INEQUALITIES FOR A SIMPLEX AND ANY POINT

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Abstract. By applying an analytic inequality and the polar moment of inertia inequality in E^n, we establish some inequalities for the volume, facet areas and distances between any point of E^n and vertices of an $n-$ simplex

1. Introduction

Geometric inequalities for convex bodies have been a very attractive subject for a long time. Specially, the inequalities for simplices which are the simplest and the most useful polytopes have been studied extensively (see [3, 7, 11, 13, 17, 19]). A great number of elegant results have been obtained. The well-known monograph [12] of Mitrinović and Pečarić and Volenec had collected what were already many impressive results before 1989. Ali [1], Gerber [4], Petty and Waterman [14], Slepian[16] established a number of inequalities for the distances from any point to vertices and the volume of a simplex. In this paper, we establish some new inequalities for a simplex and any point, which combine the facet areas with the volume and the distances from any point to vertices of a simplex, which are motivated by the well-known Klamkin's inequality for polar moment of inertia. Some of own results are just the generalizations to several dimensions of the Klamkin's inequality.

Let Ω be an $n-$ simplex in $n-$ dimensional Euclidean Space E^n with vertices $A_0, A_1, \cdots A_n$ (i.e., $\Omega = \langle A_0, A_1, \cdots A_n \rangle$) with the volume V, and $\Omega_i = \langle A_0, \cdots, A_{i-1}, A_{i+1}, \cdots, A_n \rangle$ its facet which lies in a hyperplane π_i, F_i the facet area of Ω_i (i.e., $(n-1)-$ dimensional volume), $F = \sum_{i=0}^n F_i$, R_i the distance from a point P of E^n to A_i $(i = 0, 1, \cdots, n)$.

The aim of this paper is to establish some new inequalities combining R_i, F_i $(i = 0, 1, \cdots, n)$ with V. Our main results are the following theorems.

THEOREM 1. *Let Ω be an $n-$ simplex in E^n, P any point in E^n, $R_i = \|PA_i\|$, and $F = \sum_{i=0}^n F_i$. Then*

$$\sum_{i=0}^n F_i R_i^2 \geqslant C(n) \left(\prod_{i=0}^n F_i \right)^{\frac{1}{n-1}},$$ (1.1)

$$\sum_{i=0}^{n}(F - 2F_i)R_i^2 \geqslant (n-1)C(n)\left(\prod_{i=0}^{n} F_i\right)^{\frac{1}{n-1}}, \tag{1.2}$$

where $C(n) = n\left[\dfrac{n!^2}{n^3}\right]^{\frac{1}{n-1}}$ and equalities hold if Ω is a regular $n-$ simplex and P is the center of Ω.

REMARK 1. Taking $n = 2$ in (1.1), we obtain the well-known Klamkin's inequality.

THEOREM 2. *Let* Ω *be an* $n-$ *simplex in* E^n, P *any point in* E^n, $R_i = \|PA_i\|$, *and* $F = \sum_{i=0}^{n} F_i$, *then*

$$\sum_{i=0}^{n}(F_0^2 + \cdots + F_n^2 - 2F_i^2)R_i^2 \geqslant \frac{n-1}{n+1}n^4V^2, \tag{1.3}$$

$$\sum_{i=0}^{n} F_i(F - 2F_i)R_i^2 \geqslant \frac{n-1}{n+1}n^4V^2, \tag{1.4}$$

$$\sum_{i=0}^{n}(F - 2F_i)^2 R_i^2 \geqslant \frac{(n-1)^2}{n+1}n^4V^2, \tag{1.5}$$

and equalities hold if Ω *is a regular* $n-$ *simplex and* P *is the center of* Ω.

REMARK 2. According to the symmetry of inequalities (1.2) and (1.3), they can be described respectively as follows.

$$\sum_{i=0}^{n}(R_0^2 + R_1^2 + \cdots + R_n^2 - 2R_i^2)F_i \geqslant (n-1)C(n)\left(\prod_{i=0}^{n} F_i\right)^{\frac{1}{n-1}}, \tag{1.2'}$$

$$\sum_{i=0}^{n}(R_0^2 + R_1^2 + \cdots + R_n^2 - 2R_i^2)F_i^2 \geqslant \frac{n-1}{n+1}n^4V^2. \tag{1.3'}$$

Suppose that $r_i(i = 0, 1 \cdots, n)$ is the radius of the ith escribed hypersphere of Ω, h_i the altitude of Ω from the vertex A_i, i.e., the distance from A_i to π_i. According to the known fact [12]

$$r_i = \frac{nV}{F - 2F_i}, \quad h_i = \frac{nV}{F_i},$$

it is clear the inequalities (1.4) and (1.5) can be written in the following equivalent forms:

THEOREM 3. *Suppose that* Ω *is an* $n-$ *simplex with the radii of escribed spheres* $r_0, r_1, \cdots, r_n$ *and with altitudes* $h_0, h_1, \cdots, h_n$, *respectively. Then*

$$\sum_{i=0}^{n} \frac{R_i^2}{h_i r_i} \geqslant \frac{n-1}{n+1}n^2,$$

$$\sum_{i=0}^{n} \frac{R_i^2}{r_i^2} \geqslant \frac{(n-1)^2}{n+1}n^2,$$

and equalities hold if Ω *is a regular* $n-$ *simplex and* P *is the center of* Ω.

This paper, except for the introduction, is divided into three sections. In Section 2, we establish an analytic inequality in E^n, and discuss its applications. In Section 3, we give some polar moment of inertia inequalities to prove above theorems. In section 4, we prove these theorems by using the obtained results from the section 2 and section 3. Finally, we present a conjecture for the distance from any point to every vertices of an $n-$ simplex.

2. An analytic inequality and its applications

In paper [9], to study Pedoe inequality, Leng and Tang proved an inequality as follows:

If $x_i > 0$ $(i = 1, 2, \cdots, n), n \geqslant 2$, $0 < x_i < \frac{1}{2}$ and $\sum_{i=1}^{n} x_i = 1$, then

$$\sum_{i=1}^{n} \prod_{\substack{j=1 \\ j \neq i}}^{n} \left(\frac{1}{x_j} - 2 \right) \geqslant n(n-2)^{n-1} \tag{2.1}$$

Chen, Yong and Xia in [2] gave an elegant proof of the inequality (2.1). Huang [5], Shi [15] and Xu [18] have studied this inequality and its applications. Summing and extending their results, we obtain the following generalization of the inequality (2.1) (Theorem 4).

First, suppose that $x_1, x_2, \cdots, x_n$ are n positive real numbers. We define the k-th elementary symmetric polynomial E_k of $x_1, x_2, \cdots, x_n$ by

$$E_k(x) = E_k(x_1, \cdots, x_n) = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \prod_{j=1}^{k} x_{i_j} \qquad k = 1, \cdots, n.$$

THEOREM 4. *Let k, n be the natural numbers, $k \leqslant n-1$, $x_i > 0$, $i = 1, 2, \cdots, n$, $\sum_{i=1}^{n} x_i = 1$. If $m \leqslant n - k + 1$, then*

$$E_k \left(\frac{1}{x_1} - m, \cdots, \frac{1}{x_n} - m \right) \geqslant \binom{n}{k} (n-m)^k, \tag{2.2}$$

and equality holds if $x_1 = x_2 = \cdots = x_n = \frac{1}{n}$.

Taking $m = 2$ in Theorem 4, we infer the following corollary.

COROLLARY 1. *Let $x_i > 0$, $i = 1, 2, \cdots, n$, $n \geqslant 2$, $\sum_{i=1}^{n} x_i = 1$. Then, for $k \in \{1, 2, \cdots, n-1\}$,*

$$E_k \left(\frac{1}{x_1} - 2, \cdots, \frac{1}{x_n} - 2 \right) \geqslant \binom{n}{k} (n-2)^k, \tag{2.3}$$

and equality holds if $x_1 = x_2 = \cdots = x_n = \frac{1}{n}$.

It should be noted that Corollary 1 is not true if $k = n$ (when $k = n = 3$, there is a counter example).

Taking $k = n - 1$ in Corollary 1, we obtain the following inequality (the condition $0 < x_i < \frac{1}{2}, i = 1, 2, \cdots, n$, have been eliminated).

COROLLARY 2. *Let $x_i > 0$, $i = 1, 2, \cdots, n$, $n \geqslant 2$, $\sum_{i=1}^{n} x_i = 1$, then*

$$\sum_{i=1}^{n} \prod_{\substack{j=1 \\ j \neq i}}^{n} \left(\frac{1}{x_j} - 2 \right) \geqslant n(n-2)^{n-1} \tag{2.4}$$

and equality holds if $x_1 = x_2 = \cdots = x_n = \frac{1}{n}$.

To prove Theorem 4, we need the following lemma.

LEMMA 1. *Let k, n be natural numbers, $n \geqslant 2$, $k \leqslant n - 1$, $m > 0$. If $x_i > 0$, $y_i > 0$, $i = 1, 2, \cdots, n$, $x_1 + x_2 = y_1 + y_2$, $x_j = y_j$, $j = 3, 4, \cdots, n$, then*

$$E_k \left(\frac{1}{x_1} - m, \cdots, \frac{1}{x_n} - m \right) - E_k \left(\frac{1}{y_1} - m, \cdots, \frac{1}{y_n} - m \right)$$

$$= \frac{x_1 + x_2}{x_1 x_2 y_1 y_2} (x_2 - y_1)(y_1 - x_1) E_{k-1} \left(\frac{1}{x_1 + x_2} - m, \frac{1}{x_3} - m, \cdots, \frac{1}{x_n} - m \right). \tag{2.5}$$

Proof. Put $P = E_k \left(\frac{1}{x_1} - m, \cdots, \frac{1}{x_n} - m \right) - E_k \left(\frac{1}{y_1} - m, \cdots, \frac{1}{y_n} - m \right)$. By the definition of E_k, it follows that

$$E_k(x_1, ..., x_n) = x_1 x_2 E_{k-2}(x_3, ..., x_n) + (x_1 + x_2) E_{k-1}(x_3, ..., x_n) + E_k(x_3, ..., x_n), \tag{2.6}$$

$$E_k(x_1, \cdots, x_n) = x_1 E_{k-1}(x_1, \cdots, x_n) + E_k(x_2, \cdots, x_n). \tag{2.7}$$

Applying (2.6) and (2.7), and noting that $y_1 y_2 - x_1 x_2 = (x_2 - y_1)(y_1 - x_1)$, we have

$$P = \left[(\frac{1}{x_1} - m)(\frac{1}{x_2} - m) - (\frac{1}{y_1} - m)(\frac{1}{y_2} - m) \right] E_{k-2} \left(\frac{1}{x_3} - m, \cdots, \frac{1}{x_n} - m \right)$$

$$+ \left[(\frac{1}{x_1} - m) + (\frac{1}{x_2} - m) - (\frac{1}{y_1} - m) - (\frac{1}{y_2} - m) \right] E_{k-1} \left(\frac{1}{x_3} - m, \cdots, \frac{1}{x_n} - m \right)$$

$$= \left(\frac{1}{x_1 x_2} - \frac{1}{y_1 y_2} \right) \left[(x_1 + x_2) \left(\frac{1}{x_1 + x_2} - m \right) E_{k-2} \left(\frac{1}{x_3} - m, \cdots, \frac{1}{x_n} - m \right) \right.$$

$$\left. + (x_1 + x_2) E_{k-1} \left(\frac{1}{x_3} - m, \cdots, \frac{1}{x_n} - m \right) \right]$$

$$= \frac{x_1 + x_2}{x_1 x_2 y_1 y_2} (x_2 - y_1)(y_1 - x_1) E_{k-1} \left(\frac{1}{x_1 + x_2} - m, \frac{1}{x_3} - m, \cdots, \frac{1}{x_n} - m \right). \quad \square$$

Proof of the Theorem 4. We use induction on k.
When $k = 1, n \geqslant m$, according to the *Cauchy* inequality, we obtain

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} \geqslant \frac{n^2}{x_1 + x_2 + \cdots + x_n} = n^2,$$

$$E_1 \left(\frac{1}{x_1} - m, \cdots, \frac{1}{x_n} - m \right) \geqslant n^2 - mn.$$

Hence (2.2) holds for $k = 1$.

Let us assume that (2.2) holds for $k-1 (k \geqslant 2)$, namely, for $n \geqslant (k-1)+m-1 = k+m-2$ $(k \geqslant 2)$, (2.2) holds. We will prove that (2.2) holds for k, namely, for $n \geqslant k+m-1$ $(k \geqslant 2)$, (2.2) holds.

If $x_1 = x_2 = \cdots = x_n = \frac{1}{n}$, then

$$E_k\left(\frac{1}{x_1}-m,\cdots,\frac{1}{x_n}-m\right) = E_k(n-m,\cdots,n-m) = \binom{n}{k}(n-m)^k,$$

(2.2) holds. Otherwise, there exist a real number x_i larger than $\frac{1}{n}$ and a real number smaller than $\frac{1}{n}$ among $x_1,\cdots,x_n$. Without loss of generality, we can assume that $x_1 < \frac{1}{n} < x_2$. Put

$$y_1 = \frac{1}{n}, \quad y_2 = x_1 + x_2 - \frac{1}{n}, \quad y_j = x_j, j = 3,4,\cdots,n.$$

According to Lemma 1, we have

$$E_k\left(\frac{1}{x_1}-m,\cdots,\frac{1}{x_n}-m\right) - E_k\left(\frac{1}{y_1}-m,\cdots,\frac{1}{y_n}-m\right) = \frac{x_1+x_2}{x_1x_2y_1y_2}\left(x_2-\frac{1}{n}\right) \times$$
$$\times \left(\frac{1}{n}-x_1\right)E_{k-1}\left(\frac{1}{x_1+x_2}-m,\frac{1}{x_3}-m,\cdots,\frac{1}{x_n}-m\right). \quad (2.8)$$

Since $n \geqslant k+m-1$, $n-1 \geqslant k+m-2$. By inductive hypothesis, we have

$$E_{k-1}\left(\frac{1}{x_1+x_2}-m,\frac{1}{x_3}-m,...,\frac{1}{x_n}-m\right) \geqslant \binom{n-1}{k-1}(n-1-m)^{k-1} \geqslant 0. \quad (2.9)$$

From (2.8) and (2.9), we infer

$$E_k\left(\frac{1}{x_1}-m,\cdots,\frac{1}{x_n}-m\right) \geqslant E_k\left(\frac{1}{y_1}-m,\cdots,\frac{1}{y_n}-m\right).$$

Suppose that there exists some y_i not equal to $\frac{1}{n}$ among $y_2,\cdots,y_n$, then we continue the above process, and at most $n-1$ times adjustment, we obtain

$$E_k\left(\frac{1}{x_1}-m,\cdots,\frac{1}{x_n}-m\right) \geqslant E_k(n-m,\cdots,n-m) = \binom{n}{k}(n-m)^k.$$

Hence (2.2) holds for k, as desired. $\square$

To prove the theorems of Section 1, we give the following lemma by applying the Theorem 4.

LEMMA 2. *Let* $\Omega = \langle A_0, A_1, \cdots A_n \rangle$ *be an* $n-$ *simplex with facet areas* F_i, $(i = 0, 1, \cdots, n)$, *and let*

$$\lambda_i = \frac{F - 2F_i}{F_i}, \quad \mu_i = \frac{F_0^2 + \cdots + F_n^2 - 2F_i^2}{F_i^2}.$$

Then

$$\sum_{i=0}^{n} \prod_{\substack{j=0 \\ j \neq i}}^{n} \lambda_j \geqslant (n+1)(n-1)^n, \tag{2.10}$$

$$\sum_{i=0}^{n} \prod_{\substack{j=0 \\ j \neq i}}^{n} \mu_j \geqslant (n+1)(n-1)^n, \tag{2.11}$$

and equalities hold if Ω *is a regular* $n-$ *simplex and* P *is the center of* Ω.

Proof. Taking $x_i = \dfrac{F_i}{F} (i = 0, 1, \cdots, n)$ and applying Corollary 2 for these $n+1$ positive real numbers $x_0, x_1, \cdots, x_n$, the inequality (2.10) follows.

Taking $x_i = \dfrac{F_i^2}{F_0^2 + F_1^2 \cdots + F_n^2}$ and applying Corollary 2, the desired (2.11) holds. □

3. The polar moment of inertia inequality in E^n

To prove the theorems in Section 1, we establish the polar moment of inertia inequality in E^n, which are generalizations to several dimensions of the well-known polar moment of inertia inequality of Klamkin [6]. In this section, let $\sum = \{A_i(\lambda_i), i = 0, 1, \cdots, N\}$ be a mass-point system in E^n, the points A_i are endowed with corresponding weights $\lambda_i > 0 \ (i = 0, 1, \cdots, N, \text{ and } N > n)$, respectively.

THEOREM 5. *Let* $\Omega = \langle A_0, A_1, \cdots A_n \rangle$ *be an* $n-$ *simplex, and* $\Omega_i = \langle A_0, \cdots, A_{i-1}, A_{i+1}, \cdots, A_n \rangle$ *be its facet with area* F_i, P *any point in* E^n, $R_i = \|PA_i\|$ $(i = 0, 1, \cdots, n)$. *If* $\lambda_0, \lambda_1, \cdots, \lambda_n$ *are* $n+1$ *real positive constants, then*

$$\left(\sum_{i=0}^{n} \lambda_i \right) \left(\sum_{i=0}^{n} \lambda_i R_i^2 \right)^{n-1} \geqslant n^{n-4}(n!)^2 \left(\prod_{j=0}^{n} \lambda_j \sum_{i=0}^{n} \lambda_i^{-1} F_i^2 \right) \tag{3.1}$$

and the equality holds if the inertial ellipsoid of $\sum$ *is a hyperspere and* P *is the centroid of* $\sum$.

To prove Theorem 5, we need the following Zhang and Yang's inequality for mass-point system, which has played an important role in recent investigations of geometric inequalities of finite point sets [8, 10, 12].

LEMMA 3. (Zhang and Yang [20]). *Let* $\sum = \{A_i(\lambda_i), \ i = 0, 1, \cdots, N\}$ $(N \geqslant n)$ *be a mass-point system in* E^n, *and denote by* $V_{i_0 i_1 \cdots i_k}$ *the* $k-$ *dimensional volume of the* $k-$ *simplex spanned by the points* $A_{i_0}, A_{i_1}, \cdots, A_{i_k}$ *of* $\sum (0 \leqslant i_0 < i_1 < \cdots < i_k \leqslant N)$. *Put*

$$M_k = \sum_{0 \leqslant i_0 < i_1 < \cdots < i_k \leqslant N} \lambda_{i_0} \lambda_{i_1} \cdots \lambda_{i_k} V_{i_0 i_1 \cdots i_k}^2, \quad M_0 = \sum_{i=0}^{N} \lambda_i.$$

If $k, l \in \{0, 1, \cdots, n\}$, $k < l$, then

$$\frac{M_k^l}{M_l^k} \geqslant \frac{[(n-l)!(l!)^3]^k}{[(n-k)!(k!)^3]^l} \cdot (n!M_0)^{l-k}, \tag{3.2}$$

and equality holds if the inertial ellipsoid of $\sum$ is a hypersphere.

For the proof of Lemma 3 the reader is referred to [12]. We also need the following lemma.

LEMMA 4. *Let $A_0, A_1, \cdots, A_N$ be $N + 1$ given points in E^n, $a_{ij} = \|A_i A_j\|$ $(0 \leqslant i < j \leqslant N)$, P any point in E^n, $R_i = \|PA_i\|$ $(i = 0, 1, \cdots, N)$. If $\lambda_0, \lambda_1, \cdots, \lambda_N$ are real positive constants, then*

$$\left(\sum_{i=0}^{N} \lambda_i\right)\left(\sum_{i=0}^{N} \lambda_i R_i^2\right) \geqslant \sum_{0 \leqslant i < j \leqslant N} \lambda_i \lambda_j a_{ij}^2, \tag{3.3}$$

and equality holds if P is the centroid of $\sum$.

For the proof of Lemma 4 the reader is referred to [12].

Proof of Theorem 5. Taking $k = 1$, $l = n-1$, $\lambda_i = m_i > 0$ in the inequality (3.2) of Lemma 3, it implies

$$M_1 = \sum_{0 \leqslant i < j \leqslant n} \lambda_i \lambda_j a_{ij}^2, \quad M_{n-1} = \lambda_0 \lambda_1 \cdots \lambda_n \cdot \sum_{i=0}^{n} \lambda_i^{-1} F_i^2,$$

and

$$\left(\sum_{0 \leqslant i < j \leqslant n} \lambda_i \lambda_j a_{ij}^2\right)^{n-1} \geqslant n^{n-4}(n!)^2 \left(\sum_{i=0}^{n} \lambda_i\right)^{n-2} M_{n-1}. \tag{3.4}$$

Combining Lemma 4 with (3.4), it yields that

$$\left[\left(\sum_{i=0}^{n} \lambda_i\right)\left(\sum_{i=0}^{n} \lambda_i R_i^2\right)\right]^{n-1} \geqslant \left(\sum_{0 \leqslant i < j \leqslant n} \lambda_i \lambda_j a_{ij}^2\right)^{n-1} \geqslant n^{n-4}(n!)^2 \left(\sum_{i=0}^{n} \lambda_i\right)^{n-2} M_{n-1},$$

divided by $(\sum_{i=0}^{n} \lambda_i)^{n-2}$, therefore, the Theorem 5 is proved. $\square$

4. Proofs of the Theorems

Proof of the Theorem 1. Taking $\lambda_i = F_i$ $(i = 0, 1, \cdots, n)$ in Theorem 5, (1.1) follows immediately. Let

$$\sigma = \prod_{j=0}^{n}(F - 2F_j) \cdot \sum_{i=0}^{n} \frac{F_i}{F - 2F_i}.$$

Then (2.10) of Lemma 2 can be written in the following equivalent form:

$$\sigma \geqslant (n+1)(n-1)^n \cdot \prod_{i=0}^{n} F_i. \tag{4.1}$$

Taking $\lambda_i = F - 2F_i (i = 0, 1, \cdots, n)$ in Theorem 5 and noting that $\sum_{i=0}^{n} \lambda_i = (n-1)F$, we infer

$$\left[\sum_{i=0}^{n}(F - 2F_i)R_i^2\right]^{n-1} \geqslant \frac{n^{n-4}n!^2}{n-1} \cdot F^{-1}\left(\prod_{i=0}^{n}(F - 2F_i)\right)\left(\sum_{i=0}^{n}\frac{F_i^2}{F - 2F_i}\right). \tag{4.2}$$

Without loss of generality, we may assume that

$$F_0 \geqslant F_1 \geqslant \cdots \geqslant F_n > 0.$$

Then

$$0 < F - 2F_0 \leqslant F - 2F_1 \leqslant \cdots \leqslant F - 2F_n,$$

it follows

$$(F - 2F_0)^{-1}F_0 \geqslant (F - 2F_1)^{-1}F_1 \geqslant \cdots \geqslant (F - 2F_n)^{-1}F_n > 0.$$

Applying Chebishev's inequality, we get

$$\begin{aligned}
\sum_{i=0}^{n}\frac{F_i^2}{F - 2F_i} &= \sum_{i=0}^{n} F_i[F_i(F - 2F_i)^{-1}] \\
&\geqslant \frac{1}{n+1}\left(\sum_{i=0}^{n} F_i\right)\left(\sum_{i=0}^{n}\frac{F_i}{F - 2F_i}\right) \\
&= \frac{1}{n+1}F \cdot \sum_{i=0}^{n}\frac{F_i}{F - 2F_i}.
\end{aligned} \tag{4.3}$$

From (4.2) and (4.3), we have

$$\left(\sum_{i=0}^{n}(F - 2F_i)R_i^2\right)^{n-1} \geqslant \frac{n^{n-4}n!^2}{n^2 - 1} \cdot \sigma. \tag{4.4}$$

Combining (4.4) with (4.1), we obtain the desired inequality (1.2). $\square$

Proof of the Theorem 2. Let

$$\delta = \prod_{i=0}^{n}\left(F_0^2 + \cdots + F_n^2 - 2F_i^2\right) \cdot \sum_{i=0}^{n}\frac{F_i^2}{F_0^2 + \cdots + F_n^2 - 2F_i^2}.$$

The inequality (2.11) of Lemma 2 can be rewritten in the form

$$\delta \geqslant (n+1)(n-1)^n \prod_{i=0}^{n} F_i^2. \tag{4.5}$$

Taking $\lambda_i = (F_0^2 + \cdots + F_n^2 - 2F_i^2)$ in the in Theorem 5 and applying (4.5), we infer

$$\left[\sum_{i=0}^{n} (F_0^2 + ... + F_n^2 - 2F_i^2)R_i^2 \right]^{n-1} \geq \frac{n!^2 n^{n-4}}{n-1} \cdot \frac{\delta}{F_0^2 + ... + F_n^2}$$

$$\geq \frac{n!^2 n^{n-4}}{n-1} \cdot (n+1)(n-1)^n \cdot \frac{\prod_{i=0}^{n} F_i^2}{\sum_{i=0}^{n} F_i^2}. \qquad (4.6)$$

On the other hand, taking for $N = n+1$, $l = n$, $k = n-1$, and $m_i = \frac{F_i^2}{n+1}$ in Lemma 2, we find

$$\frac{\prod_{i=0}^{n} F_i^2}{\sum_{i=0}^{n} F_i^2} \geq \frac{n^{3n}}{n!^2(n+1)^n} V^{2(n-1)}. \qquad (4.7)$$

Combining (4.6) with (4.7), the desired inequality (1.3) follows.

Now, taking $\lambda_i = F_i(F - 2F_i) > 0$ in Theorem 5 and noting the obvious fact

$$\sum_{i=0}^{n} \lambda_i = F^2 - 2\sum_{i=0}^{n} F_i^2 \leq (n-1) \sum_{i=0}^{n} F_i^2,$$

and applying (4.1) and (4.7), we deduce

$$\left[\sum_{i=0}^{n} F_i(F - 2F_i)R_i^2 \right]^{n-1} \geq \frac{n^{n-4} \cdot n!^2 \cdot \left(\prod_{i=0}^{n} F_i \right) \cdot \sigma}{\sum_{i=0}^{n} \lambda_i}$$

$$\geq \frac{n!^2 n^{n-4} \cdot (n+1)(n-1)^n \cdot \prod_{i=0}^{n} F_i^2}{(n-1)\sum_{i=0}^{n} F_i^2} \geq \left[\frac{n^4(n-1)}{n+1} V^2 \right]^{n-1},$$

the inequality (1.4) of the Theorem 2 is proved.

Further, taking $\lambda_i = (F - 2F_i)^2$ in Theorem 5, noting that

$$\sum_{i-0}^{n} \lambda_i = (n-3)F^2 + 4\sum_{i=0}^{n} F_i^2 \leq (n-1)^2 \cdot \sum_{i=0}^{n} F_i^2$$

and applying *Cauchy* inequality and (4.1) and (4.7), we deduce

$$
\left[\sum_{i=0}^{n} (F - 2F_i)^2 R_i^2 \right]^{n-1} \geqslant \frac{n^{n-4} \cdot n!^2 \cdot \prod_{i=0}^{n} (F - 2F_i)^2 \cdot \sum_{i=0}^{n} \lambda_i^{-1} F_i^2}{\sum_{i=0}^{n} \lambda_i}
$$

$$
\geqslant \frac{n!^2 n^{n-4} \sigma^2}{(n+1)(n-1)^2 \sum_{i=0}^{n} F_i^2}
$$

$$
\geqslant \frac{n^{n-4} \cdot n!^2 \cdot (n+1)(n-1)^{2(n-1)} \cdot \prod_{i=0}^{n} F_i^2}{\sum_{i=0}^{n} F_i^2}
$$

$$
\geqslant \left[\frac{n^4(n-1)}{n+1} V^2 \right]^{n-1}.
$$

We complete the proof of (1.5) of Theorem 2. □

Finally, we can conjecture that the following linear inequality for R_i $(i = 0, 1, \cdots, n)$ is true.

$$
\sum_{i=0}^{n} (F - 2F_i) R_i \geqslant (n-1) n^2 \sqrt{\frac{n}{n+1}} \cdot V.
$$

It is not difficult to prove the above inequality if $n = 2$.

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