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KANTOROVICH TYPE REVERSE INEQUALITIES FOR OPERATOR NORM

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Abstract. In this paper, we shall extend Bourin's theorem for unitarily invariant norm in the framework of operator theory on a Hilbert space by applying the Mond-Pečarić method for convex functions. Moreover we obtain the operator norm version. Among others, we show that if A and Z are positive operators on a Hilbert space H such that $0 < mI \leqslant Z \leqslant MI$ for some scalars $0 < m < M$, then for each $\alpha > 0$

$$\|(AZ^pA)^{\frac{1}{p}}\| \leqslant \alpha\, r(ZA^{\frac{2}{p}}) + \beta(m,M,p,\alpha)\|A\|^{\frac{2}{p}} \qquad \text{for all } p > 1$$

for some suitable constant $\beta(m,M,p,\alpha)$, where $\|\cdot\|$ is the operator norm and $r(\cdot)$ is the spectral radius.

1. Introduction

Let A and B be two $n \times n$ matrices. The spectral radii of AB and BA are equal. If the product AB is normal, then we have

$$|||AB||| \leqslant |||BA|||$$

for every unitarily invariant norm.

In [3], Bourin showed the following reverse inequality for unitarily invariant norm under a general setting:

THEOREM A. *Let A, B and Z be $n \times n$ matrices. Suppose that AB is positive semi-definite and Z is positive definite such that $0 < m1_n \leqslant Z \leqslant M1_n$ for some scalars $0 < m < M$. Then*

$$|||ZAB||| \leqslant \frac{M+m}{2\sqrt{Mm}}|||BZA|||$$

for every unitarily invariant norm $|||\cdot|||$.

We recall the celebrated Kantorovich inequality [8, 11]: If Z is positive definite such that $0 < m1_n \leqslant Z \leqslant M1_n$ for some scalars $0 < m < M$, then $(Z^{-1}x,x)(Zx,x) \leqslant \frac{(M+m)^2}{4Mm}$ for every unit vector $x \in H$. We call the constant $\frac{(M+m)^2}{4Mm}$ the Kantorovich constant. We here cite Furuta's textbook [6] as a pertinent reference to Kantorovich

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inequalities. It follows that the constant in Theorem A is just the square root of the Kantorovich constant.

As an application, he showed the following reverse inequality of the well-known inequality $r(A) \leqslant \|A\|$, where $r(\cdot)$ is the spectral radius and $\|\cdot\|$ is the operator norm.

THEOREM B. *If A is positive semidefinite and Z is positive definite such that $0 < m1_n \leqslant Z \leqslant M1_n$ for some scalars $0 < m < M$, then*

$$\|ZA\| \leqslant \frac{M+m}{2\sqrt{Mm}}\, r(ZA).$$

In this note, we shall extend Theorem B due to Bourin in the framework of operator theory on a Hilbert space by applying the Mond-Pečarić method for convex functions [10, 12]. Moreover we show the operator norm version of Theorem A. In particular, we obtain the following difference version which is parallel to Theorem B: If A and Z are positive operators on a Hilbert space H such that $0 < mI \leqslant Z \leqslant MI$ for some scalars $0 < m < M$, then

$$\|ZA\| - r(ZA) \leqslant \frac{(M-m)^2}{4(M+m)}\|A\|.$$

2. Preliminary

Let Z be a positive operator on a Hilbert space H and x a unit vector in H. By Jensen's inequality, we have the relation between the continuous power mean and the continuous arithmetic one :

$$(Zx,x) \leqslant (Z^p x,x)^{\frac{1}{p}} \qquad \text{for all } p > 1\,. \tag{1}$$

By using the Mond-Pečarić method, we have the following reverse inequality of (1) [13]:

LEMMA 1. *If Z is a positive operator on H such that $0 < mI \leqslant Z \leqslant MI$ for some scalars $0 < m < M$, then for each $\alpha > 0$*

$$(Z^p x,x)^{\frac{1}{p}} \leqslant \alpha(Zx,x) + \beta(m,M,p,\alpha) \qquad \textit{for all } p > 1 \tag{2}$$

holds for every unit vector $x \in H$, where

$$\beta(m,M,p,\alpha) = \begin{cases} \dfrac{p-1}{p}\left(\dfrac{M^p - m^p}{\alpha p(M-m)}\right)^{\frac{1}{p-1}} & \textit{if} \quad \dfrac{M^p-m^p}{pM^{p-1}(M-m)} \leqslant \alpha \\ \qquad + \dfrac{\alpha(Mm^p - mM^p)}{M^p - m^p} & \qquad\qquad \leqslant \dfrac{M^p-m^p}{pm^{p-1}(M-m)}, \\ (1-\alpha)M & \textit{if} \quad 0 < \alpha \leqslant \dfrac{M^p - m^p}{pM^{p-1}(M-m)}, \\ (1-\alpha)m & \textit{if} \quad \alpha \geqslant \dfrac{M^p - m^p}{pm^{p-1}(M-m)}. \end{cases} \tag{3}$$

Proof. For the sake of reader's convenience, we give a proof. Put $\beta = \beta(m,M,p,\alpha)$ and $f(t) = (at+b)^{\frac{1}{p}} - \alpha t$ for $a = \frac{M^p-m^p}{M-m}$ and $b = \frac{Mm^p-mM^p}{M-m}$, then we have $f'(t) = \frac{a}{p}(at+b)^{\frac{1}{p}-1} - \alpha$. It follows that the equation $f'(t) = 0$ has exactly one solution $t_0 = \frac{1}{a}(\frac{\alpha p}{a})^{\frac{p}{1-p}} - \frac{b}{a}$. If $m \leqslant t_0 \leqslant M$, then we have $\beta = \max_{m\leqslant t\leqslant M} f(t) = f(t_0)$ since $f''(t) = \frac{a^2(1-p)}{p^2}(at+b)^{\frac{1}{p}-2} < 0$ and the condition $m \leqslant t_0 \leqslant M$ is equivalent to the condition $\frac{M^p-m^p}{pM^{p-1}(M-m)} \leqslant \alpha \leqslant \frac{M^p-m^p}{pm^{p-1}(M-m)}$. If $M \leqslant t_0$, then $f(t)$ is increasing on $[m,M]$ and hence we have $\beta = \max_{m\leqslant t\leqslant M} f(t) = f(t_0) = (1-\alpha)M$ for $t_0 = M$. Similarly, we have $\beta = \max_{m\leqslant t\leqslant M} f(t) = f(t_0) = (1-\alpha)m$ for $t_0 = m$ if $t_0 \leqslant m$. Hence it follows that

$$(at+b)^{\frac{1}{p}} - \alpha t \leqslant \beta \qquad \text{for all } t \in [m,M].$$

Since t^p is convex for $p > 1$, it follows that $t^p \leqslant at+b$ for $t \in [m,M]$. By the spectral theorem, we have $Z^p \leqslant aZ+b$ and hence $(Z^p x,x) \leqslant a(Zx,x)+b$ for every unit vector $x \in H$. Therefore we have

$$\begin{aligned}(Z^p x,x)^{\frac{1}{p}} - \alpha(Zx,x) &\leqslant (a(Zx,x)+b)^{\frac{1}{p}} - \alpha(Zx,x)\\ &\leqslant \max_{m\leqslant t\leqslant M} f(t) = \beta(m,M,p,\alpha).\end{aligned}$$

By Lemma 1, we have the following estimates of both the difference and the ratio between the continuous power mean and the continuous arithmetic one:

LEMMA 2. *If Z is a positive operator on H such that $0 < mI \leqslant Z \leqslant MI$ for some scalars $0 < m < M$, then for each $p > 1$*

$$(Z^p x,x)^{\frac{1}{p}} \leqslant K(m,M,p)^{\frac{1}{p}}(Zx,x) \tag{4}$$

and

$$(Z^p x,x)^{\frac{1}{p}} - (Zx,x) \leqslant -C(m^p,M^p,\frac{1}{p}) \tag{5}$$

hold for every unit vector $x \in H$, where a generalized Kantorovich constant $K(m,M,p)$ [4, 5, 7] is defined as

$$K(m,M,p) = \frac{mM^p - Mm^p}{(p-1)(M-m)}\left(\frac{p-1}{p}\,\frac{M^p-m^p}{mM^p-Mm^p}\right)^p$$

and $C(m,M,p)$ [9, 14] is defined as

$$C(m,M,p) = (p-1)\left(\frac{M^p-m^p}{p(M-m)}\right)^{\frac{p}{p-1}} + \frac{Mm^p-mM^p}{M-m}.$$

Proof. If we choose α such that $\beta(m,M,p,\alpha) = 0$ in Lemma 1, then we have $\alpha = K(m,M,p)^{\frac{1}{p}}$. If we put $\alpha = 1$ in Lemma 1, then we have $\beta(m,M,p,1) = -C(m^p,M^p,\frac{1}{p})$.

We remark that $K(m,M,2)$ coincides with the Kantorovich constant $\frac{(M+m)^2}{4Mm}$ if $p = 2$.

3. Reverse inequality for operator norm

Firstly, we state our main theorem, which is a generalization of Theorem B.

THEOREM 1. *If A and Z are positive operators on H such that $0 < mI \leqslant Z \leqslant MI$ for some scalars $0 < m < M$, then for each $\alpha > 0$*

$$\|(AZ^pA)^{\frac{1}{p}}\| \leqslant \alpha\, r(ZA^{\frac{2}{p}}) + \beta(m,M,p,\alpha)\|A\|^{\frac{2}{p}} \qquad \text{for all } p > 1,$$

where $\beta(m,M,p,\alpha)$ is defined by (3).

Proof. For every unit vector $x \in H$, it follows that

$$\begin{aligned}
((AZ^pA)^{\frac{1}{p}}x,x) &\leqslant (AZ^pAx,x)^{\frac{1}{p}} \quad \text{by Hölder-McCarthy inequality and } 0 < \tfrac{1}{p} < 1\\
&= \left(Z^p\frac{Ax}{\|Ax\|},\frac{Ax}{\|Ax\|}\right)^{\frac{1}{p}}\|Ax\|^{\frac{2}{p}}\\
&\leqslant \left(\alpha\left(Z\frac{Ax}{\|Ax\|},\frac{Ax}{\|Ax\|}\right)+\beta(m,M,p,\alpha)\right)\|Ax\|^{\frac{2}{p}} \quad \text{by Lemma 1}\\
&= \alpha(ZAx,Ax)\|Ax\|^{\frac{2}{p}-2} + \beta(m,M,p,\alpha)\|Ax\|^{\frac{2}{p}}\\
&= \alpha\left(A^{\frac{1}{p}}ZA^{\frac{1}{p}}\frac{A^{1-\frac{1}{p}}x}{\|A^{1-\frac{1}{p}}x\|},\frac{A^{1-\frac{1}{p}}x}{\|A^{1-\frac{1}{p}}x\|}\right)\|Ax\|^{\frac{2}{p}-2}\|A^{1-\frac{1}{p}}x\|^2\\
&\qquad + \beta(m,M,p,\alpha)\|Ax\|^{\frac{2}{p}}
\end{aligned}$$

and

$$\begin{aligned}
\|Ax\|^{\frac{2}{p}-2}\|A^{1-\frac{1}{p}}x\|^2 &= (A^2x,x)^{\frac{1}{p}-1}(A^{2-\frac{2}{p}}x,x)\\
&\leqslant (A^2x,x)^{\frac{1}{p}-1}(A^2x,x)^{1-\frac{1}{p}} = 1 \quad \text{by } 0 < 1-\tfrac{1}{p} < 1.
\end{aligned}$$

By combining two inequalities above, we have

$$\begin{aligned}
((AZ^pA)^{\frac{1}{p}}x,x) &\leqslant \alpha\,\|A^{\frac{1}{p}}ZA^{\frac{1}{p}}\| + \beta(m,M,p,\alpha)\|Ax\|^{\frac{2}{p}}\\
&= \alpha\, r(A^{\frac{1}{p}}ZA^{\frac{1}{p}}) + \beta(m,M,p,\alpha)\|Ax\|^{\frac{2}{p}}\\
&\leqslant \alpha\, r(ZA^{\frac{2}{p}}) + \beta(m,M,p,\alpha)\|A\|^{\frac{2}{p}}
\end{aligned}$$

for every unit vector $x \in H$ and hence we have the desired inequality.

REMARK 1. If A and Z are positive operators, then it follows that

$$r(ZA^{\frac{2}{p}}) \leqslant \|(AZ^pA)^{\frac{1}{p}}\| \qquad \text{for all } p > 1. \tag{6}$$

As a matter of fact, by Araki's inequality [1, 2], we have

$$r(ZA^{\frac{2}{p}}) = r(A^{\frac{1}{p}}ZA^{\frac{1}{p}}) = \|A^{\frac{1}{p}}ZA^{\frac{1}{p}}\| \leqslant \|(AZ^pA)^{\frac{1}{p}}\|$$

for all $p > 1$. Therefore, Theorem 1 can be considered as a reverse inequality to the inequality (6).

The following theorem is a variant of Theorem 1 with 2-variables.

THEOREM 2. *If A and Z are positive operators on H such that $0 < mI \leqslant Z \leqslant MI$ for some scalars $0 < m < M$, then for each $\alpha > 0$*

$$\|(AZ^pA)^{\frac{1}{q}}\| \leqslant \alpha\, r(Z^{\frac{p}{q}}A^{\frac{2}{q}}) + \beta(m^{\frac{p}{q}}, M^{\frac{p}{q}}, q, \alpha)\|A\|^{\frac{2}{q}} \qquad \text{for all } p > 1 \text{ and } q > 1,$$

where $\beta(m,M,p,\alpha)$ is defined by (3).

Proof. For every unit vector $x \in H$, we have

$$\begin{aligned} ((AZ^pA)^{\frac{1}{q}}x,x) &\leqslant (AZ^pAx,x)^{\frac{1}{q}} \quad \text{by } 0 < \tfrac{1}{q} < 1 \\ &= \left((Z^{\frac{p}{q}})^q \frac{Ax}{\|Ax\|}, \frac{Ax}{\|Ax\|}\right)^{\frac{1}{q}} \|Ax\|^{\frac{2}{q}} \\ &\leqslant \left(\alpha \left(Z^{\frac{p}{q}} \frac{Ax}{\|Ax\|}, \frac{Ax}{\|Ax\|}\right) + \beta(m^{\frac{p}{q}}, M^{\frac{p}{q}}, q, \alpha)\right) \|Ax\|^{\frac{2}{q}}. \end{aligned}$$

The rest of the proof is proved in a similar way as the proof of Theorem 1.

THEOREM 3. *Let Z be a positive operator on H such that $0 < mI \leqslant Z \leqslant MI$ for some scalars $0 < m < M$. Then for each $p > 1$*

$$\|(AZ^pA)^{\frac{1}{p}}\| \leqslant K(m,M,p)^{\frac{1}{p}}\, r(ZA^{\frac{2}{p}}) \tag{7}$$

holds for every positive operator A on H.

In addition, (7) is equivalent to (4) in Lemma 2.

Proof. By using (4) of Lemma 2 instead of Lemma 1 in the proof of Theorem 1, we obtain (7). Conversely, for every unit vector $x \in H$, if we put $A = x \otimes x$ in (7), then we have (4) of Lemma 2.

We have the following corollary as a special case of (7) in Theorem 3, which is an operator version of Theorem B:

COROLLARY 1. *If A and Z are positive operators on H such that $0 < mI \leqslant Z \leqslant MI$ for some scalars $0 < m < M$, then*

$$\|ZA\| \leqslant \frac{M+m}{2\sqrt{Mm}}\, r(ZA).$$

Proof. If we put $p = 2$ in Theorem 3, then we have

$$\|(AZ^2A)^{\frac{1}{2}}\| \leqslant K(m,M,2)^{\frac{1}{2}}\, r(ZA).$$

Since $\|(AZ^2A)^{\frac{1}{2}}\| = \|(ZA)^*(ZA)\|^{\frac{1}{2}} = \|ZA\|$ and $K(m,M,2)^{\frac{1}{2}} = \left(\frac{(M+m)^2}{4Mm}\right)^{\frac{1}{2}} = \frac{M+m}{2\sqrt{Mm}}$, we have the desired inequality.

THEOREM 4. *Let Z be a positive operator on H such that $0 < mI \leqslant Z \leqslant MI$ for some scalars $0 < m < M$. Then for each $p > 1$*

$$\|(AZ^pA)^{\frac{1}{p}}\| \leqslant r(ZA^{\frac{2}{p}}) - C(m^p, M^p, \frac{1}{p})\|A\|^{\frac{2}{p}} \tag{8}$$

holds for every positive operator A on H.

In addition, (8) is equivalent to (5) in Lemma 2.

Proof. By using (5) of Lemma 2 instead of Lemma 1 in the proof of Theorem 1, we obtain (8). Conversely, for every unit vector $x \in H$, if we put $A = x \otimes x$ in (8), then we have (5) of Lemma 2.

We have the following corollary as a special case of (8) in Theorem 4.

COROLLARY 2. *If A and Z are positive operators on H such that $0 < mI \leqslant Z \leqslant MI$ for some scalars $0 < m < M$, then*

$$\|ZA\| - r(ZA) \leqslant \frac{(M-m)^2}{4(M+m)}\|A\|. \tag{9}$$

Proof. If we put $p = 2$ in Theorem 4, then we have (9) since $C(m^2, M^2, \frac{1}{2}) = \frac{(M-m)^2}{4(M+m)}$.

The following corollary is an operator norm version of Theorem A.

COROLLARY 3. *If A and B are two operators such that $AB \geqslant 0$ is positive and Z is a positive operator on H such that $0 < mI \leqslant Z \leqslant MI$ for some scalars $0 < m < M$, then*

$$\|ZAB\| \leqslant \frac{M+m}{2\sqrt{Mm}}\|BZA\|.$$

Proof. By Corollary 1 it follows that

$$\|ZAB\| \leqslant \frac{M+m}{2\sqrt{Mm}}\, r(ZAB) = \frac{M+m}{2\sqrt{Mm}}\, r(BZA) \leqslant \frac{M+m}{2\sqrt{Mm}}\, \|BZA\|.$$

REMARK 2. It follows that Corollary 3 implies Corollary 1. In fact, if we replace A by $A^{\frac{1}{2}}$ and B by $A^{\frac{1}{2}}$ in Corollary 3 respectively, then we have

$$\|ZA\| \leqslant \frac{M+m}{2\sqrt{Mm}}\|A^{\frac{1}{2}}ZA^{\frac{1}{2}}\| = \frac{M+m}{2\sqrt{Mm}}\, r(A^{\frac{1}{2}}ZA^{\frac{1}{2}}) = \frac{M+m}{2\sqrt{Mm}}\, r(ZA).$$

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REFERENCES

[1] H. ARAKI, *On an inequality of Lieb and Thirring*, Letters in Math. Phys., **19** (1990), 167–170.

[2] R. BHATIA, *Matrix Analysis*, Springer, New York, 1997.
[3] J.-C. BOURIN, *Compressions, Dilations and Matrix inequalities*, Monographs in Research Group in Math, Inequal. and Appl., 2004.
[4] J. I. FUJII, M. FUJII, Y. SEO AND M. TOMINAGA, *On generalized Kantorovich inequalities*, Proc. Int. Sym. on Banach and Function Spaces, Kitakyushu, Japan, October 2-4, (2003), 205–213.
[5] M. FUJII, S. IZUMINO, R. NAKAMOTO AND Y. SEO, *Operator inequalities related to Cauchy-Schwarz and Hölder-McCarthy inequalities*, Nihonkai Math. J., **8** (1997), 117–122.
[6] T. FURUTA, *Invitation to Linear Operators*, Taylor and Francis, London and New York, 2001.
[7] T. FURUTA, *Specht ratio* $S(1)$ *can be expressed by Kantorovich constant* $K(p)$ *:* $S(1) = \exp K'(1)$ *and its application*, Math. Inequal. Appl., **6**, 3 (2003), 521–530.
[8] W. GREUB AND W. RHEINBOLDT, *On a generalization of an inequality of L.V. Kantorovich*, Proc. Amer. Math. Soc., **10** (1959), 407–415.
[9] J. MIĆIĆ, Y. SEO, S.-E. TAKAHASI AND M. TOMINAGA, *Inequalities of Furuta and Mond-Pečarić*, Math. Inequal. Appl., **2**, 1 (1999), 83–111.
[10] B. MOND AND J. E. PEČARIĆ, *Convex inequalities in Hilbert space*, Houston J. Math., **19**, (1993), 405–420.
[11] M. NAKAMURA, *A remark on a paper of Greub and Rheiboldt*, Proc. Japan Acad., **36**, (1960), 198–199.
[12] J. E. PEČARIĆ, T. FURUTA, J. MIĆIĆ AND Y. SEO, *Mond-Pečarić Method in Operator Inequalities* 1, Element, Zagreb, 2005.
[13] M. TOMINAGA, *Reverse Estimations of Operator Inequalities with Convexity*, Thesis, 2002.
[14] T. YAMAZAKI, *An extension of Specht's theorem via Kantorovich inequality and related results*, Math. Inequal. Appl. **3**, 1 (2000), 89–96.

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