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GENERAL INCLUSION RELATIONS FOR ABSOLUTE SUMMABILITY

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Abstract. We obtain sufficient conditions for the series $\sum a_n$, which is absolutely summable of order k by a triangular matrix method A, $1 < k \leqslant s < \infty$, to be such that $\sum a_n$ is absolutely summable of order s by a triangular matrix B. As corollaries, we obtain a number of inclusion theorems.

In a recent paper, the author [1] obtained necessary conditions for a series summable $|A_k|$, $1 < k \leqslant s < \infty$, to imply that the series is summable $|B_s|$ where A and B are lower triangular matrices. In this paper we obtain sufficient conditions for a series summable $|A_k|$, $1 < k \leqslant s < \infty$, to imply that the series is summable $|B_s|$. Using these results we obtain a number of corollaries.

Let T be a lower triangular matrix, $\{s_n\}$ a sequence.

Then

$$T_n := \sum_{\nu=0}^{n} t_{n\nu} s_\nu .$$

A series $\sum a_n$ is said to be summable $|T|_k, k \geqslant 1$ if

$$\sum_{n=1}^{\infty} n^{k-1} |T_n - T_{n-1}|^k < \infty. \tag{1}$$

We may associate with T two lower triangular matrices $\overline{T}$ and $\hat{T}$ as follows:

$$\bar{t}_{n\nu} = \sum_{r=\nu}^{n} t_{nr}, \qquad n, \nu = 0, 1, 2, \ldots,$$

and

$$\hat{t}_{n\nu} = \bar{t}_{n\nu} - \bar{t}_{n-1,\nu}, \qquad n = 1, 2, 3, \ldots .$$

With $s_n := \sum_{i=0}^{n} a_i$.

$$y_n := \sum_{i=0}^{n} t_{ni} s_i = \sum_{i=0}^{n} t_{ni} \sum_{\nu=0}^{i} a_\nu = \sum_{\nu=0}^{n} a_\nu \sum_{i=\nu}^{n} t_{ni} = \sum_{\nu=0}^{n} \bar{t}_{n\nu} a_\nu$$

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and

$$Y_n := y_n - y_{n-1} = \sum_{\nu=0}^{n} (\bar{t}_{n\nu} - \bar{t}_{n-1,\nu} a_\nu) = \sum_{\nu=0}^{n} \hat{t}_{n\nu} a_\nu. \tag{2}$$

We shall call T a triangle if T is lower triangular and $t_{nn} \neq 0$ for each n. The notation $\Delta_\nu \hat{a}_{n\nu}$ means $\hat{a}_{n\nu} - \hat{a}_{n,\nu+1}$.

THEOREM 1. *Let $1 < k \leqslant s < \infty$. Let A and B be triangles satisfying*

(i) $\dfrac{|b_{nn}|}{|a_{nn}|} = O\left(\nu^{1/s-1/k}\right)$,

(ii) $(n|X_n|)^{s-k} = O(1)$,

(iii) $|a_{nn} - a_{n+1,n}| = O(|a_{nn} a_{n+1,n+1}|)$,

(iv) $\sum_{\nu=0}^{n-1} |\Delta_\nu(\hat{b}_{n\nu})| = O(|b_{nn}|)$,

(v) $\sum_{n=\nu+1}^{\infty} (n|b_{nn}|)^{s-1} |\Delta_\nu(\hat{b}_{n\nu})| = O(\nu^{s-1}|b_{\nu\nu}|^s)$,

(vi) $\sum_{\nu=0}^{n-1} |b_{\nu\nu}||\hat{b}_{n,\nu+1}| = O(|b_{nn}|)$,

(vii) $\sum_{n=\nu+1}^{\infty} (n|b_{nn}|)^{s-1} |\hat{b}_{n,\nu+1}| = O((\nu|b_{\nu\nu}|)^{s-1})$,

and

$(viii)$ $\sum_{n=1}^{\infty} n^{s-1} \left| \sum_{\nu=2}^{n} \hat{b}_{n\nu} \sum_{i=0}^{\nu-2} \hat{a}'_{\nu i} X_i \right|^s = O(1)$.

Then if $\sum a_n$ is summable $|A|_k$, it is summable $|B|_s$.

Proof. If y_n denotes the n-th term of the B-transform of a sequence $\{s_n\}$, then

$$y_n = \sum_{i=0}^{n} b_{ni} s_i = \sum_{i=0}^{n} b_{ni} \sum_{\nu=0}^{i} a_\nu = \sum_{\nu=0}^{n} a_\nu \sum_{i=\nu}^{n} b_{ni} = \sum_{\nu=0}^{n} \bar{b}_{n\nu} a_\nu.$$

$$y_{n-1} = \sum_{\nu=0}^{n-1} \bar{b}_{n-1,\nu} a_\nu.$$

$$Y_n := y_n - y_{n-1} = \sum_{\nu=0}^{n} \hat{b}_{n\nu} a_\nu, \tag{3}$$

where $s_n = \sum_{i=0}^{n} a_i$. Let x_n denote the n-th term of the A-transform of a series $\sum a_n$, then as in (3)

$$X_n := x_n - x_{n-1} = \sum_{\nu=0}^{n} \hat{a}_{n\nu} a_\nu. \tag{4}$$

Since $\hat{A}$ is a triangle, it has a unique two-sided inverse, which we shall denote by A'. Thus we may solve (4) for a_n to obtain

$$a_n = \sum_{\nu=0}^{n} \hat{a}'_{n\nu} X_\nu. \tag{5}$$

Substituting (5) into (3) yields

$$\begin{aligned}
Y_n &= \sum_{\nu=0}^{n} \hat{b}_{n\nu} a_\nu = \sum_{\nu=0}^{n} \hat{b}_{n\nu} \Big(\sum_{i=0}^{\nu} \hat{a}'_{\nu i} X_i \Big) \\
&= \sum_{\nu=0}^{n} \hat{b}_{n\nu} \Big(\sum_{i=0}^{\nu-2} \hat{a}'_{\nu i} X_i + \hat{a}'_{\nu,\nu-1} X_{\nu-1} + \hat{a}'_{\nu\nu} X_\nu \Big) \\
&= \sum_{\nu=0}^{n} \hat{b}_{n\nu} \hat{a}'_{\nu\nu} X_\nu + \sum_{\nu=1}^{n} \hat{b}_{n\nu} \hat{a}'_{\nu,\nu-1} X_{\nu-1} + \sum_{\nu=2}^{n} \hat{b}_{n\nu} \sum_{i=0}^{\nu-2} \hat{a}'_{\nu i} X_i \\
&= \hat{b}_{nn} \hat{a}'_{nn} X_n + \sum_{\nu=0}^{n-1} \hat{b}_{n\nu} \hat{a}'_{\nu\nu} X_\nu + \sum_{\nu=0}^{n-1} \hat{b}_{n,\nu+1} \hat{a}'_{\nu+1,\nu} X_\nu + \sum_{\nu=2}^{n} \hat{b}_{n\nu} \sum_{i=0}^{\nu-2} \hat{a}'_{\nu i} X_i \\
&= \frac{b_{nn}}{a_{nn}} X_n + \sum_{\nu=0}^{n-1} (\hat{b}_{n\nu} a'_{\nu\nu} + \hat{b}_{n,\nu+1} \hat{a}'_{\nu+1,\nu}) X_\nu + \sum_{\nu=2}^{n} \hat{b}_{n\nu} \sum_{i=0}^{\nu-2} \hat{a}'_{\nu i} X_i \\
&= \frac{b_{nn}}{a_{nn}} X_n + \sum_{\nu=0}^{n-1} (\hat{b}_{n\nu} a'_{\nu\nu} + \hat{b}_{n,\nu+1} a'_{\nu\nu} - \hat{b}_{n,\nu+1} a'_{\nu\nu} \\
&\qquad + \hat{b}_{n,\nu+1} \hat{a}'_{\nu+1,\nu}) X_\nu + \sum_{\nu=2}^{n} \hat{b}_{n\nu} \sum_{i=0}^{\nu-2} \hat{a}_{\nu i} X_i \\
&= \frac{b_{nn}}{a_{nn}} X_n + \sum_{\nu=0}^{n-1} \frac{\Delta_\nu(\hat{b}_{n\nu})}{a_{\nu\nu}} X_\nu + \sum_{\nu=0}^{n-1} \hat{b}_{n,\nu+1} (a'_{\nu\nu} + \hat{a}'_{\nu+1,\nu}) X_\nu + \sum_{\nu=2}^{n} \hat{b}_{n\nu} \sum_{i=0}^{\nu-2} \hat{a}'_{\nu i} X_i.
\end{aligned}$$

Using the fact that

$$a'_{\nu\nu} + \hat{a}'_{\nu+1,\nu} = \frac{1}{a_{\nu\nu}} \Big(\frac{a_{\nu\nu} - a_{\nu+1,\nu}}{a_{\nu+1,\nu+1}} \Big), \tag{6}$$

and substituting (7) into (6), we have the following

$$\begin{aligned}
Y_n &= \frac{b_{nn}}{a_{nn}} X_n + \sum_{\nu=0}^{n-1} \frac{\Delta_\nu(\hat{b}_{n\nu})}{a_{\nu\nu}} X_\nu \\
&\quad + \sum_{\nu=0}^{n-1} \hat{b}_{n,\nu+1} \Big(\frac{a_{\nu\nu} - a_{\nu+1,\nu}}{a_{\nu\nu} a_{\nu+1,\nu+1}} \Big) X_\nu + \sum_{\nu=2}^{n} \hat{b}_{n\nu} \sum_{i=0}^{\nu-2} \hat{a}'_{\nu i} X_i \\
&= T_{n1} + T_{n2} + T_{n3} + T_{n4}, \qquad \text{say .}
\end{aligned}$$

By Minkowski's inequality it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{s-1} |T_{ni}|^s < \infty, \qquad i = 1, 2, 3, 4.$$

Using (i)

$$\begin{aligned}J_1 := \sum_{n=1}^{\infty} n^{s-1}|T_{n1}|^s &= \sum_{n=1}^{\infty} n^{s-1}\left|\frac{b_{nn}}{a_{nn}}X_n\right|^s \\ &= O(1)\sum_{n=1}^{\infty} n^{s-1}(n^{1/s-1/k})^s\left|X_n\right|^s \\ &= O(1)\sum_{n=1}^{\infty} n^{k-1}|X_n|^k\left(n^{s-s/k-k+1}|X_n|^{s-k}\right).\end{aligned}$$

But $n^{s-s/k-k+1}|X_n|^{s-k} = \left(n^{1-1/k}|X_n|\right)^{s-k} = O\Big((n|X_n|)^{s-k}\Big) = O(1)$, from (ii) of Theorem. Since $\sum a_n$ is summable $|A|_k$, $J_1 = O(1)$. Using (i), (ii), (iv), (v) and Hölder's inequality,

$$\begin{aligned}J_2 := \sum_{n=1}^{\infty} n^{s-1}|T_{n2}|^s &= \sum_{n=1}^{\infty} n^{s-1}\left|\sum_{\nu=0}^{n-1}\frac{\Delta_\nu(\hat{b}_{n\nu})}{a_{\nu\nu}}X_\nu\right|^s \\ &\leqslant \sum_{n=1}^{\infty} n^{s-1}\Big\{\sum_{\nu=0}^{n-1}\nu^{1/s-1/k}|b_{\nu\nu}|^{-1}|\Delta_\nu(\hat{b}_{n\nu})||X_\nu|\Big\}^s \\ &= O(1)\sum_{n=1}^{\infty} n^{s-1}\Big(\sum_{\nu=0}^{n-1}\nu^{1-s/k}|b_{\nu\nu}|^{-s}|\Delta_\nu(\hat{b}_{n\nu})||X_\nu|^s\Big)\Big(\sum_{\nu=0}^{n-1}|\Delta_\nu(\hat{b}_{n\nu})|\Big)^{s-1} \\ &= O(1)\sum_{n=1}^{\infty}(n|b_{nn}|)^{s-1}\sum_{\nu=0}^{n-1}\nu^{1-s/k}|b_{\nu\nu}|^{-s}|\Delta_\nu(\hat{b}_{n\nu})||X_\nu|^s \\ &= O(1)\sum_{\nu=1}^{\infty}\nu^{1-s/k}|b_{\nu\nu}|^{-s}|X_\nu|^s\sum_{n=\nu+1}^{\infty}(n|b_{nn})^{s-1}|\Delta_\nu(\hat{b}_{n\nu})| \\ &= O(1)\sum_{\nu=1}^{\infty}\nu^{1-s/k}|b_{\nu\nu}|^{-s}|X_\nu|^s\nu^{s-1}|b_{\nu\nu}|^s = O(1)\sum_{\nu=1}^{\infty}\nu^{s-s/k}|X_\nu|^s \\ &= O(1)\sum_{\nu=1}^{\infty}\nu^{k-1}|X_\nu|^k(\nu^{s-s/k-k+1}|X_\nu|^{s-k}) \\ &= O(1)\sum_{\nu=1}^{\infty}\nu^{k-1}|X_\nu|^k = O(1).\end{aligned}$$

Using (ii), (iii), (vi) and (vii) and Hölder's inequality,

$$\begin{aligned}J_3 := \sum_{n=1}^{\infty} n^{s-1}|T_{n3}|^s &= \sum_{n=1}^{\infty} n^{s-1}\left|\sum_{\nu=0}^{n-1}\hat{b}_{n,\nu+1}\Big(\frac{a_{\nu\nu}-a_{\nu+1,\nu}}{a_{\nu\nu}a_{\nu+1,\nu+1}}\Big)X_\nu\right|^s \\ &\leqslant \sum_{n=1}^{\infty} n^{s-1}\Big(\sum_{\nu=0}^{n-1}|\hat{b}_{n,\nu+1}|\left|\frac{a_{\nu\nu}-a_{\nu+1,\nu}}{a_{\nu\nu}a_{\nu+1,\nu+1}}\right||X_\nu|\Big)^s\end{aligned}$$

$$= O(1)\sum_{n=1}^{\infty} n^{s-1}\Big(\sum_{\nu=0}^{n-1} |\hat{b}_{n,\nu+1}||X_\nu|\Big)^{s}$$

$$= O(1)\sum_{n=1}^{\infty} n^{s-1}\Big(\sum_{\nu=0}^{n-1} \Big(\frac{|b_{\nu\nu}|}{|b_{\nu\nu}|}\Big)|\hat{b}_{n,\nu+1}||X_\nu|\Big)^{s}$$

$$= O(1)\sum_{n=1}^{\infty} n^{s-1}\Big(\sum_{\nu=0}^{n-1} |b_{\nu\nu}|^{1-s}|\hat{b}_{n,\nu+1}||X_\nu|^{s}\Big)\Big(\sum_{\nu=0}^{n-1} |b_{\nu\nu}|\hat{b}_{n,\nu+1}|\Big)^{s-1}$$

$$= O(1)\sum_{n=1}^{\infty}(n|b_{nn}|)^{s-1}\sum_{\nu=0}^{n-1} |b_{\nu\nu}|^{1-s}|\hat{b}_{n,\nu+1}||X_\nu^{s}$$

$$= O(1)\sum_{\nu=0}^{\infty} |b_{\nu\nu}|^{1-s}||X_\nu|^{s}\sum_{n=\nu+1}^{\infty}(n|b_{nn})^{s-1}|\hat{b}_{n,\nu+1}|$$

$$= O(1)\sum_{\nu=0}^{\infty} |b_{\nu\nu}|^{1-s}||X_\nu|^{s}\nu^{s-1}|b_{\nu\nu}|^{s-1} = O(1)\sum_{\nu=0}^{\infty}\nu^{s-1}|X_\nu|^{s}$$

$$= O(1)\sum_{\nu=0}^{\infty}\nu^{k-1}|X_\nu|^{k}(\nu|X_\nu|)^{s-k} = O(1)\sum_{\nu=0}^{\infty}\nu^{k-1}|X_\nu|^{k} = O(1).$$

From $(viii)$,

$$\sum_{n=1}^{\infty} n^{s-1}|T_{n4}|^{s} = \sum_{n=1}^{\infty} n^{s-1}\Big|\sum_{\nu=2}^{n}\hat{b}_{n\nu}\lambda_\nu\sum_{i=0}^{\nu-2}\hat{a}'_{\nu i}X_i\Big|^{s} = O(1). \quad \square$$

THEOREM 2. *Let A and B be triangles satisfying*

(i) $\dfrac{|b_{nn}|}{|a_{nn}|} = O\Big(1\Big)$,

(ii) $|a_{nn} - a_{n+1,n}| = O(|a_{nn}a_{n+1,n+1}|)$,

(iii) $\sum_{\nu=0}^{n-1}|\Delta_\nu(\hat{b}_{n\nu})| = O(|b_{nn}|)$,

(iv) $\sum_{n=\nu+1}^{\infty}(n|b_{nn}|)^{k-1}|\Delta_\nu(\hat{b}_{n\nu})| = O(\nu^{k-1}|b_{\nu\nu}|^{k})$,

(v) $\sum_{\nu=0}^{n-1}|b_{\nu\nu}||\hat{b}_{n,\nu+1}| = O(|b_{nn}|)$,

(vi) $\sum_{n=\nu+1}^{\infty}(n|b_{nn}|)^{k-1}|\hat{b}_{n,\nu+1}| = O((\nu|b_{\nu\nu}|)^{k-1})$,

and

(vii) $\sum_{n=1}^{\infty} n^{k-1}\Big|\sum_{\nu=2}^{n}\hat{b}_{n\nu}\sum_{i=0}^{\nu-2}\hat{a}'_{\nu i}X_i\Big|^{k} = O(1)$.

Then, if $\sum a_n$ is summable $|A|_k$, it is summable $|B|_k$.

To prove Theorem 2, simply set s=k in Theorem 1.

COROLLARY 1. *[5] Let be $\{p_n\}$ a sequence of positive constants, B a triangle satisfying*

$$(i)\quad P_n|b_{nn}| = O\Big((p_n)\nu^{1/s-1/k}\Big),$$

$$(ii)\quad (n|X_n|)^{s-k} = O(1),$$

$$(iii)\quad \sum_{\nu=0}^{n-1} |\Delta_\nu(\hat{b}_{n\nu})| = O(|b_{nn}|),$$

$$(iv)\quad \sum_{n=\nu+1}^{\infty} (n|b_{nn})^{s-1}|\Delta_\nu(\hat{b}_{n\nu})| = O(\nu^{s-1}|b_{\nu\nu}|^s),$$

$$(v)\quad \sum_{\nu=0}^{n-1} |b_{\nu\nu}\hat{b}_{n,\nu+1}| = O(|b_{nn}|),$$

$$(vi)\quad \sum_{n=\nu+1}^{\infty} (n|b_{nn}|)^{s-1}|\hat{b}_{n,\nu+1}| = O((\nu|b_{\nu\nu}|)^{s-1}).$$

Then if $\sum a_n$ is summable $|\overline{N},p_n|_k$, it is summable $|B|_s$.

COROLLARY 2. *Let be $\{p_n\}$ a sequence of positive constants, A a triangle satisfying*

$$(i)\quad p_n/(P_n|a_{nn}|) = O(\nu^{1/s-1/k}),$$

$$(ii)\quad (n|X_n|)^{s-k} = O(1),$$

$$(iii)\quad |a_{nn} - a_{n+1,n}| = O(|a_{nn}a_{n+1,n+1}|),$$

$$(iv)\quad \sum_{\nu=0}^{n-1} |\Delta_\nu(P_{\nu-1})| = O(P_{n-1}),$$

$$(v)\quad |\Delta_\nu(P_{\nu-1})| \sum_{n=\nu+1}^{\infty} \left(\frac{np_n}{P_n}\right)^{s-1} \frac{p_n}{P_nP_{n-1}} = O\left(\nu^{s-1}\left(\frac{p_\nu}{P_\nu}\right)^s\right),$$

$$(vi)\quad \sum_{\nu=0}^{n-1} p_\nu = O(P_{n-1}),$$

$$(vii)\quad \sum_{n=\nu+1}^{\infty} n^{s-1}\left(\frac{np_n}{P_n}\right)^{s-1} \frac{p_n}{P_nP_{n-1}} = O\left(\frac{(\nu p_\nu)^{s-1}}{P_\nu^s}\right),$$

and

$$(viii)\quad \sum_{n=\nu+1}^{\infty} n^{s-1}\left(\frac{p_n}{P_nP_{n-1}}\right)^s \left|\sum_{\nu=2}^{n} P_{\nu-1} \sum_{i=0}^{\nu-2} \hat{a}'_{\nu i}X_i\right|^s = O(1).$$

Then if $\sum a_n$ is summable $|A|_k$ it is summable $|\overline{N},p_n|_s$,

Proof. With $B = (\overline{N},p_n)$, conditions (i) - $(viii)$ of Theorem 1 reduce to conditions (i) - $(viii)$, respectively of Corollary 2, since

$$\hat{b}_{n\nu} = \frac{p_nP_{\nu-1}}{P_nP_{n-1}}. \quad \square$$

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