = \sum_{v=0}^{n-1}\left(\frac{q_v}{Qv}\right)\frac{q_nQ_v}{Q_nQ_{n-1}} = \frac{q_n}{Q_nQ_{n-1}}\sum_{v=0}^{n-1}q_v = \frac{q_n}{Q_n},$$

and condition $(v)$ is satisfied. □

COROLLARY 11. *Let $B$ be a triangle satisfying*

$$(i)\quad \frac{1}{n|b_{nn}|} = O(1),$$

$$(ii)\quad \left|\frac{b_{n+1,n}-b_{nn}}{b_{nn}b_{n+1,n+1}}\right| = O(1),$$

$$(iii)\quad \sum_{r=0}^{n-2}\left|\sum_{v=2}^{n}\frac{v}{n(n+1)}\hat{b}'_{vr}\right| = O(1),$$

*and*

$$(iv)\quad \sum_{n=r+2}^{\infty}\left|\sum_{v=r+2}^{n}\frac{v}{n(n+1)}\hat{b}_{vr}\right|^k = O(1).$$

*Then, if $\sum a_n$ is summable $|B|_k$, it is summable $|C,1|_k, k \geqslant 1$.*

*Proof.* Set each $q_n = 1$ in Theorem 2. Then conditions $(i)$, $(ii)$, $(iv)$ and $(v)$ of Theorem 2 become conditions $(i)$ - $(iv)$ of Corollary 11, respectively.

It remains to show that condition $(iii)$ of Theorem 2 is satisfied.

$$\begin{aligned}\frac{Q_v^k}{(vq_v)^{k-1}}\sum_{n=v+1}^{\infty}n^{k-1}\left(\frac{q_n}{Q_n}\right)^k\frac{1}{Q_{n-1}} &= O(1)(v+1)\sum_{n=v+1}^{\infty}n^{k-1}\left(\frac{1}{n+1}\right)^k\frac{1}{n}\\ &= O(1)(v+1)\sum_{n=v+1}^{\infty}\frac{O(1)}{n(n+1)} = O(1).\end{aligned}$$

□

COROLLARY 12. *Let* $\{p_n\}, \{q_n\}$ *be positive sequences satisfying*

$$(i) \quad \frac{q_n P_n}{Q_n p_n} = O(1)$$

*and*

$$(ii) \quad \sum_{n=\nu+1}^{\infty} n^{k-1}\Big(\frac{q_n}{Q_n}\Big)^{k-1}\frac{1}{Qn} = O\Big(\frac{(\nu q_\nu)^{k-1}}{Q_\nu^k}\Big).$$

*Then, if* $\sum a_n$ *is summable* $|\overline{N}, p_n|_k$ *it is summable* $|\overline{N}, q_n|_k, k \geqslant 1$.

*Proof.* In Theorem 2 set $B = (\overline{N}, p_n)$. Then conditions $(i)$ and $(iii)$ of Theorem 2 become conditions $(i)$ and $(ii)$ of Corollary 12, respectively.

As in the proof of Corollary 1, conditions $(ii)$, $(iv)$ and $(v)$ of Theorem 2 are automatically satisfied. □

Corollary 12 can also be proved using Corollary 2, but with more effort, since more conditions need to be verified.

Corollary 12 is the sufficiency part of the theorem of [1].

COROLLARY 13. *Let* $\{q_n\}$ *be a positive sequence satisfying*

$$(i) \quad \frac{nq_n}{Q_n} = O(1)$$

*and*

$$(ii) \quad \sum_{n=\nu+1}^{\infty} n^{k-1}\Big(\frac{q_n}{Qn}\Big)^{k-1}\frac{1}{Q_{n-1}} = O\Big(\frac{(\nu q_\nu)^{k-1}}{Q_\nu^k}\Big).$$

*Then, if* $\sum a_n$ *is summable* $|C,1|_k$ *it is summable* $|\overline{N}, q_n|_k,\ k \geqslant 1$.

Corollary 13 follows immediately from Corollary 12 by setting each $p_n = 1$.

COROLLARY 14. *Let* $\{p_n\}$ *be a positive sequence satisfying*

$$\frac{P_n}{np_n} = O(1).$$

*Then, if* $\sum a_n$ *is summable* $|\overline{N}, p_n|_k$, *it is summable* $|C,1|_k,\ k \geqslant 1$.

*Proof.* In Corollary 12, set each $q_n = 1$ to obtain the above condition, and observe that condition $(ii)$ of Corollary 12 is automatically satisfied. □

Combining Corollaries 13 and 14 we have the following.

COROLLARY 15. *If* $\{p_n\}$ *is a positive sequence satisfying*

$$\frac{np_n}{P_n} = O(1) \quad \text{and} \quad \frac{P_n}{np_n} = O(1),$$

*then* $|C,1|_k$ *and* $|\overline{N}, p_n|_k$ *are equivalent for* $k \geqslant 1$.

A series will be called k-absolutely convergent if

$$\sum n^{k-1}|a_n|^k < \infty.$$

Using some of the corollaries of Theorem 1, and Theorem 2, it is possible to determine a number of matrices which sum all k-absolutely convergent series.

COROLLARY 16. *Every Rhaly generalized Cesàro matrix sums all k-absolutely convergent series.*

*Proof.* In Corollary 3 set $B = I$. Then all of the conditions are automatically satisfied. □

COROLLARY 17. *Every Rhaly p-Cesàro matrix sums all k-absolutely convergent series.*

*Proof.* Set $B = I$ in Corollary 6. Then all of the conditions are automatically satisfied. □

COROLLARY 18. *Let $(\overline{N}, p_n)$ be a weighted mean matrix with nonnegative weights and $p_0 > 0$. Then, if $\{p_n\}$ satisfies*

$$\sum_{n=\nu+1}^{\infty} \frac{n^{k-1}}{P_{n-1}} \left(\frac{p_n}{P_n}\right)^k = O\left(\frac{(\nu p_\nu)^{k-1}}{P_\nu^k}\right),$$

*then $(\overline{N}, p_n)$ sums every k-absolutely convergent series.*

*Proof.* Substitute $B = I$ in Theorem 2. Then conditions $(i)$, $(ii)$, $(iv)$, and $(v)$ are automatically satisfied, and condition $(iii)$ is the condition of this corollary. □

We shall now determine some weighted mean matrices that satisfy the condition of Corollary 18.

COROLLARY 19. *Let $p_n$ satisfy*

$$\frac{np_n}{P_n} = O(1) \quad \textit{and} \quad \frac{P_n}{np_n} = O(1).$$

*Then $(\overline{N}, p_n)$ sums every k-absolutely convergent series.*

*Proof.*

$$\begin{aligned}\frac{P_\nu^k}{(\nu p_\nu)^{k-1}} \sum_{n=\nu+1}^{\infty} \frac{n^{k-1}}{P_{n-1}} \left(\frac{p_n}{P_n}\right)^k &= O(1) P_\nu \sum_{n=\nu+1}^{\infty} \left(\frac{np_n}{P_n}\right)^{k-1} \frac{p_n}{P_n P_{n-1}} \\ &= O(1) P_\nu \sum_{n=\nu+1}^{\infty} O(1) \left(\frac{1}{P_{n-1}} - \frac{1}{P_n}\right) \\ &= O(1),\end{aligned}$$

and the condition of Corollary 18 is satisfied. □

In particular, $(C, 1)$ sums every k-absolutely convergent sequence.

COROLLARY 20. *Let $\{p_n\}$ be nonincreasing with limit $\sigma > 0$. Then $(\overline{N}, p_n)$ sums every k-absolutely convergent series.*

*Proof.* Note that $P_n \geqslant (n+1)p_n$ and $P_n \leqslant (n+1)p_0$. Thus

$$\begin{aligned}\frac{P_\nu^k}{(\nu p_\nu)^{k-1}} \sum_{n=\nu+1}^{\infty} \frac{n^{k-1}}{P_{n-1}} \left(\frac{p_n}{P_n}\right)^k &= \frac{P_\nu^k}{(\nu p_\nu)^{k-1}} \sum_{n=\nu+1}^{\infty} O(1) \frac{p_n}{P_n P_{n-1}} \\ &= O(1)\left(\frac{P_\nu}{\nu p_\nu}\right)^{k-1} = O(1)\left(\frac{p_0}{p_\nu}\right)^{k-1} \\ &= O(1),\end{aligned}$$

Since $p_\nu \geqslant \sigma > 0$. □

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